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QUADRATIC IRRATIONALS AND SYMMETRIES OF CONTINUED FRACTION

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ABSTRACT

This paper inspects origin of Continued Fractions and some interesting facts correlated to it. We have demonstrated some significant properties related to symmetries of Continued Fraction in Quadratic Irrationals which assists us to understand fascinating results. Finally, we use the theory to examine examples of continued fractions.

Keywords: *Continued Fraction, Quadratic Irrationals, Greatest Common Divisor.*

Ams Mathematics Subject Classification (2000): *11E04, 11R04, 11Y65.*

1. INTRODUCTION AND PRELIMINARIES

The origin of continued fraction is traditionally placed at the time of Euclid's Algorithm; however it is used to find the greatest common divisor of two numbers. By algebraically manipulating the algorithm, one be capable of derive the simple continued fraction of the rational as opposed to find the G.C.D of p and q. It is suspicious whether Euclid or his predecessors actually used this algorithm in such a manner but due to its close relationship to continued fraction. In 1965, continued fraction turned out to be a field in its right through the work of John Wallis (1616 -1703). In his work book Opera Mathematics (1695) Wallis laid some of the essential groundwork for continued fraction. He elucidated how to complete the n-th convergent and discovered some of the familiar properties of convergent. Euler laid modern theory to a great extent in his work De Fractionious continuous (1737). He illustrated that every irrational can be expressed as a terminating simple continued fraction. Lagrange used continued fraction to investigate the values of irrationals roots. He also verified that irrational is periodic continued fraction. The nineteenth century can probably be described as the golden age of the continued fraction.

Some of the more prominent Mathematicians to make contributions to this field include Karl Jacobi, Charles Hermit, Karl FriedrichGauss, AugustineCauchy and Thomas Stieljes. Darren C. Collins [5] discussed the definitions and notations of continued fraction and the development of the subject throughout the history. Continued fractions offer a valuable means of expressing numbers and functions.

In the beginning of 20th century, continued fraction has become more familiar in various other areas. For example, Robert M. Corliss's [4] paper examines the connection between chaos theory and continued fractions. They have also been used in computer algorithms for computing rational approximations to real numbers, as well as for solving Diophantine and Pell's equations. The continued fractions of quadratic surds are periodic according to a theorem by Langrange. Their period have differing types of symmetries. R. A Mollin set down the relationship between symmetry in the continued fraction expansion of a quadratic irrational.

The concept of continuous fraction, which is to be presented in this paper, plays an interesting and important role in certain parts of the Theory of Numbers. After introducing both finite and infinite continued fractions, we consider in detail the infinite continued fractions. In particular we will establish a key result that will tell us that the set of quadratic irrationals can be characterized as the set of numbers with periodic continued fraction to set down the relationship between symmetry in the continued fraction expansion of the quadratic irrational.

Let Q be the field of rational numbers and m be a non square positive integer. Then The set $Q(\sqrt{m}) = \{ a + b\sqrt{m}\,; a, b \in Q \}$ is a filed with respect to addition and multiplication of rational numbers and is called Quadratic irrational field which is a subfield of $\mathbb{R}$. The elements of $Q(\sqrt{m})$are called quadratic irrational numbers. A quadratic

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irrational number is an irrational number which is a root of a quadratic equation $ax^2 + bx + c = 0$; $a, b, c \in z$ and $a \neq 0$. Q. Mushtaq [11] proved that Every real quadratic irrational number can be written uniquely as $\frac{a+\sqrt{n}}{c}$ where n is a non-square positive integer and $a, \frac{(a^2-n)}{c}, c$ are relatively prime integers.

A simple continued fraction is an expression of the form

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + 1/a_{3+\dots}}}$$

With $a_0 \in Z$ and $a_n \in N$ for $n \geq 1$.

In general the continuous fraction will mean simple continued fraction unless otherwise stated. In order to represent continuous fraction in a more concise way we will use following notations: $a_0 + \frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \cdots$or $< a_1, a_2, a_3, a_4 \dots >$. The expression for a continuous fraction can either be finite or infinite number of terms. A number represented in a continued fraction is rational iff the continued fraction is finite.

1.1: Definition: Let X $=< a_1, a_2, \dots, a_n >$ and $0 < n \leq N. Then < a_1, a_2, \dots, a_n >$ is called the n-th convergent to the continued fraction thus$< a_1 >$,is called first convergent $< a_1, a_2 >$is the second convergent,$< a_1, a_2, a_3 >$ is the third convergent to the continued fraction and $< a_1, a_2, \dots, a_n >$ is called the n-th convergent of X.

1.2: Definition: For a given continued fraction$< a_1, a_2, a_3, a_4 \dots >$. Let: $\begin{pmatrix} A_0 & A_{-1} \\ B_0 & B_{-1} \end{pmatrix} = \begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix}$and define A_n and B_n recursively via $\binom{A_n}{B_n} = \begin{pmatrix} A_{n-1} & A_{n-2} \\ B_{n-1} & B_{n-1} \end{pmatrix} \binom{a_n}{1}$.

$\frac{A_n}{B_n}$is the result of evaluating the first of n terms of the continued fraction and is called the nth convergent. Let $\frac{A_m}{B_m}$ be the nth convergent of a continued fraction and suppose that the convergent converges to D then$\frac{A_0}{B_0}, \frac{A_2}{B_2}, \frac{A_4}{B_4}, \dots$ forms an increasing sequence less than D.$\frac{A_1}{B_1}, \frac{A_3}{B_3}, \frac{A_5}{B_5}, \dots$ forms an decreasing sequence greater than D.

1.1: Example: Let us evaluate$< -3, 2, 4, 5, 2 >$. The calculations are indicated by the following convenient table I. The values of a_0, a_1, a_2, a_3 and a_4 are exhibited in the second row of table 1. After the values of $A_{-2}, A_{-1}, B_{-2}$and B_{-1} have been inserted.

k	-2	-1	0	1	2	3	4
a_n			-3	2	4	5	2
A_n	0	1	-3	-5	-23	-120	-263
B_n	1	0	1	2	9	47	103

Table: 1 Convergent of Continued Fraction

The Values of all the remaining $A's$ are successively computed by the formula $A_n = a_n A_{-1}, A_{n-2}$and similarly for $B's$. Then by corollary we have that$< -3, 2, 4, 5, 2 > = \frac{A_4}{B_4} = \frac{-263}{103}$.For each integer $K \geq 0$. A_k and B_k are relatively prime.The value of a continued fraction lies between any two consecutive convergent is odd and the other even.

1.3: Definition: A continued fraction $< a_0, a_1, a_2, a_3 \dots a_N >$ is symmetric if$a_1 = a_N$, $a_2 = a_{N-1}$, $a_3 = a_{N-2}$ and so on.Let $< a_1, a_2, \dots a_n, a_n, \dots, a_1 >$ be a symmetric continued fraction with even number of partial quotients .Then P_{2n} $=p_n^2 + p_{n-1}^2$ and $q_{2n-1} = q_n^2 + q_{n-1}^2$.

1.4: Definition: A quadratic irrational is called purely periodic if the point where it begins to be periodic is the first term. For example $\sqrt{3}= [1, \overline{1, 2}]$, where the bar means this part keep repeating, so $\sqrt{3}$ is periodic, but not purely periodic where as $\sqrt{3}$+1 $= [\overline{1, 2}]$ is purely periodic.

A quadratic irrational α is called a reduced if $\alpha > 1$ and $-1 < \bar{\alpha} < 0$, where $\bar{\alpha}$ is the conjugate ate of α.The simple continued fraction of the quadratic irrational α is purely periodic if and only if α is reduced. Furthermore if α is reduced and α $=< a_0, a_1, a_2, \dots a_n >$ then the continued fraction of $-\frac{1}{\alpha} =< a_n, a_{n-1}, \dots, a_0 >$. A quadratic form $ax^2 + bxy + cy^2$ is reduced iff$ax^2 + bx + c$ is the minimal polynomial of a reduced number.

1.5: Definition: The length of a continued fraction is defined as the number of terms in a period of a periodic infinite continued fraction. It is only defined when the number represented is a quadratic irrational. We denote this function by $l(\alpha)$.

1.6: Definition: A reduced quadratic irrational γ is said to have pure skew- symmetric period if $\alpha = \langle \overline{q_0, q_1, q_2, \dots, q_{l-1}} \rangle$where $q_j = q_{l-j}$ for $j = 1,2,3, \dots, l-1$. In other words $q_1, q_2, \dots, q_{l-1}$ is a palindrome.

Let $\alpha = \langle \overline{q_0, q_1, q_2, \dots, q_{l-1}} \rangle$, a reduced quadratic irrational where $D > 0$ is a radicand. Then γ is said to have pure symmetric period if $q_j = q_{l-j-1}$ for all integer with $0 \le j \le l-1$.In other words,$q_0, q_1, \dots, q_{l-1}$ is a palindrome.

CONTINUED FRACTION ALGORITHM

Let α be a real number. We construct continued fraction as follows.To beginlet $\gamma = \alpha$, then follow these steps repeatedly:

i. Set an equal to$\lfloor \gamma \rfloor$, where $\lfloor \gamma \rfloor$ is the floor function of γ.
ii. $s = \gamma - \lfloor \gamma \rfloor$
iii. Set $\gamma = s^{-1}$

And go to step i. This process stops if $s = 0$ and in this case the continued fraction is finite and α is rational. For irrational numbers this process continues indefinitely, but may be periodic.

INFINITE CONTINUED FRACTION

We will be more concerned with infinite continued fraction and particularly with the CF that represent quadratic irrationals that is the number of the form $\frac{P+\sqrt{D}}{Q}$, where P and Q are integers and D is non square positive numbers and $Q \,/\, D - P^2)$ i.e. $P^2 \equiv b(modQ)$.

1.1: Theorem: Let α be a quadratic irrational, then the integers P, Q and D such that $\alpha = \frac{P+\sqrt{D}}{Q}$where $Q \neq 0$, $D > 0$, D is not a perfect square, and $Q \,/\, (D - P^2)$.

Recursively define: $\alpha_k = \frac{P_k+\sqrt{D}}{Q_k}$, $D = P^2{}_{i+1} + Q_i\, Q_{i+1}$

$$a_k = \lfloor \alpha_k \rfloor,\ P_{k+1} = q_k Q_k - P_k\ ,\ Q_{k+1} = \frac{D - P^2{}_{k+1}}{Q_k}$$

For $K = 0, 1, 2, \dots$ then $\alpha =< a_0, a_1, a_2, a_3 \dots >$

1.2: Example: Let $\alpha = \frac{6+\sqrt{28}}{4}$.

We set P_0 =6, $Q_0 = 4$ and $D = 28$.

Hence $a_k = \lfloor \alpha_k \rfloor = 2$ and

$P_1 = 2.4 - 6 = 2,$	$\alpha_1 = \frac{2+\sqrt{28}}{6}$
$Q_1 = \frac{28-2^2}{4} = 6,$	$a_1 = \left\lfloor \frac{2+\sqrt{28}}{6} \right\rfloor = 1$
$P_2 = 1.6 - 2 = 4,$	$\alpha_2 = \frac{4+\sqrt{28}}{2}$
$Q_2 = \frac{28-4^2}{6} = 2,$	$a_2 = \left\lfloor \frac{2+\sqrt{28}}{2} \right\rfloor = 4$
$P_3 = 4.2 - 4 = 4,$	$\alpha_3 = \frac{4+\sqrt{28}}{6}$
$Q_3 = \frac{28-4^2}{2} = 6,$	$a_3 = \left\lfloor \frac{4+\sqrt{28}}{6} \right\rfloor = 1$
$P_4 = 1.6 - 4 = 2,$	$\alpha_4 = \frac{\sqrt{28}}{4}$
$Q_4 = \frac{28-2^2}{6} = 4,$	$a_4 = \left\lfloor \frac{2+\sqrt{28}}{4} \right\rfloor = 1$
$P_5 = 1.4 - 2 = 2,$	$\alpha_5 = \frac{\sqrt{28}}{6}$
$Q_5 = \frac{28-2^2}{4} = 6\ ,$	$a_5 = \left\lfloor \frac{2+\sqrt{28}}{6} \right\rfloor = 1$

And so on, with repetition, because $P_1 = P_5$ and $Q_1 = Q_5$.

Hence we see that $\frac{6+\sqrt{28}}{4} = [2;\ \overline{1,4,1,1}, \dots]$.

2. PURE SKEW SYMMETRIC CONTINUED FRACTION

In this section we illustrate the concept of pure symmetric continued fraction.

2.1: Theorem [12]: Let $\alpha = \frac{P+\sqrt{D}}{Q}$ be a reduced quadratic irrational where $D > 0$ is a radicand and $l = l(\alpha)$. Then the following are equivalent:

i. α has a skew symmetric period.
ii. For all $j \in N$ with $j \le l-1$, $\alpha_{l-j+1}\,\alpha_j' = -1$.
iii. $\alpha\alpha_1 = -1$.
iv. If $P = P_0$ and $Q = Q_0$ in the simple continued fraction expansion of α, then $D = P^2{}_0 + Q_0\,Q_1$.

Proof: If α has pure skew symmetric period, then for all natural numbers $j \le l-1$, we have

$q_j = q_{l-j}$. Thus, for each j,

$\acute{\alpha}_j = \langle \overline{q_j,\ q_{j+1}, \dots q_{l-1}, q_0,\ q_{1,} \dots, q_{j-1}} \rangle$

$= \langle \overline{q_{l-j},\ q_{l-j-1}, \dots q_1, q_0,\ q_{l-1,} \dots, q_{l-j+1}} \rangle$

$1/\alpha_j{}' = \langle \overline{q_{l-j+1},\ q_{l-j+2}, \dots q_0, q_1, \dots, q_{l-j}} \rangle = \alpha_{l-j+1}$

Hence for all such j, $\alpha_{l-j+1}\acute{\alpha}_j = -1$

So part (i) implies part (ii). Next, assume part (ii). Then, in particular, for $j = 1$ we have part (iii), since $\alpha_l = \alpha_0 = \alpha$.

Now assume (iii) holds. Thus

$$-1 = \alpha_0\alpha_1' = \left(\frac{P_0+\sqrt{D}}{Q_0}\right)\left(\frac{P_1-\sqrt{D}}{Q_1}\right)$$

Multiplying numerator and denominator by $P_1 - \sqrt{D}$, we get

$$-1 = \alpha_0\alpha_1' = \left(\frac{P_0+\sqrt{D}}{Q_0 Q_1}\right)\left(\frac{P_1^2 - D}{P_1+\sqrt{D}}\right) = -\left(\frac{P_0+\sqrt{D}}{P_1+\sqrt{D}}\right)$$

So $P_0 = P_1$.

2.1. Example: Let $D = 385$ and $\alpha = 7+\sqrt{385}|14$, then the simple continued fraction data for α is given in table 2

j	0	1	2	3	4	5	6	7	8	9
P_j	7	7	17	19	17	15	15	17	19	17
Q_j	14	24	4	6	16	10	16	6	4	24
q_j	1	1	9	6	2	3	2	6	9	1

Table: 2 Pure Skew-Symmetric

Since $q_1 q_2 \dots q_9 = 196232691$ is a palindrome, where $l = 10$, then $\alpha\gamma$ has pure skew-symmetric period and $D = 385 = P_0^2 + Q_0 Q_1 = 7^2 + 14.24$.

$$\alpha\alpha_1' = \left(\frac{7+\sqrt{385}}{14}\right)\left(\frac{7-\sqrt{835}}{24}\right) = \frac{7^2-385}{336} = -1$$

And finally $\alpha_{l-j+1}\alpha_j' = -1$ for any natural number $\alpha \le 9$.

3. PURE SYMMETRIC CONTINUED FRACTION

No we deal with the pure symmetric continued fraction and find its result on quadratic irrationals.

3.1 Theorem [12]: Let $\alpha = P + \sqrt{D}\big|Q = \langle\overline{q_0, q_1, \dots, q_{l-1}}\rangle$ be a reduced quadratic irrational with radicand $D > 0$, then the following are equivalent

i. α has pure symmetric period
ii. $\alpha\alpha' = -1$
iii. $D = P^2 + Q^2$
iv. For any $j \in Z$ with $0 \le j \le l-1$, we have $\alpha_j' \alpha_{l-j} = -1$

3. 1 Example: Let $D = 221$ and $\alpha = 11 + \sqrt{221}\big|10$, then the simple continued data for α is given in table 3

j	0	1	2	3
P_j	11	9	5	9
Q_j	10	14	14	10
q_j	2	1	1	2

Table: 3 Pure Symmetric Period

Thus $\alpha = \langle\overline{2; 1, 1,2}\rangle$ has pure symmetric period. Furthermore,

$$\alpha\alpha_1' = \left(\frac{11+\sqrt{221}}{10}\right)\left(\frac{11-\sqrt{221}}{10}\right) = -1$$

And $D = 221 = 11^2 + 10^2$

3.1 Corollary: Let $\alpha = \frac{P+\sqrt{D}}{Q}$ be a reduced quadratic irrational with $l = l(\alpha)$
Then the following are equivalent

i. α has a skew symmetric period.
ii. $If\ l$is even, then $\alpha_{l/2}$ has pure symmetric period.
iii. $If\ l$is odd , then $\alpha_{(l-1)/2}$ has pure skew symmetric period.

Proof: If α has pure symmetric period, then by part (iv) of Theorem (3.1) $\alpha_j' \alpha_{l-j} = -1$. For any natural number $j \le l-1$. In particular, if l is even, then $\alpha_{l/2}\acute{\alpha}_{l/2} = 1.$ So $\alpha_{l/2}$has pure symmetric period. If l is odd, then $\alpha_{(l-1)/2}\acute{\alpha}_{(l+1)/2}$ = -1. Therefore,$\alpha_{(l-1)/2}$ has pure skew – symmetric period. Conversely, assume that part (ii) holds. In this case l is even,

$$\alpha_{l/2} = \langle\overline{q_{l/2}, q_{l/2+1}, \dots q_{l-2}, q_{l-1}, q_0, q_1, \dots, q_{l/2-2}, q_{l/2-1}}\rangle$$

Therefore, $q_{l/2} = q_{l/2-1}$, $q_{l/2+1} = q_{l/2-2}, \dots . q_{l-2} = q_1, q_{l-1} = q_0,$

Namely $q_j = q_{l-j-1}$ for all integers $j = 0, 1,2,3, \dots, . l-1.$ In other words, α has pure symmetric periods. If l is odd, then $\alpha(_{l-1)/2} = \langle\overline{q_{(l-1)/2}, q_{(l+1)/2}, \dots q_{l-2}, q_{l-1}, q_0, q_1, \dots, q_{(l-5)/2}, q_{l-3/2}}\rangle$

Therefore, $q_{l+1/2} = q_{l-3/2}$, $q_{l+3/2} = q_{l-5/2}, \dots . q_{l-2} = q_1, q_{l-1} = q_0,$or $q_j = q_{l-j-1}$ for all integers $j = 0, 1,2,3, \dots, . l-1,$ so α has pure symmetric period.

We conclude the paper by following remark, looking at Example (3.1) again, $\alpha_{(l+1)/2} = \alpha_3 = \langle\overline{2; 1, 1, 1,2}\rangle$has pure symmetric period, and in Example (2.1) $\alpha_{l/2} = \alpha_3 = \langle\overline{1; 1, 9, 6, 2, 3, 2, 6, 9, 1}\rangle$ has pure skew symmetric period. A natural question is,if there exist reduced quadratic irrationals that have both pure and pure skew symmetric periods? The answer is yes, but they are very extraordinary. If both occur simultaneously for a given reduced quadratic irrational α, then we must have $D = P_0^2 + Q_0^2 = P_1^2 + Q_0Q_1$, So $P_0 = P_1$, $Q_0 = Q_1$ and $q_0 = q_1$Hence $l(\alpha) = 1$. One can check example:$D = 650 = 2.5^2.13$and $\alpha = 25 + \sqrt{650}\big|5$, then the simple continued data for α has $l = 1$ with $\alpha = \langle\overline{10}\rangle$, and $D = 25^2 + 5^2 = P_0^2 + Q_0^2$ and Let $D = 26 = 5^2.1$, then $\alpha = 5 + \sqrt{26} = 5^2.1$, so$\alpha = 5 + \sqrt{26} = \langle\overline{10}\rangle$.

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