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ON SOME PLANAR GRAPHS RELATED TO KNOTS AND LINKS<br>ElRokh, Ashraf*<br>Department of Mathematics, Faculty of Science, Menoua University, Egypt

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#### Abstract

In this paper ,we study planar graphs associated with knots and links when the number of vertices $n=8$. We find the equivalence classes of non -isomorphic but minimal types related to the same link structure.


Keywords: Graphs ,Knots, Links,

## 1. INTRODUCTION

In [1] we studied the equivalence problem when $\mathbb{I} \quad$. Take for a given knot K or a link L a special projection into a plane such that there are only double points as multiple points. The vertices of the graphs $\Gamma(\mathrm{L})$ are the point of the projection and the edges are the arcs between them. In the sequel we will use in general the word link and specify explicitly to a knot if the considered graph-theoretically object is indeed a 1 - component link, i.e. a knot. Of course a link can have very different associated graphs, but the essential point of interest is tofind the simplest one. Let us call a planar graph $\Gamma$ a minimal link-graph iff there is a link L with $\Gamma=\Gamma(\mathrm{L})$ and there is no other graph $\Gamma^{\prime}$ with a smaller number of vertices which also represent the same link $L$, this means that the $\Gamma^{\prime}$ one gets as a projection of $L$ too.
n the so- called lists of prime knots or prime links there are minimal diagrams of prime knots and prime links for the first number of crossings. A link-diagram means a plane projection of the link saying with respect to the projection point which part of the space curve is over or under the just crossed part. It clear that every such list (Atlas) gives minimal link-graphs if we interpret the under- and over- crossings as vertices. Harary found examples for link-graphs which are not isomorphic but are associated to the same link. We will study this question in a systematic manner to find link-graphs which are non-isomorphic associated graphs for links with crossing numbern $=8$.

Which graphs Гcan be link-graphs?
A graph $G$ is a underlying graph of a link $L$ (link graph) if and only if $G$ has the following properties:

1. $G$ is a finite connected graph
2. G is planar.
3. $G$ has the homogeneous vertex degree 4.

For minimal link-graphs there are further restrictions!
4. The graph has no loops!
5. In the graph can not be any bridge point! This means a vertex such that the deletion of it and its edges makes the remainder graph disconnected.


From this follows that the following graphs with properties 1.-3.are not minimal link-graphs.

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Remark: The properties 1.-5.are necessary for a graph to be a minimal link graph. Otherwise in case of vertex numbers $\mathrm{n} \leq 8$ all graphs of these properties are minimal Link-Graphs. We have no proof that holds for arbitrary vertex number.

In graph theory some polynomial invariants are know. Via the notation of a minor of graph G.W. Tutte[12] discovered a fruitful polynomial $\mathrm{T}(\mathrm{G} ; \mathrm{x}, \mathrm{y}$ ) of two variable $\mathrm{x}, \mathrm{y}$ which bears now his name.

Let $G=(v, E)$ be a given graph and e any arbitrary one of its edges. $G$ e denoted the reduction of $G$ with respect to $e$. This means that one has to delete e from $G$ whereby the remainder will not be changed.
$G e$ denoted the contraction of $G$ with respect to $e$. This means that one has to contract the edge onto one of its vertices.
The following recursive formula holds:

1. $T(G: x, y)=1$ for $G(v, E)$ with $E$ empty.
2. $T(G: x, y)=x T(G \backslash e ; x, y)$ for a bridge $e$.
3.T $(G: x, y)=y T(G \backslash e ; x, y)$ for a loop $e$.
3. $T(G: x, y)=T(G / e ; x, y)+T(G \backslash e ; x, y)$ otherwise.

El-Rokh[1] studied planar graphs with a vertex number $n \leq 7$ associated with knots links to find equivalence classes on non- isomorphic nut minimal types related to the same link structurer, and showed that :

Proposition: 1 Every link-graph with $n$ vertices has $2 n$ edges and $n+2$ countries
Proof. [1]
Proposition: 2 Every link-graph with 5 vertices has at least one multiple edge.
Proof. [1]
Remark: For the convenience of the reader now we list the minimal graphs for the number of vertices $n=3,4$, 5,6 and 7. For the corresponding konts and links we use the enumeration after the list in [1]. We remember the already known Harary example of non-isomorphic minimal kont-graph for the knot $3_{1} \cup 4_{1}$.: What's about link graphs without any multiple edge? Also knot-graphs can occur.

## 2. PRELIMINARIES

Basic definitions:
Definition: 1. Let K be a polygonal knot and let $\pi$ be a projection of K into the plane such that:
(i) The only multiple points of $\pi \mathrm{K}$ are double points, and the number of them is finite.
(ii) No double point of $\pi \mathrm{K}$ is the image of a vertex of $K$, and (iii) No projection [meeting conditions (i) and (ii)] of a knot equivalent to K has fewer double point than $\pi \mathrm{K}$.

Condition (i) and (ii) means that $K$ is in "regular position" (see[13] for terminology), while condition (iii) implies $\pi K$ is "minimal".

Definition: 2. A subspace $K \subseteq R^{\wedge}(3)$ is a knot if it is the image of a smooth injective map $f: S \rightarrow R^{3}$, with df/d $\boldsymbol{d}$ never zero.(smooth means $\mathbf{d}^{\mathrm{n}}$ fd $\theta^{\mathrm{n}}$ exists for all n )

Definition: 3. A link L is an embedding of a topological sum offinitely many copies of a circle $\mathrm{S}^{1}$ into three dimensional topological space $R^{3}, L: S^{1} \sqcup S^{1} \sqcup S^{1} \ldots \sqcup S^{1} \rightarrow R^{3}$. The restriction of $L$ to one of the copies of $S^{1}$ is called a component of $L$.

## Examples:


un-link $=S^{1} \ldots$ U $S^{1} 2$ - chainBorromean rings
Note that each component of a link is a Knot.
Definition: 4. A diagram of a knot or link is a projection of the latter into a plane with marking of each crossing (under-crossing or over crossing) in the image of the projection. That means a knot diagram is a picture of a projection of a knot onto a plane, a diagram in $\mathrm{R}^{2}$, is made up of a number of arcs and crossings. At a crossing one arc is the over pass and the other to make up an under pass. and it not allowed the following:


Definition: 5. Graph of a diagram of a knot or link is the graph consisting of the crossings as the vertices of the graphs and the arcs between two crossings as the edges. With other words:

A graph $\Gamma$ is a link-graph (knot-graph) if there is a link $L$ (a knot $K$ )such that a suitable projection ofL(ofK) give $\Gamma$. We write $\Gamma(\mathrm{L})$ resp. $\Gamma(\mathrm{K})$.

## Definition: 6. Isomorphic graphs

Two graphs $G$ and $H$ are said to be isomorphic (written $G \sim H$ ) if there are bijec- tions $\theta: V(G) \rightarrow V(H)$ and $\phi: E(G) \rightarrow E(H)$ such that $\psi G(e)=u v$ if and only if $\psi H(\phi(e))=\theta(u) \theta(v)$, such a pair $(\theta, \phi)$ of mappings is called an isomorphism between G and H .

## Definition: 7. Equivalent knots

A knot $K_{0}$ is an equivalent to a knot $K n$ if there exists a sequence of knots $K_{1}, K_{2}, \ldots, K_{n-1}$ such that $K_{i}$ is an ele mentary deformation of $K_{i-1}$ for $1<i \leq n$.

## Definition: 8: Prime knot[11]

Prime knot is a indecomposable Knot, in the following sense, it is a non-trivial knot which cannot be written as the knot sum of two non-trivial knots. Knots that are not prime are said to be composite. It can be a nontrivial problem to determine whether a given knot is prime or not.A nice family of prime knots are the torus knots. These are formed by wrapping a circle around a tours $p$ times in one direction and $q$ times in the other direction, where $p$ and $q$ are co-prime integers. The simplest prime knot is the trefoil with three crossing. The trefoil is actually a $(3,2)$ tours knot. Thefigure -eight knot, with four crossing, is the simplest non-tours knot.

Definition: 9: Prime link[9], [10]
A prime link is a link that cannot be represented as a knot sum of other links.Doll and Hoste(1991) list polynomials for oriented links of nine or fewer crossings, and Rolfsen (1976) gives a table of links with small numbers of components and crossings.

Definition: 10: Alternating knot
An alternating knot is a knot which possesses a knot diagram in which crossings alternating between under-and overpasses. Not all knot diagram of alternating knots need be alternating diagram. The trefoil knot and figure-eight knot are alternating knots, as are all prime knots with seven or fewer crossings.

Definition: 11: Reidmeister moves [8], [7]
In the mathematical area of knot theory, a Reidemeister moves refers to one of three local moves on a link diagram. In 1926, Kutt Reidemeister [7] and independently, in 1927 J.W. Alexander and G.B. Briggs [8], demonstrated that two knots diagrams belonging to the same Knot, up to planar is otopy, can be related by a sequence of three Reidemeister moves. Each moves operates on a small region of the diagram and it one of three types:

1. Twist and untwist in either direction
2. Move one loop completely over another
3. Move a string completely over or under a crossing


Type IType II Type III
Definition: 11. $r$ - related link graphs (representation related)
$\Gamma_{1} \sim^{r} \Gamma_{2}$ : There are links $L_{1}$ and $L_{2}$ with $\Gamma_{1} \simeq \Gamma\left(L_{1}\right), \Gamma_{2} \simeq \Gamma\left(L_{2}\right)$ and $L_{1} \simeq L_{2}$.
This means: $\Gamma_{1}$ and $\Gamma_{2}$ occur as link-graphs for the same link structure. This relation is to wide ! namely, every two knot graphs would be r- related !

We have to make the relation somewhat finer. We claim :

1. version: $\Gamma_{1}$ or $\Gamma_{2}$ should be minimal representations.
2. version: $\Gamma_{1}$ and $\Gamma_{2}$ should be minimal representations.

The r- relation in version 2 means: We are interested in all minimal representations for the same link structure .
The r- relation in version 2 is an equivalence relation.
The r - relation in version 1 is in general not transitive.
If the equivalence class of the r-relation in version 2 is for a knot K or a link L an 1-point set then the knot graph $\Gamma$ (or the link graph) in a minimal representation for K (or L ) is a knot(link) invariant.

For a crossing numbers $\mathbf{n} \leq 7$ in the most cases a graph in a minimal representation of a knot (link) is an invariant of the $\operatorname{knot}(\operatorname{link})$.

## 3. MAIN RESULTS

For $\mathrm{n}=8$ we deduced the possible link-graphs by inspecting the adjacency matrices and application of isomorphism arguments and decision of planarity.

To obtain the type of link graphs we look for the adjacency matrix. The adjacency matrix of the link graph is a square matrix of size $n \times n$ which is symmetric, has integer values and for every row every column the sum is 4 and the elements of the main diagonal are zero.

The first row of the adjacency matrix could be one of the following:
02010001, $02200000,01010101,01000201,03000001$, or 01111000 Forthe firstpossibility 02010001 , the second row will be 20200000, 20011000, 20101000, 20010100, 20100001 or 20100100. Now the third rowcorre-sponding to row (1): 02010001, and row(2): 20200000 will be $02010001,02020000,02011000$ or 02001010 . Going the same way the fourth row will be 10102000,10101100 or 10100101 . Our choice of the first four rows yields the following linear system of equations:
$x+y+z=2$
$x+u+v=4$
$y+u+w=4$
$z+v+w=2$

| 0 | 2 | 0 | 1 | 0 | 0 | 0 | 1 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 2 | 0 | 2 | 0 | 0 | 0 | 0 | 0 |
| 0 | 2 | 0 | 1 | 0 | 0 | 0 | 1 |
| 1 | 0 | 1 | 0 | 2 | 0 | 0 | 0 |
| 0 | 0 | 0 | 2 | 0 | x | y | z |
| 0 | 0 | 0 | 0 | x | 0 | u | v |
| 0 | 0 | 0 | 0 | y | u | 0 | w |
| 1 | 0 | 1 | 0 | z | v | w | 0 |

Whose general solution is $x=4-u-v, y=v, z=u-2, w=4-u-v$ where $u, v$ are arbitrary integers. Our constraints on adjacency matrix show that we have onlyfinite number of solution. We study all the cases:
$u=2, v=0 \Rightarrow x=2, y=0, z=0$
and $w=0$
$u=3, v=1 \Rightarrow x=0, y=1, z=1$
and $w=0$
$u=2, v=1 \Rightarrow x=1, y=1, z=0$
and $w=1$
$u=2, v=2 \Rightarrow x=0, y=2, z=0$
and $w=0$
Repeating this method with the other possibilities, we get the list off all non-isomorphic link graphs as follows:

## For a knot-graph:



Fig ( $\left.8_{5}\right)$ Fig $\left(8_{6}\right)$ Fig $\left(8_{7}\right)$ Fig $\left(8_{8}\right)$

$\operatorname{Fig}\left(8_{9}\right)$ Fig $\left(8_{10}\right) F i g\left(8_{11}\right) F i g\left(8_{12}\right)$

$\operatorname{Fig}\left(8_{13}\right)$ Fig $\left(8_{14}\right) \operatorname{Fig}\left(8_{15}\right)$ Fig $\left(8_{16}\right)$

$\operatorname{Fig}\left(8_{17}\right) \operatorname{Fig}\left(8_{18}\right) \operatorname{Fig}\left(8_{19}\right) \operatorname{Fig}\left(8_{20}\right)$

## For a link-graph:


$\operatorname{Fig}\left(8{ }_{5}^{2}\right) F i g\left(8{ }_{6}^{2}\right) F i g\left(8{ }_{7}^{2}\right) F i g\left(8{ }_{8}^{2}\right)$

$\operatorname{Fig}\left(8{ }_{9}^{2}\right) \operatorname{Fig}\left(8{ }_{10}^{2}\right) \operatorname{Fig}\left(8_{11}^{2}\right) \operatorname{Fig}\left(8{ }_{12}^{2}\right)$

$\operatorname{Fig}\left(8{ }_{13}^{2}\right) F i g\left(8{ }_{14}^{2}\right) F i g\left(81_{1}^{3}\right) F i g\left(8{ }_{2}^{3}\right)$


Fig $\left(8{ }_{3}^{3}\right)$ Fig $\left(8{ }_{4}^{3}\right)$ Fig $\left(8{ }_{5}^{3}\right)$ Fig $\left(8{ }_{6}^{3}\right)$
Now we ask our main question: Which of these minimal link-graphs give the same link but are not isomorphic? With other words we need to find the r-equivalence classesof the link-graphs:

Harary [2] showed that the graph of a knot is not a knot invariant give some examples. We give some generalization to graphs of knots and links for the casen $=8$, we have the prime knots $8_{6}, 8_{12}, 8_{8}, 8_{14}, 8_{11}$ have exactly 2 minimal representation graphs as follows:

1-

$\operatorname{Fig}\left(G_{1}\right) \operatorname{Fig}\left(G_{2}\right)$
These graphs are not isomorphic, since the first graph has 4 double edges while the secondgraph has 5 double edges. Clearly, we see that $G_{1}, G_{2}$ have the same Tutte polynomial .

The r-equivalence class consists of the two non-isomorphic graphs for the same knot $8_{6}$ structure.
2-


These graphs are not isomorphic, since the first graph has 3 double edges while the second graph has 4 double edges.
The r-equivalence class consists of the three non-isomorphic graphs for the same knot $8_{12}$ structure.
3-


These graphs are not isomorphic graphs for the same reason above. Namely, they have different number of multiple edges.

The r-equivalence class consists of the two non-isomorphic graphs for the same knot $8_{8}$ structure.

4-


These graphs are not isomorphic, since the first graph has 2 double edges while the second graph has 3 double edges.
The r-equivalence class consists of the three non-isomorphic graphs for the same knot $8{ }_{14}$ structure.

5-

e graphs are not isomorphic, since the first graph has 3 double edges while the second graph has 4 double edges.
The r-equivalence class consists of the two non-isomorphic graphs for the same knot $8_{11}$ structure.
And the other wise for another prime knots have a unique representation graph.

## For a link:

The prime links $8{ }_{9}^{2}$, and $8{ }_{12}^{2}$, have exactly 2 minimal representation graphs, but the primelinks $8{ }_{5}^{2}$, and $8{ }_{7}^{2}$, have exactly 3 minimal representation graphs, finally the prime link $8 \frac{2}{8}$, have exactly 4 minimal representation graphs as follows:

1-


These graphs are not isomorphic for the same reason above, namely they have different number of multiple edges.
The r-equivalence class consists of the two non-isomorphic graphs for the same knot $8{ }_{9}^{2}$ structure.

## 2-



These graphs are not isomorphic, we notice that they have 3 double edges but the lengths of paths between vertices adjacent to double edges are different.

The r-equivalence class consists of the two non-isomorphic graphs for the same knot $8{ }_{12}^{2}$ structure.

## 3-



While 2 graphs have 4 double edges, they are not isomorphic as there are two adjacent double edges in first but no adjacent double edges exist in the second graph.

The r-equivalence class consists of the three non-isomorphic graphs for the same knot8 ${ }_{6}^{2}$ structure.

4-


The r-equivalence class consists of the three non-isomorphic graphs for the same knot $8{ }_{5}^{2}$ structure.
5-


The r-equivalence class consists of the three non-isomorphic graphs for the same knot $8{ }_{7}^{2}$ structure.

6-


Four graphs are not isomorphic in pairs. All the graphs have 2 double edges. For the first and fourth they have vertices on the double edges whose path lengths are 5 and 4 respectively, the second and third don't satisfy such property. We conclude the first and fourth graphs are not isomorphic to each other and not isomorphic to the second and third, to show that the second and third are not isomorphic, we notice that lengths of path between the two double edges of both of them are 3 but the second graph has an edge connecting the two double edges, while the third graph has no such edge. Hence they are not isomorphic.

The r-equivalence class consists of the four non-isomorphic graphs for the same knot $8{ }_{8}^{2}$ structure.
The Converse is also true as shown in [2], [7].

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