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FURTHER RESULTS ON HARMONIC MEAN GRAPHS

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ABSTRACT

A Graph G = (V, E) with p vertices and q edges is called a Harmonic mean graph if it is possible to label the vertices $x \in V$ with distinct labels f(x) from 1, 2… q+1 in such a way that when each edge e=uv is labeled with f(uv) = $\left\lceil \frac{2f(u)f(v)}{f(u)+f(v)} \right\rceil (or) \left\lfloor \frac{2f(u)f(v)}{f(u)+f(v)} \right\rfloor$, then the edge labels are distinct. In this case f is called Harmonic Mean Labeling of G. In this paper we prove that mC_n, $mC_n \cup P_k$, $mC_n \cup C_k$, $mC_n \cup PC_k$, $nk_3 \cup C_m$, $nk_3 \cup PC_m$, $P_m \times P_3$ are Harmonic mean graphs. Also we prove that the graph obtained by joining two copies of cycle C_n by a path of arbitrary length is a Harmonic mean graph.

Keywords: *Graph, Harmonic mean graph, path, cycle, planar grid, union of graphs, mG.*

1. INTRODUCTION

The graph considered here will be finite, undirected and simple. Terms not defined here are used in the sense of Harary [1]. The symbols V (G) and E(G) will denote the vertex set and edge set of a graph G. The square G^2 of a graph G has V (G^2) = V (G), with u, v adjacent in G^2 whenever $d(u,v) \le 2$ in G. The union of two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is a graph $G = G_1 \cup G_2$ with vertex set $V = V_1 \cup V_2$ and edge set $E = E_1 \cup E_2$. The Cartesian product of two graphs $G_1 = (V_1\ E_1)$ and $G_2 = (V_2, E_2)$ is a graph $G = (V, E) = G_1 \times G_2$ with $V = V_1 \times V_2$ and two vertices $u = (u_1, u_2)$ and $v = (v_1, v_2)$ are adjacent in $G_1 \times G_2$ whenever ($u_1 = v_1$ and u_2 is adjacent to u_2 or $u_2 = v_2$ and u_1 is adjacent to v_1). The product $P_m x P_n$ is called a planar grid and $P_m \times P_2$ in called ladder graph. mG denotes the disjoint union of m copies of the graph G. Let G_1, $G_2 \ldots G_n$ $n \ge 2$ be n copies of a fixed graph G. The graph obtained by adding an edge between G_i and G_{i+1}, i =1, 2…..n-1 is a called path union of G [7].

S. Somasundaram and R. Ponraj introduced Mean labeling of Graphs in [2]. We introduced Harmonic Mean labeling of Graphs in [3] and studied their behavior in [4] and [5]. In this paper we discuss Harmonic mean labeling behavior for union of two graphs like $C_m \cup P_n$, $mC_n \cup P_k$, $mC_n \cup C_k$, $mC_n \cup PC_k$ etc.

Here we shall use frequent reference to the following definition and theorems.

Definition 1.1: A Graph G with p vertices and q edges is called a harmonic mean graph if it is possible to label the vertices $x \in V$ with distinct labels f(x) from 1, 2….q+1 in such a way that when each edge e = uv is labeled with

f(e = uv) = $\left\lceil \frac{2f(u)f(v)}{f(u)+f(v)} \right\rceil$ (or) $\left\lfloor \frac{2f(u)f(v)}{f(u)+f(v)} \right\rfloor$ then the edge labels are distinct. In this case f is called a Harmonic mean labeling of G.

Theorem 1.2[4]: nK_3, $nK_3 \cup P_m$, m >1, $nk_3 \cup C_m$, $m \ge 3$ are Harmonic mean graphs.

Theorem 1.3 [4]: mC_4, $mC_4 \cup P_n$, n >1, $mC_4 \cup C_n$, $n \ge 3$, $nk_3 \cup mC_4$ are Harmonic mean graphs.

Theorem 1.4 [3]: Ladders are Harmonic mean graphs.

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Theorem 1.5 [6]: The graph $C_n^{(2)}$ is a Harmonic mean graph.

2. MAIN RESULT

Theorem 2.1: mC_n is a Harmonic mean graph.

Proof: Let the vertex set of mC_n be $V = V_1 \cup V_2 \cup \ldots \cup V_m$ where $V_i = \{v_i, v_i^2, v_i^3 \ldots . v_i^m\}$. Now define a function $f : V(mC_n) \rightarrow \{1, 2, \ldots, q+1\}$ by $f(v_i^j) = m(i-1)+j$, $1 \leq i \leq m$, $1 \leq j \leq n$. If a and a+1 are two integers, then the Harmonic mean lies between a and a+1, $a < \frac{2a(a+1)}{2a+1} < a+1$.

Consider a graph with vertices n_i-3, n_i-2, n_i-1, n_i.

Figure: 1

For the edges joining the vertices n_i-3 and n_i-2 we may assign the edge label n_i-3. Similarly for the edge joining the vertices n_i-2 and n_i-1 we may assign the edge label n_i-1 and for the edge joining the vertices n_i-1 ad n_i we may assign the edge label n_i.

Since $n_i-3 < \frac{2n_i(n_i-3)}{2n_i-3} < n_i$, we may assign the edge label n_i-2 for the edges joining the vertices n_i-3 and n_i. Since mC_n has distinct edge labels, it is a Harmonic Mean graph.

Example 2.2: The following figure shows the Harmonic mean labeling of $3C_6$.

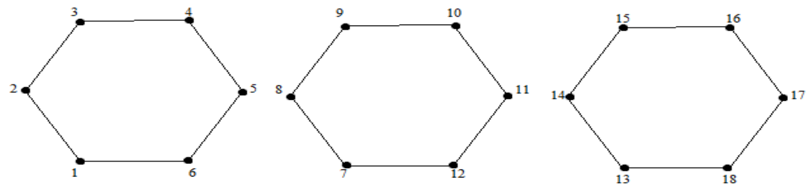

Figure: 2

Now we investigate Harmonic mean labeling of union of mC_n with path and cycle.

Theorem 2.3: $mC_n \cup P_k$ is a Harmonic mean graph for $m \geq 1$, $n \geq 3$ and $k > 1$.

Proof: Let the vertex set of mC_n be $V = V_1 \cup V_2 \cup \ldots \cup V_m$

where
$V_i = \{v_i^1, v_i^2, v_i^3 \ldots . v_i^n\}$ and the edge set be $E = E_1 \cup E_2 \cup \ldots \cup E_m$

where
$E_i = \{e_i^1, e_i^2, e_i^3 \ldots \ldots e_i^n\}$. Let P_k be the path $u_1 u_2 \ldots . u_k$.

Define a function $f : V(mC_n \cup P_k) \rightarrow \{1, 2, \ldots, q+1\}$

by $f(v_i^j) = n(i-1)+j$, $1 \leq i \leq m$, $1 \leq j \leq n$

$f(u_k) = mn+i$, $1 \leq i \leq k$.
Edge labels are shown below

The set of labels of the edges of mC_n is {1, 2, 3… mn} and the set of labels of the edges of P_k is (mn+1, mn+2… mn+k-1}.

Hence $mC_n \cup P_k$ is a Harmonic mean graph.

Example 2.4: A Harmonic mean labeling of $3C_5 \cup P_7$ is given below

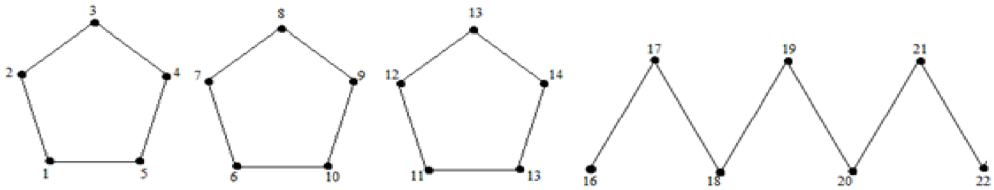

Figure: 3

Next we have

Theorem 2.5: $mC_n \cup C_k$ is a Harmonic mean graph for $m \geq 3$ and $k \geq 3$.

Proof: Let mC_n be m copies of the cycle C_n and C_k be cycle with k vertices. Let the vertex set of mC_n be $V= V_1 \cup V_2 \cup \ldots \cup V_m$

where $V_i = \{v_i^1, v_i^2, \ldots, v_i^n\}$ and the edge set be $E = E_1 \cup E_2 \cup \ldots \cup E_m$

where $E_i = \{ e_i^1, e_i^2, e_i^3, \ldots, e_i^n\}$.

Let $u_1u_2 \ldots u_ku_1$ be the cycle C_k.

Define a function $f: V(mC_n \cup C_k) \rightarrow \{1, 2 \ldots q+1\}$ by $f(v_i^j) = n(i-1)+j$, $1 \leq i \leq m$, $1 \leq i \leq m$

$$f(u_i) = mn+i,\ 1 \leq i \leq k.$$

Hence $mC_n \cup C_k$ is a Harmonic mean graph.

Example 2.6: Harmonic mean labeling pattern of $3C_5 \cup C_6$ is given in the following figure.

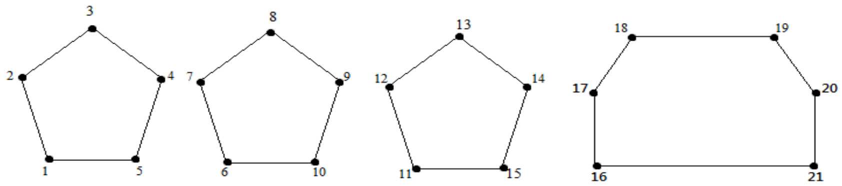

Figure: 4

The same argument as in Theorem 2.3 and Theorem 2.5 gives the following

Theorem 2.7: $mC_n \cup PC_k$ is a Harmonic mean graph for $n, k \geq 3$ and $m, p > 1$. Now we have

Theorem 2.8: $nK_3 \cup mC_p$, is a Harmonic mean graph for $p > 3$ and $n, m > 1$

Proof: Let the vertex set of nK_3 be $V= V_1 \cup V_2 \cup \ldots \cup V_n$ where $V_i = \{v_i^1, v_i^2, v_i^3\}$.

Let the vertex set of mC_p be $U = U_1 \cup U_2 \cup U_3 \cup \ldots \cup U_m$ where $U_k = \{u_k^1, u_k^2, \ldots, u_k^n\}$.

Define a function $f : V(nk_3 \cup mC_n) \rightarrow \{1,2, \ldots, q+1\}$ by $f(v_i^j) = 3(i-1)+j$, $1 \leq i \leq n$, $1 \leq j \leq 3$ and $f(u_k^j) = p(k-1)+3n+l$, $1 \leq k \leq m$, $1 \leq l \leq p$.

Hence $nk_3 \cup mC_p$ is a Harmonic mean graph.

Example 2.9: Harmonic mean labeling of $4k_3 \cup 2C_5$ is given below

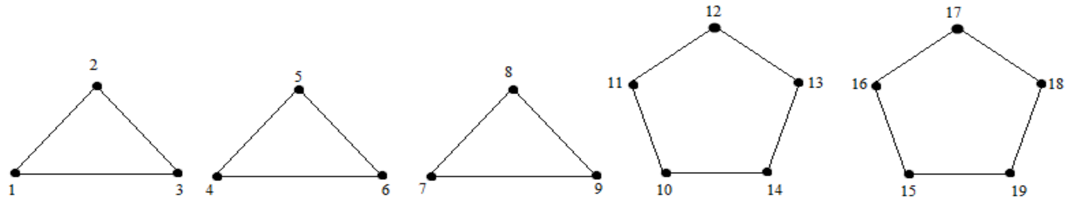

Figure: 5

Next we prove the following

Theorem 2.10: Two copies of cycle C_m sharing a common edge is a Harmonic mean graph.

Proof: Let the cycle C_m be $u_1u_2\ldots u_mu_1$. Consider two copies of cycle C_m.

Let G be a graph obtained from two copies of cycle C_m sharing common edge. Then G has 2n-2 vertices and 2n-1 edges. Let us take $e = u_{m-1}\, u_m$ be the common edge between two copies of C_m.

Define a function $f : V(G) \rightarrow \{1,2, \ldots, q+1\}$by

$f(u_i) = i,\ 1 \le i \le m\text{-}2$

$f(u_m) = m\text{-}1,\ f(u_{m-1}) = m$

$f(v_i) = i\text{+}1,\ m\text{+}1 \le i \le 2m\text{-}2.$

Hence G is a Harmonic mean graph.

The following example illustrates the above theorem.

The labeling pattern of two copies of C_7 sharing a common edge is shown below.

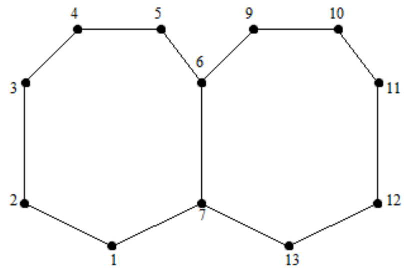

Figure: 6

Now we investigate the Harmonic mean labeling of a planar grid for a particular case of n.

Theorem 2.11: The planar gird $P_m \times P_3$ is a Harmonic mean graph.

Proof: Let the vertex set of P_m x P_3 be $V(P_m \times P_3) = \{a_{ij}:1 \le i \le m,\ 1 \le j \le 3\}$ and the edge set be

$E(P_m \times P_3) = (a_{i(j-1)}\ a_{ij};1 \le i \le m\ ,\ 2 \le j \le 3\} \cup \{\ a_{(i-1)}\ a_{ij}:\ 2\le i \le m,\ 1\le j \le 3\}$

Define f: $V\ (P_m \times P_3) \rightarrow \{1, 2\ldots\ q+1\}$

$f(a_{ij}) = i,\ i=1,\ 1 \le j \le 3$

$f(a_{ij}) = f(a_{(i-1)3}) +2+j,\ 2 \le i \le m,\ 1 \le j \le n,\ 1 \le j \le 3$

Edges are labeled with

$f(a_{11}\ a_{21}) =1$

$f(a_{ij}\, a_{i(j+1)}) = 5\,(i\text{-}1)\text{+}j\text{+}1,\ i = 1,\ 1 \le j \le 2$

$f(a_{ij}\, a_{i(j+1)}) = 5\,(i\text{-}1)\text{+}j\ ,\ 2 \le j \le m,\ 1 \le j \le 2$

$f(a_{ij}\, a_{i(j+1)j}) = 5\,(i\text{-}1)\text{+}j\text{+}2,\quad 1 \le i \le m\text{-}1,\ 1 \le j \le 3$

Here all the edges are labeled with distinct labels.

Hence $P_m \times P_3$ is a Harmonic mean graph for $m \ge 2$.

Example 2.12: Harmonic mean labeling of $P_4 \times P_3$ is shown in the following figure.

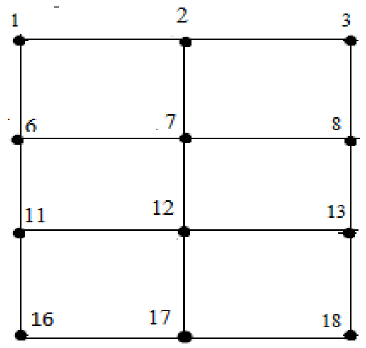

Figure: 7

Next we prove

Theorem 2.13: The graph obtained by joining two copies of cycle C_n by a path P_m is a Harmonic mean graph for all m and n

Proof: Let G be a graph obtained by joining two copies of cycle C_n by a path P_m Let $u_1u_2\ldots.u_n$ be the vertices of first copy of cycle C_n and $v_1v_2\ldots..v_n$ be the vertices of second copy of cycle C_n.

Let P_m be the path $w_1w_2\ldots..w_m$ with $u_1 = w_1$ and $v_1 = w_m$

Define a function $f:V(G) \rightarrow \{1,2\ldots..q+1\}$ by

$f(u_i) = i,\ 1 \le i \le n$

$f(v_1) = n+3$

$f(v_i) = n+3+i,\ 2 \le i \le n$

$f(w_1) = n$

$f(w_j) = n+j-1,\ 2 \le j \le m$

Hence G is a Harmonic mean graph

Example 2.14: The following example shows the graph G obtained by joining two copies of the cycle C_5 by a path P_4.

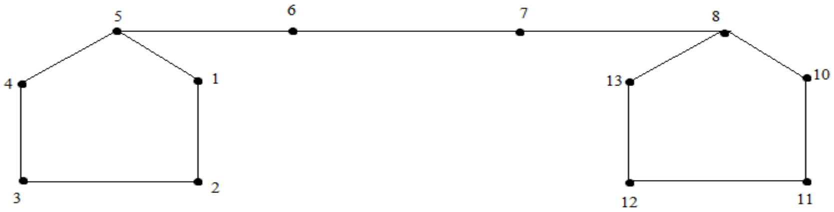

Figure: 8

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