International Journal of Mathematical Archive-4(1), 2013, 1-10
Available online through www.ijma.info ISSN 2229 – 5046

INTUITIONISTIC EUCLIDEAN N-NORMED LINEAR SPACE

S. Jeevitha*
Department of Mathematics KTVR Knowledge park for Engineering and Technology, Coimbatore-19, Tamilnadu, India

(Received on: 25-10-12; Revised & Accepted on: 18-01-13)

ABSTRACT

This paper introduces the notion of Cauchy sequence, convergent sequence and completeness in Intuitionistic Euclidean-n-normed linear space (i-e-n-NLS).

Key words: *Continuous t-norm, t-co-norm, Fuzzy n-normed linear space, Intuitionistic Fuzzy n-normed linear space, Intuitionistic Euclidean n-normed linear space.*

AMS Subject Classification No: *54E50, 46S40, 03E72.*

1. INTRODUCTION

S. Gähler [6] introduced the theory of n-norm on a linear space. For a systematic development of n-normed linear space one may refer to [8, 10, 11, 12]. The detailed theory of fuzzy normed linear space can be found in [2, 3, 4, 7, 8, 10]. The origin and the development of intuitionistic fuzzy set theory can be found in [1, 5, 6, 9, 16]. In [15] they have discussed the notion of fuzzy n-normed linear space, intuitionistic fuzzy n-normed linear space, Cauchy sequence and convergent sequence in generalized Cartesian product of intuitionistic fuzzy n-normed linear space.

The purpose this paper is to introduce the notion of intuitionistic Euclidean n-normed linear space as a further generalization of Cartesian product of intuitionistic fuzzy n-normed linear space and also we provide some result on it.

2. PRELIMINARIES

This section is devoted to the collection of basic definitions and results which will be needed in the sequel.

Definition: 2.1 Let X be a real linear space of dimension greater than one and let $\|\bullet,\bullet\|$ be a real valued function on $X \times X$ satisfying the following conditions:

(1) $\|x, y\| = 0$ if and only if x and y are linearly dependent

(2) $\|x, y\| = \|y, x\|$

(3) $\|\alpha x, y\| = |\alpha| \|x, y\|$, where $\alpha \in R$ (set of real numbers)

(4) $\|x, y + z\| \le \|x, y\| + \|x, z\|$.

$\|\bullet,\bullet\|$ is called a 2-norm on X and the pair $(X, \|\bullet,\bullet\|)$ is called a 2-normed linear space.

Definition: 2.2 Let $n \in N$ (natural numbers) and X be a real linear space of dimension $d \ge n$. (Here we allow d to be infinite). Areal valued function $\|\bullet,...,\bullet\|$ on $\underbrace{X \times ... \times X}_{n} = X^n$ satisfying the following four properties:

(1) $\|x_1, x_2,..., x_n\| = 0$ if and only if $x_1, x_2,..., x_n$ are linearly dependent

(2) $\|x_1, x_2,..., x_n\|$ is invariant under any permutation

(3) $\|x_1, x_2,..., \alpha x_n\| = |\alpha| \|x_1, x_2,..., x_n\|$, where $\alpha \in R$ (set of real numbers)

(4) $\|x_1, x_2,.., x_{n-1}, y + z\| \le \|x_1, x_2,.., x_{n-1}, y\| + \|x_1, x_2,.., x_{n-1}, z\|$

is called n-norm on X and the pair $(X, \|\bullet,...,\bullet\|)$ is called an n-normed linear space.

Corresponding author: *S. Jeevitha*, *Department of Mathematics KTVR Knowledge park for Engineering and Technology, Coimbatore-19, Tamilnadu, India*

Definition: 2.3 A sequence $\{x_n\}$ in an n-normed linear space $(X, \|\bullet,\ldots,\bullet\|)$ is said to converge to an $x \in X$ (in n-norm) whenever $\lim_{n\to\infty}\|x_1, x_2, \ldots, x_{n-1}, x_n - x\| = 0$.

Definition: 2.4 A sequence $\{x_n\}$ in an n-normed linear space $(X, \|\bullet,\ldots,\bullet\|)$ is called Cauchy sequence if $\lim_{n,k\to\infty}\|x_1, x_2, \ldots, x_{n-1}, x_n - x_k\| = 0$.

Definition: 2.5 An n – normed linear space is said to be complete if every Cauchy sequence in it is convergent.

Definition: 2.6 Let X be a linear space over a field F . A fuzzy subset N of X^n x R (set of real numbers) is called fuzzy n – norm on X iff

(N1) For all $t \in R$ with $t \le 0$, $N(x_1, x_2 \cdots x_n, t) = 0$

(N2) For all $t \in R$ with $t > 0$, $N(x_1, x_2 \cdots x_n, t) = 1$ if and only if $x_1, x_2 \cdots x_n$ are linearly dependent.

(N3) $N(x_1, x_2 \cdots x_n, t)$ is invariant under any permutation of $x_1, x_2 \cdots x_n$

(N4) For all $t \in R$ with $t > 0$,

$$N(x_1, x_2 \cdots cx_n, t) = N\left(x_1, x_2 \cdots cx_n, \frac{t}{|c|}\right), \text{ if } c \ne 0, c \in F \text{ (field).}$$

(N5) For all s, t ∈ R, $N(x_1, x_2 \cdots x'_n, s+t) \ge \min\{N(x_1, x_2 \cdots x_n, s), N(x_1, x_2 \cdots x'_n, t)\}$

(N6) $N(x_1, x_2 \cdots x_n)$ is a non - decreasing function of $t \in R$ and $\lim_{t\to\infty} N(x_1, x_2 \cdots x_n) = 1.$

Then (X, N) is called a fuzzy n-normed liner space or in short f-n-NLS.

Definition: 2.7 A binary operator $* : [0,1] \times [1,0] \to [1,0]$ is continuous t-norm if * satisfies the following conditions:

(1) * is commutative and associative
(2) * is continuous
(3) a*1=a for all a $\in [1,0]$
(4) $a * b \le c * d$ whenever $a \le c$ and $b \le d$ and $a, b, c, d \in [1,0]$.

Definition: 2.8 A binary operator $\lozenge : [0,1] \times [1,0] \to [1,0]$ is continuous co-norm if * satisfies the following conditions:

(1) ◊ is commutative and associative
(2) ◊ is continuous
(3) a ◊ 0=a for all a $\in [1,0]$
(4) $a \lozenge b \le c \lozenge d$ whenever $a \le c$ and $b \le d$ and $a, b, c, d \in [1,0]$.

Remark:

(a) For any $r_1, r_2 \in (0,1)$ with $r_1 > r_2$, there exist $r_3, r_4 \in (0,1)$ such that $r_1 * r_3 \ge r_2$ and $r_1 \ge r_4 \lozenge r_2$.

(b) For any $r_5 \in (0,1)$, there exist $r_6, r_7 \in (0,1)$ such that $r_6 * r_6 \ge r_5$ and $r_7 \lozenge r_7 \le r_5$.

Definition: 2.9 Let $*$ and $*'$ be two continuous t – norms. Then $*'$ dominates$*$, and we write $*' \gg *$, if for all $x_1, x_2, y_1, y_2 \in [0,1]$, $(x_1 *' x_2) * (y_1 *' y_2) \le (x_1 * y_1) *' (x_2 * y_2)$

Definition: 2.10 Let E be any set. An intuitionistic fuzzy set A of E is an object of the form $A = \{(x, \mu_A(x), \nu_A(x)) \mid x \in E\}$, where the functions $\mu_A : E \to [0,1]$ and $\nu_A : E \to [0,1]$ Denote the degree of membership and the non-membership of the element $x \in E$ respectively and for every $x \in E$, $0 \le \mu_A(x) + \nu_A(x) \le 1$.

Definition: 2.11 If A and B are two intuitionistic fuzzy sets of a non-empty set E, then $A \subseteq B$ if for all $x \in E, \mu_A(x) \le \mu_B(x)$ and $\nu_A(x) \ge \nu_B(x)$: A=B if and only if for all $x \in E, \mu_A(x) = \mu_B(x)$ and $\nu_A(x) = \nu_B(x)$;

$$\overline{A} = \{(x, \mu_A(x), \nu_A(x)) \mid x \in E\};$$
$$A \cap B = \{(x, \min(\mu_A(x), \mu_B(x)), \max(\nu_A(x), \nu_B(x))) \mid x \in E\};$$
$$A \cup B = \{(x, \max(\mu_A(x), \mu_B(x)), \min(\nu_A(x), \nu_B(x))) \mid x \in E\}.$$

Definition: 2.12 Let A and B be intuitionistic fuzzy sets in E_1 and E_2 respectively. Then the generalized Cartesian product $A \times_{T,S} B = \{((x, y), T(\mu_A(x), \mu_B(y)), S(\nu_A(x), \nu_B(y))) \mid x \in E_1 \text{ and } y \in E_2\}$, T denotes the t-norm and S denotes the t-co-norm.

Definition: 2.13 An intuitionistic fuzzy n-normed linear space (or) in short i-f-n-NLS is an object of the form $A = \{(X, \mu(x_1, x_2, \cdots, x_n, t), \nu(x_1, x_2, \cdots, x_n, t)) \mid (x_1, x_2, \cdots, x_n) \in X^n\}$, where X is a linear space over a field F, $*$ is a continuous t –norm, $\Diamond$ is a continuous t -co-norm and μ, ν are fuzzy sets on $X^n \times (0, \infty)$, μ denotes the degree of membership and γ denotes the degree of of non – membership $x_1, x_2, \cdots, x_n \in X^n \times (0, \infty)$ satisfying the following conditions:

i. $\mu(x_1, x_2, \cdots, x_n, t) + \nu(x_1, x_2, \cdots, x_n, t) \le 1;$
ii. $\mu(x_1, x_2, \cdots, x_n, t) > 0;$
iii. $\mu(x_1, x_2, \cdots, x_n, t) = 1$ if and only if $x_1, x_2, \cdots, x_n$ are linearly independent ;
iv. $\mu(x_1, x_2, \cdots, x_n, t)$ is invarient under any permutation of $x_1, x_2, \cdots, x_n$;
v. $\mu(x_1, x_2, \cdots, cx_n, t) = \mu(x_1, x_2, \cdots, x_n, \frac{t}{|c|})$ if $c \ne 0, c \in F(\text{field});$
vi. $\mu(x_1, x_2, \cdots, x_n, s) * \mu(x_1, x_2, \cdots, x'_n, t) \le \mu(x_1, x_2, \cdots, x_n + x'_n, s + t);$
vii. $\mu(x_1, x_2, \cdots, x_n, t) : (0, \infty) \to [0,1]$ is continuous in t;
viii. $\nu(x_1, x_2, \cdots, x_n, t) > 0;$
ix. $\nu(x_1, x_2, \cdots, x_n, t) = 0$ if and only if $x_1, x_2, \cdots, x_n$ are linearly dependent ;
x. $\nu(x_1, x_2, \cdots, x_n, t)$ is invarient under any permutation of $x_1, x_2, \cdots, x_n$;
xi. $\nu(x_1, x_2, \cdots, cx_n, t) = \nu(x_1, x_2, \cdots, x_n, \frac{t}{|c|})$ if $c \ne 0, c \in F(\text{field});$
xii. $\nu(x_1, x_2, \cdots, x_n, s) \Diamond \nu(x_1, x_2, \cdots, x'_n, t) \ge \nu(x_1, x_2, \cdots, x_n + x'_n, s + t);$
xiii. $\nu(x_1, x_2, \cdots, x_n, t) : (0, \infty) \to [0,1]$ is continuous in t;

Definition: 2.14 The 5- tuple $(R^n, \Phi, \Psi, *, \Diamond)$ intuitionistic fuzzy Euclidean n-normed space if $*$ is a t –norm, $\Diamond$ is a t co –norm and (Φ, Ψ) is an intuitionistic fuzzy Euclidean n-norm defined by $\Phi(x, t) = \prod_{j=1}^{n} \mu(x_j, t)$ and $\Psi(x, t) = \prod_{j=1}^{n} \nu(x_j, t)$. Where $x = x_1, x_2, \cdots, x_n$. $\prod_{j=1}^{n} a_j = a_1 *' \cdots *' a_n *' >> *'$, $\prod_{j=1}^{n} a_j = a_1 \Diamond \cdots \Diamond a_n, t > 0$ and (μ, ν) is an intuitionistic fuzzy norm with respect to $*$ and $\Diamond$.

Corollary: Suppose that hypotheses of Definition (2.14) are satisfied. If $*$ is normal and $\Diamond = \max$, then $(R^n, \Phi, \Psi, *, \Diamond)$ is an intuitionistic fuzzy normed space.

Proof: For (a),let $\Psi(x, t) = \max_{j=1}^{n} \nu(x_j, t) = \nu(x_k, t)$ in which $1 \le k \le n$. since $\Phi(x, t) = \prod_{j=1}^{n} \mu(x_j, t) \le \mu(x_k, t)$ then we have $\Phi(x, t) + \Psi(x, t) \le \Phi(x_k, t) + \Psi(x_k, t) \le 1.$

The properties of (ii)-(v),(vi)-(xi) and (xii) are immediate from the definition. For triangle inequalities (vi) and (xii), suppose that $x, y \in X$ and $t, s > 0$.

$$
\begin{aligned}
\Phi(x, y) * \Phi(y, s) &= \prod_{j=1}^{n} \mu(x_j, t) * \prod_{j=1}^{n} \mu(y_j, t) \\
&= (\mu(x_1, t) *' \ldots *' \mu(x_n, t)) * (\mu(y_1, t) *' \ldots *' \mu(y_n, t)) \\
&\leq (\mu(x_1, t) * \mu(y_1, t)) *' \ldots *' (\mu(x_n, t) * \mu(y_n, t)) \\
&\leq \mu(x_1 + y_1, t + s) *' \ldots *' \mu(x_n + y_n, t + s) \\
&= \prod_{j=1}^{n} \mu(x_j + y_j, t + s) = \Phi(x + y, t + s)
\end{aligned}
$$

The proof for (xii) is similar to (vi).

Example: Let $(x, \|\bullet, \bullet, \ldots\ldots, \bullet\|)$ and be an n-normed linear space. Define $a * b = \min\{a, b\}$ and $a \Diamond b = \max\{a, b\}$ for all $a, b \in [1, 0]$, $\Phi(x, t) = \prod_{j=1}^{n} \mu(x_j, t) = \dfrac{t}{t + \|x_1, x_2, \ldots, x_n\|}$, $\Psi(x, t) = \coprod_{j=1}^{n} \nu(x_j, t) = \dfrac{\|x_1, x_2, \ldots, x_n\|}{t + \|x_1, x_2, \ldots, x_n\|}$.

Then A= $\left\{(x . \Phi(x, t), \Psi(x, t)) \,\middle|\, x = (x_1, x_2, \ldots, x_n \in X^n\right\}$ is an i-f-e-n-NLS.

Proof:

(i) Clearly $\Phi(x, t) + \Psi(x, t) \leq 1$.

(ii) Obviously $\Phi(x, t) > 0$.

(iii) $\Phi(x, t) = 1 \Leftrightarrow \prod_{j=1}^{n} \mu(x_j, t) = 1 \Leftrightarrow \dfrac{t}{t + \|x_1, x_2, \ldots, x_n\|} = 1$

$\Leftrightarrow t = t + \|x_1, x_2, \ldots, x_n\| \Leftrightarrow \|x_1, x_2, \ldots, x_n\| = 0$

$\Leftrightarrow$ $x_1, x_2, \ldots, x_n$ are linearly dependent.

(iv)
$$
\begin{aligned}
\Phi(x, t) &= \prod_{j=1}^{n} \mu(x_j, t) = \frac{t}{t + \|x_1, x_2, \ldots, x_n\|} \\
&= \frac{t}{t + \|x_1, x_2, \ldots, x_n, x_{n-1}\|} \\
&= \mu(x_1, x_2, \ldots, x_n, x_{n-1}, t) \\
&= \ldots
\end{aligned}
$$

$$
\Phi\left(x, \frac{t}{|c|}\right) = \prod_{j=1}^{n} \mu\left(x_j, \frac{t}{|c|}\right)
$$

$$
= \mu\left(x_1, x_2, \ldots x_n, \frac{t}{|c|}\right) = \frac{\frac{t}{|c|}}{\frac{t}{|c|} + \|x_1, x_2, \ldots, x_n\|} = \frac{\frac{t}{|c|}}{\frac{t + |c|\ \|x_1, x_2, \ldots, x_n\|}{|c|}}
$$

$$
= \frac{t}{t + |c|\ \|x_1, x_2, \ldots, x_n\|} = \frac{t}{t + \|x_1, x_2, \ldots, cx_n\|} = \mu(x_1, x_{2,} \ldots, cx_n, t)
$$

$$
= \Phi(cx, t)
$$

(v) Without loss of generality assume that

$$\Phi(x',t) \le \Phi(x,s)$$

$$\Rightarrow \prod_{j=1}^{n} \mu(x'_j,t) \le \prod_{j=1}^{n} \mu(x_j,s)$$

$$\Rightarrow \frac{t}{t+\|x_1,x_2,\dots,x'_n\|} \le \frac{s}{s+\|x_1,x_2,\dots,x_n\|}$$

$$\Rightarrow t(s+\|x_1,x_2,\dots,x_n\|) \le s(t+\|x_1,x_2,\dots,x'_n\|)$$

$$\Rightarrow t\|x_1,x_2,\dots,x_n\| \le s\|x_1,x_2,\dots,x'_n\|$$

$$\Rightarrow \|x_1,x_2,\dots,x_n\| \le \frac{s}{t}\|x_1,x_2,\dots,x'_n\|$$

Therefore,

$$\|x_1,x_2,\dots,x_n\| + \|x_1,x_2,\dots,x'_n\| \le \frac{s}{t}\|x_1,x_2,\dots,x_n\| + \|x_1,x_2,\dots,x'_n\|$$

$$\le \left(\frac{s}{t}+1\right)\|x_1,x_2,\dots,x'_n\|$$

$$\le \left(\frac{s+t}{t}\right)\|x_1,x_2,\dots,x'_n\|$$

But

$$\|x_1,x_2,\dots,x_n + x'_n\| \le \|x_1,x_2,\dots,x_n\|\|x_1,x_2,\dots,x'_n\| \le \left(\frac{s+t}{t}\right)\|x_1,x_2,\dots,x'_n\|$$

$$\Rightarrow \frac{\|x_1,x_2,\dots,x_n + x'_n\|}{s+t} \le \frac{\|x_1,x_2,\dots,x'_n\|}{t}$$

$$\Rightarrow 1+\frac{\|x_1,x_2,\dots,x_n + x'_n\|}{s+t} \le 1+\frac{\|x_1,x_2,\dots,x'_n\|}{t}$$

$$\Rightarrow \frac{s+t+\|x_1,x_2,\dots,x_n + x'_n\|}{s+t} \le \frac{t+\|x_1,x_2,\dots,x'_n\|}{t}$$

$$\Rightarrow \frac{s+t}{s+t+\|x_1,x_2,\dots,x_n + x'_n\|} \ge \frac{t}{t+\|x_1,x_2,\dots,x'_n\|}$$

$$\Rightarrow \mu(x_1,x_2,\dots,x_n + x'_n, s+t) \ge \min\{\mu(x_1,x_2,\dots,x_n,s),\mu(x_1,x_2,\dots,x'_n,t)\}$$

(vi) Clearly $\Phi(x,t)$ is continuous in t.

(vii) Clearly $\Psi(x,t) > 0$.

(viii) $\Psi(x,t) > 0 \Leftrightarrow \coprod_{j=1}^{n} \nu(x_j,t) = 0$

$$\Leftrightarrow \frac{\|x_1,x_2,\dots,x_n\|}{t+\|x_1,x_2,\dots,x_n\|} = 0$$

$$\Leftrightarrow \|x_1,x_2,\dots,x_n\| = 0$$

$\Leftrightarrow$ $x_1,x_2,\dots,x_n$ are linearly dependent.

(ix) $\Psi(x,t) = \coprod_{j=1}^{n} \nu(x_j,t) = \dfrac{\|x_1,x_2,\dots,x_n\|}{t+\|x_1,x_2,\dots,x_n\|}$

$$= \frac{\|x_1,x_2,\dots,x_n,x_{n-1}\|}{t+\|x_1,x_2,\dots,x_n,x_{n-1}\|}$$

$$= \nu(x_1, x_2, ..., x_n, x_{n-1}, t)$$
$$= ...$$

(x) $$\Psi(cx, t) = \coprod_{j=1}^{n} \nu(cx_j, t) = \nu(x_1, x_2, ..., cx_n, t)$$

$$= \frac{\|x_1, x_2, ..., cx_n\|}{t + \|x_1, x_2, ..., cx_n\|} = \frac{|c| \ \|x_1, x_2, ..., x_n\|}{t + |c| \ \|x_1, x_2, ..., x_n\|} = \frac{\|x_1, x_2, ..., x_n\|}{\frac{t}{|c|} + \ \|x_1, x_2, ..., x_n\|} ==$$

$$= \nu\left(x_1, x_2, ..., x_n, \frac{t}{|c|}\right) = \Psi\left(x, \frac{t}{|c|}\right) =$$

(xi) Without loss of generality assume,

$$\Psi(x, s) \le \Psi(x', t)$$

$$\Rightarrow \coprod_{j=1}^{n} \nu(x_j, s) \le \coprod_{j=1}^{n} \nu(x'_j, t)$$

(1) $$\Rightarrow \frac{\|x_1, x_2, ..., x_n\|}{s + \|x_1, x_2, ..., x_n\|} \le \frac{\|x_1, x_2, ..., x'_n\|}{t + \|x_1, x_2, ..., x'_n\|}$$

$$\Rightarrow \|x_1, x_2, ..., x_n\|(t + \|x_1, x_2, ..., x'_n\|) \le \|x_1, x_2, ..., x'_n\|(s + \|x_1, x_2, ..., x_n\|)$$

$$\Rightarrow t\|x_1, x_2, ..., x_n\| \le s\|x_1, x_2, ..., x''_n\|$$

Now

$$\frac{\|x_1, x_2, ..., x_n + x'_n\|}{s + t + \|x_1, x_2, ..., x_n + x'_n\|} - \frac{\|x_1, x_2, ..., x'_n\|}{t + \|x_1, x_2, ..., x'_n\|} \le \frac{\|x_1, x_2, ..., x_n\| + \|x_1, x_2, ..., x'_n\|}{s + t + \|x_1, x_2, ..., x_n\| + \|x_1, x_2, ..., x'_n\|} - \frac{\|x_1, x_2, ..., x'_n\|}{t + \|x_1, x_2, ..., x'_n\|}$$

$$= \frac{t\|x_1, x_2, ..., x_n\| - s\|x_1, x_2, ..., x'_n\|}{\left(s + t + \|x_1, x_2, ..., x_n\| + \|x_1, x_2, ..., x'_n\|\right)\left(t + \|x_1, x_2, ..., x'_n\|\right)}$$

By (1)

$$\frac{\|x_1, x_2, ..., x_n + x'_n\|}{s + t + \|x_1, x_2, ..., x_n + x'_n\|} \le \frac{\|x_1, x_2, ..., x'_n\|}{t + \|x_1, x_2, ..., x'_n\|}$$

Similarly

$$\frac{\|x_1, x_2, ..., x_n + x'_n\|}{s + t + \|x_1, x_2, ..., x_n + x'_n\|} \le \frac{\|x_1, x_2, ..., x_n\|}{t + \|x_1, x_2, ..., x_n\|}$$

$$\Rightarrow \nu(x_1, x_2, ..., x_n + x'_n, s + t) \le \max\{\nu(x_1, x_2, ..., x_n, s), \nu(x_1, x_2, ..., x'_n, t)\}$$

(xii) Clearly $\Psi(x, t)$ is continuous in t.This A is an i-f-e-n-NLS.

Definition: 2.15 A sequence $\{x_n\}$ in an i-f-e-n-NLS A is said to converge to x if given $r > 0$, $t > 0, 0 < r < 1$ there exists an integer $n_0 \in N$ such that $\Phi(x_n - x, t) = \prod_{j-i}^{n} \mu(x_j - x, t) > 1 - r$ and

$$\Psi(x_n - x, t) = \coprod_{j=1}^{n} \nu(x_j - x, t) < r, \text{for all } n \ge n_0.$$

Theorem: In an i-f-e-n-NLS A, a sequence $\{x_n\}$ converges to x denoted by $x_n \xrightarrow{(\Phi,\Psi)} x$ if and only if $\Phi(x_n - x, t) \to 1$ and $\Psi(x_n - x, t) \to 0$, as $n \to \infty$.

Proof: Fix $t > 0$. Suppose $\{x_n\}$ converges to x in A. Then for a given $r, 0 < r < 1,$ there exists an integer $n_0 \in N$ such that $\Phi(x_n - x, t) = \prod_{j-i}^{n} \mu(x_j - x, t) > 1 - r$ and $\Psi(x_n - x, t) = \coprod_{j=1}^{n} \nu(x_j - x, t) < r$. Thus $1 - \Phi(x_n - x, t) = 1 - \prod_{j-i}^{n} \mu(x_j - x, t) < r$ and $\Psi(x_n - x, t) = \coprod_{j=1}^{n} \nu(x_j - x, t) < r$, and hence $\Phi(x_n - x, t) \to 1$ and $\Psi(x_n - x, t) \to 0$, as $n \to \infty$. Conversely, if for each $t > 0,$ $\Phi(x_n - x, t) \to 1$ and $\Psi(x_n - x, t) \to 0$, as $n \to \infty$, then for every $r, 0 < r < 1,$ there exists an integer n_0 such that $1 - \Phi(x_n - x, t) < r$ and $\Psi(x_n - x, t) < r,$ for all $n \geq n_0$. Thus $\Phi(x_n - x, t) = \prod_{j-i}^{n} \mu(x_j - x, t) > 1 - r$ and $\Psi(x_n - x, t) = \coprod_{j=1}^{n} \nu(x_j - x, t) < r,$ for all $n \geq n_{0.}$ Hence $\{x_n\}$ converges to x in A.

Definition: 2.16 A sequence $\{x_n\}$ is an i-f-e-n-NLS A is said to be Cauchy sequence if given $\varepsilon > 0,$ with $0 < \varepsilon < 1, t > 0$ there exists an integer $n_0 \in N$ such that $\Phi(x_n - x_k, t) = \prod_{j-i}^{n} \mu(x_j - x_k, t) > 1 - \varepsilon$ and $\Psi(x_n - x_k, t) = \coprod_{j=1}^{n} \nu(x_j - x_k, t) < \varepsilon$ for all $n, k \geq n_{0.}$

Theorem: In an i-f-e-n-NLS A, every convergent sequence is a Cauchy sequence.

Proof: Let $\{x_n\}$ be convergent sequence in A. Suppose $\{x_n\}$ converges to x. Let $t > 0$ and $\varepsilon \in (0,1)$. Choose $r \in (0,1)$ such that $(1 - r) * (1 - r) > 1 - \varepsilon$ and $r \lozenge r < \varepsilon$. Since $\{x_n\}$ converges to x, we have an integer n_0 such that $\Phi\left(x_n - x, \frac{t}{2}\right) = \prod_{j-i}^{n} \mu\left(x_j - x, \frac{t}{2}\right) > 1 - r$ and $\Psi\left(x_n - x, \frac{t}{2}\right) = \coprod_{j=1}^{n} \nu\left(x_j - x, \frac{t}{2}\right) < r$.

Now,

$$\begin{aligned}
\Phi(x_n - x_k, t) &= \prod_{j-i}^{n} \mu(x_j - x_k, t) = \prod_{j-i}^{n} \mu\left(x_j - x + x - x_k, \frac{t}{2} + \frac{t}{2}\right) \\
&\geq \mu\left(x_1, x_2, ..., x_{n-1}, x_n - x, \frac{t}{2}\right) * \mu\left(x_1, x_2, ..., x_{n-1}, x - x_k, \frac{t}{2}\right) \\
&> (1 - r) * (1 - r), \text{ for all } n, k \geq n_0 \\
&> 1 - \varepsilon, \text{ for all } n, k \geq n_0
\end{aligned}$$

and

$$\begin{aligned}
\Psi(x_n - x_k, t) &= \coprod_{j=1}^{n} \nu(x_j - x_k, t) = \coprod_{j=1}^{n} \nu\left(x_j - x + x - x_k, \frac{t}{2} + \frac{t}{2}\right) \\
&\leq \nu\left(x_1, x_2, ..., x_{n-1}, x_n - x, \frac{t}{2}\right) * \nu\left(x_1, x_2, ..., x_{n-1}, x - x_k, \frac{t}{2}\right) \\
&< r \lozenge r \\
&< \varepsilon, \text{ for all } n, k, \geq n_{0.}
\end{aligned}$$

Therefore $\{x_n\}$ is a Cauchy sequence in A.

Definition: 2.17 An i-f-e-n-NLS A is said to be complete if every Cauchy sequence in A is convergent.

The following example shows that there may exist Cauchy sequence in an i-f-e-n-NLS which is not convergent.

Example: Let $(X, \|\bullet,\bullet,\ldots,\bullet\|)$ be an n-normed linear space and define $a * b = \min\{a, b\}$ and $a \lozenge b = \max\{a, b\}$, for all $a, b \in [0,1]$, $t > 0$. $\Phi(x,t) = \prod_{j=1}^{n} \mu(x_j, t) = \frac{t}{t + \|x_1, x_2, \ldots, x_n\|}$ and

$$\Psi(x,t) = \prod_{j=1}^{n} \nu(x_j, t) = \frac{\|x_1, x_2, \ldots, x_n\|}{t + \|x_1, x_2, \ldots, x_n\|}.$$

Then $A = \{(X, \Phi(x,t), \Psi(x,t)) \mid X = (x_1, x_2, \ldots, x_n) \in X^n\}$ is an i-f-e-n-NLS by example.

Let $\{x_n\}$ be a sequence in A.Then

(a) $\{x_n\}$ is a Cauchy sequence in $(X, \|\bullet,\bullet,\ldots,\bullet\|)$ if and only if $\{x_n\}$ is a Cauchy sequence in A.

(b) $\{x_n\}$ is a convergent sequence in $(X, \|\bullet,\bullet,\ldots,\bullet\|)$ if and only if $\{x_n\}$ is a convergent sequence in A.

Proof: **(**a) $\{x_n\}$ is a Cauchy sequence in $(X, \|\bullet,\bullet,\ldots,\bullet\|)$

$$\Leftrightarrow \lim_{n,k\to\infty} \|x_1, x_2, \ldots, x_{n-1}, x_n - x_k\| = 0$$

$$\Leftrightarrow \lim_{n,k\to\infty} \prod_{j=1}^{n} \mu(x_j - x_k, t) = \lim_{n,k\to\infty} \frac{t}{t + \|x_1, x_2, \ldots, x_{n-1}, x_n - x_k\|} = 1$$

$$\text{and} \lim_{n,k\to\infty} \prod_{j=1}^{n} \nu(x_j - x_k, t) = \frac{\|x_1, x_2, \ldots, x_n - x_k\|}{t + \|x_1, x - x_2, \ldots, x_n - x_k\|} = 0$$

$$\Leftrightarrow \Phi(x_n - x_k, t) \to 1 \text{ and } \Psi(x_n - x_k, t) \to 0, \text{ as } n, k \to \infty$$

$$\Leftrightarrow \Phi(x_n - x_k, t) > 1 - r \text{ and } \Psi(x_n - x_k, t) < r,\ r \in (0,1), \text{ for all } n, k \ge n_{0.}$$

$\Leftrightarrow \{x_n\}$ is a Cauchy sequence in A.

(b) $\{x_n\}$ is a convergent sequence in $(X, \|\bullet,\bullet,\ldots,\bullet\|)$.

$$\Leftrightarrow \lim_{n\to\infty} \|x_1, x_2, \ldots, x_{n-1}, x_n - x\| = 0$$

$$\Leftrightarrow \lim_{n\to\infty} \prod_{j=1}^{n} \mu(x_j - x, t) = \lim_{n\to\infty} \frac{t}{t + \|x_1, x_2, \ldots, x_{n-1}, x_n - x\|} = 1$$

$$\text{and} \lim_{n\to\infty} \prod_{j=1}^{n} \nu(x_j - x, t) = \frac{\|x_1, x_2, \ldots, x_n - x\|}{t + \|x_1, x - x_2, \ldots, x_n - x\|} = 0$$

$$\Leftrightarrow \Phi(x_n - x, t) \to 1 \text{ and } \Psi(x_n - x, t) \to 0, \text{ as } n \to \infty$$

$$\Leftrightarrow \Phi(x_n - x, t) > 1 - r \text{ and } \Psi(x_n - x, t) < r,\ r \in (0,1), \text{ for all } n \ge n_{0.}$$

$\Leftrightarrow \{x_n\}$ is a convergent sequence in A.

Thus if there exists an n-normed linear space $(X, \|\bullet,\bullet,\ldots,\bullet\|)$ which is not complete, then the intuitionistic fuzzy Euclidean n-norm induced by such a crisp n-norm $\|\bullet,\bullet,\ldots,\bullet\|$ on an incomplete n-normed linear space X is an incomplete intuitionistic fuzzy Euclidean n-normed linear space.

Theorem: Let A be an i-f-e-n-NLS, such that every Cauchy sequence in A has a convergent subsequence. Then A is complete.

Proof: Let $\{x_n\}$ be a Cauchy sequence in A and $\{x_n\}$ be a subsequence of $\{x_n\}$ that converges to x. We prove that $\{x_n\}$ converges to x. Let $t > 0$ and $\varepsilon \in (0,1)$.choose $r \in (0,1)$ such that $(1-r) * (1-r) > 1-\varepsilon$ and $r \lozenge r < \varepsilon$. since $\{x_n\}$ is a Cauchy, there exists an integer $n_0 \in N$ such that

$$\Phi\left(x_n - x_k, \frac{t}{2}\right) = \prod_{j-i}^{n} \mu\left(x_j - x_k, \frac{t}{2}\right) > 1-r \text{ and } \Psi\left(x_n - x_k, \frac{t}{2}\right) = \coprod_{j=1}^{n} \nu\left(x_j - x_k, \frac{t}{2}\right) < r, \text{ for all } n, k \geq n_0.$$

Since $\{x_{n_k}\}$ converges to x, there is a positive $i_k > n_0$ such that

$$\Phi\left(x_n - x, \frac{t}{2}\right) = \prod_{j-i}^{n} \mu\left(x_j, x_{i_k} - x, \frac{t}{2}\right) > 1-r \text{ and } \Psi\left(x_n - x, \frac{t}{2}\right) = \coprod_{j=1}^{n} \nu\left(x_j, x_{i_k} - x, \frac{t}{2}\right) < r$$

Now,

$$\begin{aligned}
\Phi(x_n - x, t) &= \prod_{j-i}^{n} \mu\left(x_j - x_{i_k} + x_{i_k} - x, \frac{t}{2} + \frac{t}{2}\right) \\
&\geq \mu\left(x_1, x_2, ..., x_{n-1}, x_n - x_{i_k}, \frac{t}{2}\right) * \mu\left(x_1, x_2, ..., x_{n-1}, x_{i_k} - x, \frac{t}{2}\right) \\
&> (1-r) * (1-r) > 1-\varepsilon
\end{aligned}$$

and

$$\begin{aligned}
\Psi(x_n - x, t) &= \coprod_{j=1}^{n} \nu\left(x_j - x_{i_k} + x_{i_k} - x, \frac{t}{2} + \frac{t}{2}\right) \\
&\leq \nu\left(x_1, x_2, ..., x_{n-1}, x_n - x_{i_k}, \frac{t}{2}\right) \lozenge \nu\left(x_1, x_2, ..., x_{n-1}, x_{i_k} - x, \frac{t}{2}\right) \\
&< r \lozenge r < \varepsilon
\end{aligned}$$

Therefore $\{x_n\}$ converges to x in A and hence it is complete.

3. REFERENCES

[1] K. T. Atanassov, Intuitionistic fuzzy sets VII ITKR's seesion, Sofia, June 1983 (Deposed in Central Sci-Techn. Library of Bulg. Acad. of Sci., 1967/84) (in Bulgrain).

[2] T. Bag and S.K Samanta, finite dimensional fuzzy normed linear spaces, J. Fuzzy Math. 11 (2003), no. 3, 687-705.

[3] S. C. Chang and J. N Mordesen, Fuzzy linear operators and fuzzy normed linear spaces Bull. Calcutta Math. Soc. 86 (1994), no. 5, 429-436.

[4] M. S. El Naschie, On the verifications of heterotic strings theory and $\varepsilon^{(\infty)}$ theory. Chaos, Solitons and Fractals, 11 (2000), pp. 397-407.

[5] C. Felbin, Finite-dimensional fuzzy normed linear space, Fuzzy Sets and Systems 48 (1992), no. 2, 239-248.

[6] S. Gähler, Linear 2-normierte Raume, Math. Nachr. 28 (1964), 1-43.

[7] A. George and P. Veeramani, On some result in fuzzy metric space. Fuzzy Sets and Systems, 64 (1994), pp. 395-399.

[8] H. Gunawan and M. Mashadi, On n-normed spaces, Int. J. Math. Math. Sci. 27 (2001), no. 10, 631-639.

[9] A. K. Katsaras, Fuzzy topological vector spaces. II., Fuzzy Sets and Systems 12 (1984), no. 2, 143-154.

[10] S. S Kim and Y. J. Cho, Strict convexity in linear n-normed spaces, Demonstration Math. 29 (1996), no. 4, 739-744.

[11] S. V. Krishna and K.K.M. Sharma, Separation of fuzzy normed linear spaces, Fuzzy Sets and Systems 63 (1994), no. 2, 207-217.

[12] J. H. Park, Intuitionistic fuzzy metric spaces Chaos Solitons Fractals 22 (2004), no. 5, 1039-1036.

[13] R. Saadati and J. H. Park, On the Intuitionistic Fuzzy Topological Spaces. Choas, Solitons and Fractals 27 (2006), pp. 331-344.

[14] B. Schweizer and A. Sklar, Satistical metric spaces. Pacific J. Math., 10 (1960), pp. 314-334.

[15] S. Vijayabalaji, N. Thilaigovindan and Young Bae Jun, Intuitionistic Fuzzy n-normed linear space, Bull. Korean Math. Soc. 44(2007), no. 2, pp. 291-308.

[16] L. A. Zadeh, Fuzzy sets. Inform. and control, 8 (1965), pp. 338-353.

Source of support: Nil, Conflict of interest: None Declared