International Journal of Mathematical Archive-3(11), 2012, 4708-4710
Available online through www.ijma.info ISSN 2229 – 5046

NORMED LINEAR SPACES FOR ADJOINT OPERATOR

[1]KHAJA MOINUDDIN*, [2]N. GOPAL REDDY and [3]M. VIJAYA KUMAR

[1]ASST PROFESSOR IN MATHEMATICS, MALULANA AZAD NATIONAL URDU UNIVERSITY
[2]RETD, PROFESSOR IN MATHEMATICS, OSMANIA UNIVERSITY.
[3]ASST PROFESSOR IN MATHEMATICS, BHOPAL, INDIA

(Received on: 27-10-12; Revised & Accepted on: 08-11-12)

ABSTRACT

In this present paper, the definition adjoint of the operator on normed linear spaces. It is shown that if (X, || ||) *and* (Y,|| ||) *are two normed linear spaces and* T: X →Y *be a strongly (weakly) bounded linear operator, then* T * : Y* →X* *(adjoint of* T) *is strongly (weakly) bounded linear operator and* $\| T \|_{\alpha}^{\star} = \| T^* \|_{\alpha}^{\star}$ *, for each* α∈(0,1].

Keywords: *Adjoint operator; Dual space; linear operator; norm.*

1. INTRODUCTION

The idea of normed linear space was around since the 1906 disseration of Frechet ,and a precursory analysis of it can be traced in the works of Eduard Helly and Hasns Prior to 1922. The modern definition was given first by Stefan Banach then under took a comprehensive analysis of such spaces which culminated in his ground breaking 1932 treatise. For instance , we think of the magnitude of a positive real number x as the length of interval (0,1] and that of –x as the length of [-x,0) Indeed , it is easily verified that the absolute value of a vector in R^nas the distance between this vectors and the origin, and as would expect ,$x \to d_2\,(x, 0)$ defines a a norm on R^n.The adjoint of an operator is generalization of the niotion of the Hermitian Conjugate of complex matrix to linear operator on complex Hilbert spaces. In this paper the adjoint of linear operator M will be indicated by M^*, as is common mathematics. The field of optimaization uses linear operators and their adjoints extensively.

2. SOME PRELIMINARY RESULTS

Definition 2.1: A mapping η: R →[0,1] is called a real number, whose α-level set is denoted by $[\eta]_a = \{t:\ \eta(t) \geq \alpha\ \}$, if it satisfies two axioms:
(a) There exists $t_0 \in R$ such that $\eta(t_0) = 1$.
(b) For each α ∈(0, 1]: $[\eta]_a = [\eta_{\alpha}^{-}, \eta_{\alpha}^{+}\ ]$, where $-\infty < \eta_{\alpha,}^{-} \leq \eta_{\alpha}^{+} < +\infty$.

1. The set of all realnumbers is denoted by F.

2. If $\eta \in F$ and If $\eta(t) = 0$whenever $t < 0$, then If η is called a non-negative real number and F^{+} stands for the set of all non-negative real numbers.

3. The number0 stands for the number satisfying $\bar{0}$(t) = 1 if $t = 0$ and $\bar{0}$((t) = 0 if $t \neq 0$ clearly , $\bar{0} \in F^{+}$. Since to each $r \in R$, one can consider $r \in F$ defined by $r(t) = 1$ if $t = r$ and $r(t) = 0$ if $t \neq 0$, R can be embedded in F.

Definition 2.2. Let X be a linear space over R. Let ‖‖: $X \to R^{+}$ .That satisfies the following properties is called a norm on X. For x,yε X.1.$\| x \| = 0$ *if and only if* $x = 0$2. (Absolute Homogendity)$\| \lambda x \| = l\lambda l\ \| x \|\ \forall \lambda \in R$3.(sub additivity))$\| x + y \| \leq \| x + y \|$ if ‖ ‖ is a norm on X . Then we say that (X, ‖ ‖) is a Normed linear space .if ‖ ‖only satisfies the requirements. (ii) and (iii). The (X, ‖ ‖) is called semi normed linear space

Definition 2.3. Let (X,‖‖) and $(Y, \|\|)$ be two normed linear spaces. An operator $T : X \to Y$ is said to be strongly continuous at x_0 e X if for a given $\varepsilon > 0, \exists \delta > 0$such that$\| Tx - Tx_o \|_{\alpha}^{2} < \varepsilon$.

Whenever $\| x - x_o \|_{\alpha}^{2} < \delta, \forall \alpha \in (0.1]$ (0,1]. If T is strongly continuous at all points of X, then T is said to be strongly continuous on X.

Definition 2.4. Let $(X,\ \|\|)$ and $(Y, \|\|)$ be two normed linear spaces. An operator-T: $X \to Y$ is said to be weakly continuous at $x_0 \in X$ if for a given $\varepsilon > 0, \exists\delta \in F^{+},, \exists\delta \in \bar{0}$ such that
$\| Tx - Tx_o \|_{\alpha}^{1} < \varepsilon$ whenever $\| x - x_o \|_{\alpha}^{2} < \delta_{\alpha}^{2}$,
$\| Tx - Tx_o \|_{\alpha}^{2} < \varepsilon$ *whenever* $\| x - x_o \|_{\alpha}^{1} < \delta_{\alpha}^{2}$

Where$[\delta]_{\alpha} = [\delta_{\alpha}^{1}, \delta_{\alpha}^{2}], \alpha \in (0,1]$

Definition 2.5: Let $(X,\|\|)$ and $(Y, \|\|)$ be two normed linear spaces and $T : X \to Y$ be a linear operator. T said to be strongly bounded if there exists a real number k > 0 such that$\| Tx \| \leq \eta \| x \|\ \forall\ x (\neq 0) \in X$.

Notation 2.1. Denote $B(X, Y)$ = Set of all strongly bounded linear operators defined form (X, ‖ ‖) to $(Y, \|\|)$.

Definition 2.6. Let (X‖) and $(Y,\|\|)$be two normed linear spaces and $T : X \to Y$ be a linear operator. T is said to be weakly bounded if there exists a interval $\eta \in F^{+}\ \eta > \ddot{\bar{0}}$, such that$\| Tx \| \leq \eta \| x \|\ \forall\ x (\neq 0) \in X$.

Corresponding author: *[1]KHAJA MOINUDDIN*
[1]ASST PROFESSOR IN MATHEMATICS, MALULANA AZAD NATIONAL URDU UNIVERSITY

Remark 2.1. T is strongly (weakly) bounded linear operator from $(X \| \|)$ to $(Y, \| \|)$ iff T is a bounded linear operator from $(X, \| \|_\alpha^1$ to $(Y, \| \|_\alpha^{-2})$) and from,. $(X, \| \|_\alpha^2$ to $(Y, \| \|_\alpha^{-1})$

Remark 2.2. If T is strongly bounded then it is weakly bounded but not conversely.

Notation 2.2. Denote $B(X, Y) =$ Set of all weakly bounded linear operators defined form $(X, \| \|)$ $(Y, || ||)$

Theorem 2.1. Let $T : X \to Y$ be a linear operator where $(X \| \|)$ and $(Y, || ||)$ are two normed linear spaces. Then T is strongly (weakly) continuous iff it is strongly (weakly) bounded.

Theorem 2.2. A linear operator T from $(X, \| \|)$ and $(Y, || ||)$ is strongly (weakly) continuous iff it is strongly (weakly) continuous at a point.

Theorem 2.3. The set $B(X, Y)$ $(B'(X, Y))$ of all strongly (weakly) bounded linear operators from a normed linear space $(X, \| \|)$ to a normed linear space $(Y, \| \|^{\sim})$ is a linear space with respect to usual linear operators.

Definition 2.7. A strongly (weakly) bounded linear operator defined from a normed linear space $(X, \| \|)$ to $(R, \| \|)$, is called a strongly (weakly) bounded linear functional.

Where, the function $\| \| : R \to [0,1]$ is defined by

$$\| r \| (t) = \begin{cases} 1 \; if \; t = |r| \\ 0 \; otherwise \end{cases}$$

Then $\| r \|$ is a norm on R and a -level sets of $\| r \|$are given by $[\| r \|]_a = [r , r]$, $0 < a \le 1$.

Denote by X^* (X') the set of all strongly (weakly) bounded linear functional over $(X, \| \|)$. We call X^* (X') the first strong (weakly) dual space of X.

3. THE MAIN RESULTS

In this section Hahn- Banach theorem is given which will be used in our main results.

Theorem 3.1. (Hahn- Banach [1]). Let $(X, \| \|)$ be a normed linear space and Z be a subspace of X. Let f be a strongly bounded linear functional defined on$(Z, \| \|)$. Then there exists a strongly bounded linear functional Λ on X such that $\Lambda|_Z = f$ and $| f |_Z < \| \Lambda \|_X$.

Lemma 3.1. [2]. Let $(X, \| \|)$ be a normed linear space and $x_0 \in X$, then the exists a strongly bounded linear functional $\Lambda \in X^*$ such that $\| \Lambda \|_\alpha^* = \bar{1}$ and $\Lambda x_0 = \|x_0\|\alpha, \forall \alpha \in (0,1]$.

Definition 3.1. Let$(X, \| \|)$.and $(Y, \| \|)$ are two normed linear spaces and $T \in B(X, Y)(B'(X, Y))$Operator $T^* : Y^* \to X^*$ *is defined* $(T^*\Lambda)(x) = \Lambda(Tx), \Lambda \in Y^*$operator T^* *is* called adjoint operator T.

Theorem 3.2. Let $(X, \| \|)$ and $(Y, \| \|)$ are two normed linear spaces. If $T \in B(X, Y)$ then $T^* \in B(Y^*, X^*)$ and $\| T \|_\alpha^* = \| T^* \|_\alpha^*$ $\forall \alpha \in (0,1]$

Proof. For each $\Lambda \in Y^*$ we get $T^* \Lambda = \Lambda T$ hence $T^*\Lambda$ is linear and continuous. So for each $\Lambda \in Y^*$ we have $T^*\Lambda \in X^*$. Clearly, T^* is linear. Since T is strongly bounded linear, the exists $k(> 0) \in R$ such that

$$\|TX\| \le \overline{K\|X\|}, \forall x(\neq 0) \in X$$

$$\frac{\|T(x)\|_\alpha^1}{\|x\|_\alpha^2} \le K \; and \; \frac{\|T(x)\|_\alpha^2}{\|x\|_\alpha^1} \le K, \forall x(\neq 0) \in X \text{ define}$$

$$\| T \|_\alpha^{*1} = \underset{x \in X, X \neq 0}{sup} \frac{\| T(x) \|_\alpha^1}{\| x \|_\alpha^2} \; (\le K)$$

$$\| T \|_\alpha^{*2} = \underset{x \in X, X \neq 0}{sup} \frac{\| T(x) \|_\alpha^2}{\| x \|_\alpha^1} \; (\le K)$$

$$\| Tx \|_\alpha^1 \le \| T \|_\alpha^{*1} \| x \|_\alpha^2 \; \forall x(\neq 0) \in X$$

$$\| Tx \|_\alpha^2 \le \| T \|_\alpha^{*2} \| x \|_\alpha^1 \; \forall x(\neq 0) \in X$$

Then $(\| \|_\alpha^{*1} : \alpha \in (0,1]\}$and $\{(\| \|_\alpha^{*2} : \alpha \in (0,1]\} \}$ are, respectively, ascending and descending families of norms. Also from above it follows that $\{[\| T \|_\alpha^{*1}, \| T \|_\alpha^{*2}] : 0 < \alpha \le 1\}$ is a family of nested bounded closed intervals of real numbers.

Let $x \in X$ and $\Lambda \varepsilon Y^*$ we have

$\| (T^*\Lambda)(x) \|_\alpha^{*1} = \| \Lambda Tx \|_\alpha^1 \le [\Lambda] \| Tx \|_\alpha^1 \le [\Lambda] \| T \|_\alpha^{*1} \| x \|_\alpha^2 \le \forall \alpha \in (0, 1]$,

So, T * continuous and $\| T^* \|_\alpha^{*1} \le \| T \|_\alpha^{*1}, \forall \alpha \in (0.1]$, (ii)

For $\varepsilon > 0, so$ $\| T \|_\alpha^*$ implies that there exists a x such that $\| x \|_\alpha = 1$ and $\| Tx \|_\alpha^1 > \| Tx \|_\alpha^{*1} - \varepsilon$

By lemma 3.1 the exists $\Lambda \in Y^*$ such $\| \Lambda \|_\alpha^{*1} = 1$ and $\Lambda(Tx) = \| Tx \|_\alpha^1$
$|(T^*\Lambda)(x)| = |\Lambda(Tx)| = \| Tx \|_\alpha^1 > \| Tx \|_\alpha^{*1} - \varepsilon, \forall \alpha \in (0.1]$

So

$\| T^{*}x \|_{\alpha}^{*1} \geq \| T^{*} \|_{\alpha}^{*1}, \forall\alpha \in (0,1], > \quad \| Tx \|_{\alpha}^{*1} - \varepsilon,$ (iii)

(ii) and (iii) imply that

$$\| T^{*}x \|_{\alpha}^{*1} = \| Tx \|_{\alpha}^{*1}, \forall\alpha \in (0,1]$$

Similarly we

$$\| T^{*}x \|_{\alpha}^{*2} = \| Tx \|_{\alpha}^{*2}, \forall\alpha \in (0,1]$$

So,

$$\| T \|_{\alpha}^{*} = \| T^{*} \|_{\alpha}^{*}, \forall\alpha \in (0,1]$$

Theorem 3.3. Let (X,‖ ‖) and $(Y, \| \|)$ be two normed linear spaces. If $T \in B'(X, Y)$ then $T^* \in B'(Y^{*}X^{*})$ and

$$\| T \|_{\alpha}^{*} = \| T^{*} \|_{\alpha}^{*}, \forall\alpha \in (0,1]$$

Proof. The proof is similar to Theorem 3.2 and therefore, it is omitted.

REFERENCES

1. E. B. Davies**,** *Spectral Theory and Differential operators,* Cambridge University Press, 1995.

2. T. Kato**,** *Perturbation Theory of Linear Operators,* CIM, Springer 1981.

3 A. Pazy**,** *Semigroups of Linear Operators and Applications to Partial Differential Equations***,** Springer, 1983.

4. M. Reed, B. Simon**,** *Methods of Modern Mathematical Physics,* Vol. 1-4, Academic Press, 1971-1978.

5. Y.K. Yosida**,** *Functional Analysis,* CIM, Springer 1980**.**

Source of support: Nil, Conflict of interest: None Declared