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CONVERGENCE OF MANN ITERATES AND IMPLICIT ITERATES

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ABSTRACT

In this paper we prove some results on uniform convexity in every direction. Further we prove strong and weak convergence of Mann iterates and implicit iterates for finite family of mappings in Banach spaces which are uniformly convex in every direction.

Keywords and phrases: *UCED, Banach Space, λ-firmly no expansive maps.*

1. INTRODUCTION

Let X be a real Banach space. The closed unit ball and unit sphere of X are denoted by B(X) = {x ∈ X: ‖ x‖ ≤ 1} and S(X) = {x ∈ X : ‖ x‖ = 1} respectively. The function δ_X: [0, 2] → [0, 1] defined by

$$\delta_X(\varepsilon)=\inf\left\{1-\left\| \frac{x+y}{2} \right\| : x,y\in B(X),\|x-y\|\geq\varepsilon\right\}$$

for any $\varepsilon \in [0, 2]$, is called modulus of convexity of Banach space X . If $\delta_X(\varepsilon) > 0$, ∀ ε > 0, then the Banach space X is uniformly convex and the Banach space X is strictly convex when $\delta_X(2) = 1$. For each ε > 0, the modulus of convexity in direction of z∈S(X) is defined by

$$\delta_X(\varepsilon,z)=\inf\left\{1-\left\|\frac{x+y}{2}\right\| : x,y\in B(X), x-y=\lambda z,|x-y|\geq\varepsilon\right\}.$$

A Banach space X is called uniformly convex in every direction if for any z ∈ S(X) and $\forall\ \varepsilon > 0$, $\delta(\varepsilon, z) > 0$. It is clear that every uniformly convex Banach space is uniformly convex in every direction and every uniformly convex in every direction Banach space is strictly convex.

Let C be a nonempty closed convex subset of X. Then a mapping T: C → C is called no expansive if

‖T x − T y ‖ ≤ ‖ x – y‖, for all x, y ∈ C. A no expansive mapping T: C → C is said to be asymptotically regular on C if $\lim_{n\to\infty} \|T^n x - T^{n+1} x\| = 0$, for any x ∈ C.

A mapping T on C is said to be λ-firmly no expansive if there exists a $\lambda \in (0, 1)$ such that ‖Tx –Ty‖ ≤ ‖(1− λ)(x − y) + λ (T x − Ty) ‖, for all x, y ∈ C. It has been observed [2, Proposition 1.2] that every strongly no expansive mapping T: C → C on a nonempty weakly compact convex subset C of a Banach space X is asymptotically regular.

The existence of fixed points for a no expansive mapping is independently proved by Browder [1], Gohde [4] and Kirk [6]. Wei-Shih Du, *et. al.* [3] proved the following result: Let C = $\cup_{k=i}^{n} C_k$ be a Finite union of nonempty weakly compact convex subsets C_k of a uniformly convex in every direction Banach space X, and T: C→ C is λ-firmly no expansive for some $\lambda \in (0, 1)$, then T has an fixed point in C.

For a mapping T on C, we consider the following iteration scheme: $x_1 \in C$,

$$x_{n+1} = \alpha_n T[\beta_n T x_n + (1-\beta_n)x_n] + (1-\alpha_n)x_n,\ \forall n \geq 1, \qquad (1)$$

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where $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in [0,1]. Such an iteration scheme was introduced by Ishikawa [5]. We can also see the Mann iteration scheme [8]: $x_1 \in C$,

$$x_{n+1} = \alpha_n T x_n + (1-\alpha_n)x_n,\ \forall\, n \geq 1, \tag{2}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in [0,1]. Gang-Eun Kim [7] proved the convergence of Mann and Ishikawa iteration methods to fixed points in a uniformly convex Banach space.

Mann iterative process for common fixed points of a finite family of mappings $\{T_j : j \in J\}$ is as follows:

$$x_n = \alpha_n x_{n-1} + (1-\alpha_n)T_n x_{n-1},\ n \in N, \tag{3}$$

where $T_n = T_{n(\mathrm{mod}N)}$ and the function modN takes values in $J = \{1, 2, 3, \ldots.N\}$ the set of first N natural numbers; Xu and Ori [9] introduced the following implicit iterative process:

$$x_n = \alpha_n x_{n-1} + (1-\alpha_n)T_n x_n,\ n \in N, \tag{4}$$

where $T_n = T_{n(\mathrm{mod}N)}$ for the common fixed points of the finite family $\{T_j : j \in J\}$. In 2008 Zhao [10] *et. al.* introduced the following implicit iteration scheme for the common fixed points of the finite family $\{T_j : j \in J\}$;

$$x_n = \alpha_n x_{n-1} + \beta_n T_n x_{n-1} + \lambda_n T_n x_n,\ n \in N, \tag{5}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\lambda_n\}$ are sequences in (0,1). If N = 1, the iteration (5) reduces to

$$x_n = \alpha_n x_{n-1} + \beta_n T x_n + \lambda_n T x_n,\ n \in N, \tag{6}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\lambda_n\}$ are three sequences satisfying $\alpha_n + \beta_n + \lambda_n = 1$, $0 < a \leq \alpha_n, \beta_n, \lambda_n \leq b < 1$, a and b being any two real numbers. In first section of this paper, we derive some consequences from the definition of uniform convexity in every direction.

In section two we show that the iterates $\{x_n\}$, defined by (2) converges weakly or strongly to a fixed point in a uniformly convex in every direction Banach space X.

In section three, we prove the convergence of the implicit iteration defined by (6) to a fixed point in uniformly convex in every direction Banach space X.

2. SOME CONSEQUENCES OF UCED BANACH SPACE

Theorem 2.1 Let X be UCED Banach space. Then we have the following:

(a) For any r and $\in$ with $r \geq \in > 0$ and elements $x, y \in X$ with $\|x\| \leq r$, $\|y\| \leq r$, $\|x-y\| \geq \in$ and there is $z \in S(X)$ such that $x - y \in \mathrm{span}(\{z\})$ there exists a $\delta = \delta(z, \in/r) > 0$ such that

$$\left\|\frac{x+y}{2}\right\| \leq r[1-\delta]$$

(b) For any r and $\in$ with $r \geq \in > 0$ and elements $x, y \in X$ with $\|x\| \leq r, \|y\| \leq r, \|x-y\| \geq \in$ and there is $z \in S(X)$ such that $x - y \in \mathrm{span}(\{z\})$, then there exists a $\delta = \delta(z, \in/r) > 0$ such that

$$\|tx + (1-t)y\| \leq r[1 - 2\min\{t, 1-t\}\,\delta] \text{ for all } t \in (0, 1).$$

Proof. (a) Suppose $\|x\| \leq r, \|y\| \leq r, \|x-y\| \geq \in$ and there is $z \in S(X)$ such that $x - y \in \mathrm{span}(\{z\})$, then we have $\left\|\frac{x}{r}\right\| \leq 1$ $\left\|\frac{y}{r}\right\| \leq 1$ and $\left\|\frac{x}{r} - \frac{y}{r}\right\| \geq \frac{\in}{r} > 0$, and $\frac{x}{r} - \frac{y}{r} \in \mathrm{span}(\{z\})$. By the dentition of UCED there exist $\delta = \delta(z, \frac{\varepsilon}{r}) > 0$ such that $\left\|\frac{x+y}{2}\right\| \leq 1 - \delta$, this implies

$$\left\|\frac{x+y}{2}\right\| \leq r(1-\delta)$$

(b) When t = ½, $\|tx+(1-t)y\| = \left\|\frac{x+y}{2}\right\| \le r(1-\delta)$ (by (a)). If t ∈ (0, 1/2] we have

$$\|tx + (1-t)y\| = \|tx+(1-t)y\| = \|t(x+y)+(1-2t)y\|$$
$$\le 2t\left\|\frac{x+y}{2}\right\| + (1-2t)\|y\|$$

From part (a) there exist $\delta = \delta(z, \in/r) > 0$ such that

$$\left\|\frac{x+y}{2}\right\| \le r(1-\delta)$$

we have

$$\|tx + (1-t)y\| \le 2tr[1-\delta] + (1-2t)r = r[1-2t\delta]$$

If t ∈ [1/2, 1) we have,

$$\begin{aligned}\|tx + (1-t)y\| &= \|(2t-1)x + (1-t)(x+y)\|\\ &\le (2t-1)x + 2(1-t)\ \|(x+y)/2\|\\ &\le (2t-1)r + 2(1-t)r(1-\delta)\\ &= r[1-2(1-t)\delta].\end{aligned}$$

Therefore, we have

$$\|tx + (1-t)y\| \le r[1 - 2\min\{t, 1-t\}\delta].$$

Theorem 2.2. Let X be a UCED Banach space and let $\{t_n\}$ be a sequence of real numbers in (0,1) bounded away from 0 and 1. Let $\{x_n\}$ and $\{y_n\}$ be two sequences in X such that $\limsup_n \|x_n\| \le a$, $\limsup_n \|y_n\| \le a$ and $\limsup_n \| t_nx_n + (1-t_n)y_n\| = a$ for some $a \ge 0$ and there is $z \in S(X)$ with $x_n - y_n \in \text{span}(\{z\})$ for each n, then $\lim_n \|x_n - y_n\| = 0$.

Proof. a = 0 is a trivial case. a>0, suppose for contradiction that $\{x_n - y_n\}$ doesnot converge to 0. Then there exists a subsequence $\{x_{ni} - y_{ni}\}$ of $\{x_n - y_n\}$ such that $\inf_i \|x_{ni} - y_{ni}\| > 0$. $0 < t_n < 1$ and there exist two positive numbers α and β such that $0 < \alpha \le t_n \le \beta < 1$ for all n ∈ N. Because $\limsup_n \|x_n\| \le a$ and $\limsup_n \|y_n\| \le a$, we may assume $r \in (a, a+1)$ for a subsequence $\{n_i\}$ such that $\|x_{ni}\| \le r$, $\|y_{ni}\| \le r$, a < r. Choose $r \ge \in > 0$ such that $2\alpha(1-\beta)\delta(z, \in/r) < 1$ and $\|x_{ni}-y_{ni}\| \ge \in > 0$ and there is z ∈ S(X) such that $x_{ni} - y_{ni} \in \text{span}(\{z\})$ for all i ∈ N. From the above theorem we have

$$\begin{aligned}\|tx_{ni} + (1-t)y_{ni}\| &\le r[1-2t_{ni}(1-t_{ni})\delta(z, \varepsilon/r)]\delta(z,\in/r)]\\ &\le r[1-2\alpha(1-\beta)\delta(z, \in/r)]\\ &< r \end{aligned} \tag{9}$$

Hence contradiction. So we get

$$\lim_n \|x_n - y_n\| = 0.$$

3. CONVERGENCE OF SEQUENCES IN UCED BANACH SPACES

Theorem 3.1 Let E be a uniformly convex in every direction Banach space satisfying Opial's condition and let $C = \cup_{i=1}^{n} C$ be a union of nonempty weakly compact convex subset Ci of E and ~~let~~ $T:C$ be a λ-firmly nonexpansive for some $\lambda \in (0, 1)$ and $tTx + (1-t)x \in C$ for all $x \in C$. and $t \in (0, 1)$. Let $\{\alpha_n\}$ be a real sequence satisfying $0 < a \le \alpha_n \le b < 1$ for all n ∈ N. Pick $x_1 \in C$ and define

$$x_{n+1} = \alpha_n T x_n + (1-\alpha_n)x_n \quad \text{for all } n \in N.$$

for above sequence $\{x_n\}$ there is $z \in S_X$ such that $x_n - Tx_n \in \text{span}(\{z\})$ for all n∈N. Then sequence $\{x_n\}$ converges to a fixed point of T.

Proof. The existence of fixed point follows from Wei-Shih Du *et. al.*. Let w be a fixed point of T. Then

$$\begin{aligned}\|x_{n+1} - w\| &= \|\alpha_n T x_n + (1-\alpha_n)x_n - w\|\\ &\le \alpha_n \|T x_n - T w\| + (1-\alpha_n)\|x_n - w\|\\ &\le \alpha_n \|x_n - w\| + (1-\alpha_n)\|x_n - w\|\\ &= \|x_n - w\| \quad \text{for all } n \ge 1\end{aligned} \tag{10}$$

So $\{x_n\}$ is bounded and $\lim_{n\to\infty}\|x_n - w\|$ exists. Put $c = \lim_{n\to\infty}\| x_n - w\|$. Since T is λ-firmly nonexpansive for some $\lambda \in (0, 1)$ we obtain

$$\|T x_n - w\| \leq \| (1 - \lambda) (x_n - w) + \lambda (T x_n - w) \|$$
$$\leq (1 - \lambda) \|x_n - w \| + \lambda \|x_n - w\|$$

and

$$\|T x_n - w\| \leq \|x_n - w\| \tag{11}$$

Taking $\limsup_{n\to\infty}$ in both sides we obtain

$$\limsup_{n\to\infty} \|T x_n - w\| \leq \limsup_{n\to\infty} \|x_n - w \| = c.$$

Further, we have

$$\lim_{n\to\infty}\left\|\alpha_n\left(Tx_n - w\right)+\left(1-\alpha_n\right)\left(x_n - w\right)\right\| = \lim_{n\to\infty}\left\|x_{n+1} - w\right\| - 0$$

If $0 < a \leq \alpha_n \leq b < 1$ by Theorem 2.2 we have $\lim_{n\to\infty} \|T x_n - x_n\| = 0$. Now we have to show that the iterates $\{x_n\}$ converges weakly to a fixed point of T. We know that $\lim_{n\to\infty} \|x_n - w\|$ exists, $w \in F(T)$. Let w_1 and w_2 be two weak subsequential limits of the sequence {xn}. We claim that the conditions $x_{ni} \to w_1$ and $x_{nj} \to w_2$ implies $w_1 = w_2 = F(T)$.

We first show $w_1, w_2 \in F(T)$. In fact if $T w_1 = w_1$ then by Opial's condition we have

$$\limsup_{n\to\infty} \| x_{ni} - w_1 \| < \limsup_{n\to\infty} \|x_{ni} - T w_1\|$$

Since T is λ-firmly nonexpansive for some $\lambda \in (0, 1)$, we obtain

$$\limsup_{i\to\infty} \|T_{ni} - T w_1 \| \leq \limsup_{i\to\infty}\|(1 - \lambda)(xni - w_1) + \lambda (T x_{ni} - T w_1)\|$$

$$\leq (1 - \lambda) \limsup \|x_{ni} - w_1 \| + \lambda \limsup \|T x_{ni} - T w_1 \| \tag{12}$$

and thus

$$\limsup_{i\to\infty}\|x_{ni} - T w_1\| \leq \limsup_{i\to\infty}\|T x_{ni} - T w_1\| \leq \limsup_{i\to\infty} \| x_{ni} - w_i\|$$

hence we have $\qquad \limsup_{i\to\infty}\| x_{ni} - T w_1\| \leq \limsup_{i\to\infty} \| x_{ni} - w_1 \|$

This is contradiction. Hence we have $T w_1 = w_1$. Similarly, we have $T w_2 = w_2$. Next, we have to show that $w_1 = w_2$.

If not, i.e. $w_1 = w_2$ by Opial's condition.

$$\lim_{n\to\infty} \| x_n - w_1\| = \lim_{n\to\infty} \| x_{ni} - w_1\| < \lim_{i\to\infty} \| x_{ni} - w_2\| = \lim_{n\to\infty} \| x_n - w_2\| = \lim_{j\to\infty} \| x_n - w_2\|$$
$$< \lim_{j\to\infty} \| x_{nj} - w_1\| = \lim_{n\to\infty} \| x_n - w_1\|.$$

This is a contradiction. Hence we have $w_1 = w_2$. This implies that $\{x_n\}$ converges weakly to a fixed point of T.

Theorem 3.2 Let E be a uniformly convex in every direction Banach space. Let $C = \bigcup_{i=1}^{n} C_i$ be a union of nonempty weakly compact convex subsets C_i of E with $C_i \subset C_{I+1}$. Suppose that $T: C \to C$ is λ-firmly nonexpansive for some $\lambda \in (0, 1)$ such that T (C) is contained in compact subset of C. Then for any initial data $x_1 \in C$ the iterates $\{x_n\}$ defined by

$$x_{n+1} = \alpha_n T x_n + (1 - \alpha_n)x_n \text{ f or all } n \in N,$$

where $\{\alpha_n\}$ are choosen so that $\alpha_n \in [a, b]$ for some $0 < a \leq b < 1$, and for above sequence $\{x_n\}$ there is $z \in S(X)$ suchthat $x_n - T x_n \in \text{span}(\{z\})$ for all $n \in N$, converges strongly to a fixed point T .

Proof. $\{x_n\}$ is well defined. The existence of fixed point follows from Wei -Shih Du etal. By Mazur's theorem $\overline{co}$ $(\{x_1\} \cup T (C))$ is compact subset of C containing $\{x_n\}$. There exist a subsequence $\{x_m\}$ of the sequence $\{x_n\}$ and a point $z \in C$ suchthat $x_m \to z$. As in proof of above theorem $\{x_n - Tx_n\} \to 0$ as $n \to \infty$. Since T is λ-firmly nonexpansive for some $\lambda \in (0, 1)$, we obtain

$$\|z - Tz\| = \|z - x_m + x_m - Tx_m + Tx_m - Tz\| \le \|z - x_m\| + \|x_m - Tx_m\| + \|Tx_m - Tz\|$$

$$\le 2\|z - x_m\| + \|x_m - Tx_m\| \to 0 \text{ as } m \to \infty.$$

Hence $Tz = z$.

$$\|x_n+1 - z\| = \|\alpha_n T x_n + (1 - \alpha_n)x_n - z\| \le \alpha_n|T|\, x_n - z\| + (1-\alpha_n)\|x_n - z\| = \alpha_n\|x_n - z\| + (1 - \alpha_n)\|x_n - z\| = \|x_n - z\|.$$

This implies

$$\|x_{n+1} - z\| \le \|x_n - z\| \text{ f or all } n \ge 1.$$

So $\{x_n\}$ is bounded and $\lim_{n\to\infty} \|x_n - z\|$ exists. Hence we have $\lim_{n\to\infty} \|x_n - z\| = 0$.

Theorem 3.3 Let E be a uniformly convex in every direction Banach space and Let $C = \cup_{i=1}^{n} C_i$ be a union of nonempty weakly compact convex subsets Ci of E with $C_i \subset C_i+1$. Suppose that $T: C \to C$ is λ-firmly nonexpansive for some $\lambda \in (0, 1)$ then for any initial data $x_1$in C the iterates $\{x_n\}$ defined by

$$x_{n+1} = \alpha_n T x_n + (1 - \alpha_n) x_n \text{f or all } n \in N.$$

where $\{\alpha_n\}$ are choosen so that $\alpha_n \in [a, b]$ for some $0 < a \le b < 1$ and for above sequence $\{x_n\}$ there is $z \in S(X)$ such that $x_n - Tx_n \in \text{span}(\{z\})$ for all $n \in N$, converges weakly to a fixed point of T.

Proof $\{x_n\}$ is well defined. The existence of fixed point follows from Wei-Shih Du *et al.* C is weakly compact subset of E. There exist a subsequence $\{x_m\}$ of the sequence $\{x_n\}$ and a point $z \in C$ such that $x_m \to z$. as in proof of theorem (1) $\{x_n - Tx_n\} \to 0$ as $n \to \infty$. Since T is λ - firmly nonexpansive for some $\lambda \in (0, 1)$, we obtain from Theorem 3.2 $z = Tz$. and $\lim_{n\to\infty}\|x_n - z\|$ exists. Hence we have weak-$\lim_{n\to\infty}\|x_n - z\| = 0$. Hence sequence $\{x_n\}$ converges weakly to a fixed point of T.

4. CONVERGENCE OF IMPLICIT ITERATION

Zhao et al. introduced the following implicit iterate for finite family of mappings $\{T_j: j \in J\}$ is as follows

$$x_n = \alpha_n x_{n-1} + \beta_n T_n x_{n-1} + \gamma_n T_n x_n, \ n \in N \tag{13}$$

where $T_n = T_{n \bmod N}$. If $N = 1$, the iteration (13) reduces to

$$x_n = \alpha_n x_n - 1 + \beta_n T x_n - 1 + \gamma_n T x_n, \ n \in N \tag{14}$$

where $\{\alpha_n\},\{\beta_n\},\{\gamma_n\}$ are three sequence satisfying $\alpha_n + \beta_n + \gamma_n = 1$, $0 < a \le \alpha_n, \beta_n, \gamma_n \le b < 1$.

Lemma 4.1. Let E be a UCED Banach space and K be a nonempty weakly compact convex subset of E. Let $T: K \to K$ be a λ- firmly nonexpansive mappings with nonempty fixed point set F. Let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ be three real sequences satisfying $\alpha_n + \beta_n + \gamma_n = 1$, $0 < a \le \alpha_n, \beta_n, \gamma_n \le b < 1$. From arbitrary $x_0 \in K$, define the sequence $\{x_n\}$ by (14) for the above sequence $\{x_n\}$ there is $z \in S_X$such that $x_n - Tx_n \in \text{span}(\{z\})$ for all $n \in N$. Then
(i) $\lim_{n\to\infty} \|x_n - p\|$ exist for each $p \in F$.
(ii) $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$

Proof. The existence of fixed point follows from Wei-Shih Du et al. Let $p \in F$.

$$\begin{aligned}
\|x_n - p\| &= \|\alpha_n x_{n-1} + \beta_n T x_{n-1} + \gamma_n T x_n - p\| \\
&= \|\alpha_n(x_{n-1} - p)\| + \|\beta_n(T x_{n-1} - p) + \gamma_n(T x_n - p)\| \\
&\le \alpha_n\|x_{n-1} - p\| + \beta_n\|Tx_{n-1} - p\| + \gamma_n\|Tx_n - p\| \\
&\le \alpha_n\|x_{n-1} - p\| + \beta_n\|x_{n-1} - p\| + \gamma_n\|x_n - p\| \\
&= (\alpha_n + \beta_n)\|x_{n-1} - p\| + \gamma_n\|x_n - p\|
\end{aligned}$$

$$(1 - \gamma_n)\|x_n - p\| \le (1 - \gamma_n)\|x_n - p\|$$
$$\|x_n - p\| \le \|x_{n-1} - p\|$$

because $\gamma_n \in (0, 1)$ Thus $\lim_{n\to\infty}\|x_n - p\|$ exists for each $p \in F$.

(ii) From (i) $\lim_{n\to\infty} \| x_n - p \| = d$. Then

$$\lim_{n\to\infty}\|x_n - p\| = \lim_{n\to\infty}\left\|(1-\gamma_n)\left[\frac{\alpha_n}{1-\gamma_n}(x_{n-1}-p)+\frac{\beta_n}{1-\gamma_n}(Tx_{n-1}-p)\right]+\gamma_n(Tx_n - p)\right\|.$$

Since T is λ-firmly nonexpansive so nonexpansive and F = φ, we have$\|T x_n - p\| \le \| x_n - p\|$ f or each p Taking lim sup on both sides, we obtain $\limsup_{n\to\infty} \| T x_n - p\| \le \limsup_{n\to\infty} \| x_n - p \| = d$. Now, we have

$$\limsup_{n\to\infty}\left\|\frac{\alpha_n}{1-\gamma_n}(x_{n-1}-p)+\frac{\beta_n}{1-\gamma_n}(Tx_{n-1}-p)\right\| \le \limsup_{n\to\infty}\left[\frac{\alpha_n}{1-\gamma_n}\| x_{n-1}-p\| + \frac{\beta_n}{1-\gamma_n}\| Tx_{n-1}-p\|\right]$$
$$=\limsup_{n\to\infty}\left[\frac{\alpha_n+\beta_n}{1-\gamma_n}\right]\|x_{n-1}-p\|$$
$$=\limsup_{n\to\infty}\|x_{n-1}-p\|$$
$$= d.$$

Using Lemma (a), we get $\lim_{n\to\infty}\left\|\frac{\alpha_n}{1-\gamma_n}(x_{n-1}-p)+\frac{\beta_n}{1-\gamma_n}(Tx_{n-1}-p)-(Tx_n - p)\right\| = 0.$

This means that

$$\lim_{n\to\infty}\left\|\frac{\alpha_n x_{n-1}+\beta_n Tx_{n-1}-(1-\gamma_n)Tx_n}{1-\gamma_n}\right\| = 0.$$

Since $0 < a \le \gamma_n \le b < 1$ we have $1 - a \ge 1 - \gamma_n \ge 1 - b$. Hence we get,

$$\frac{1}{1-a} \le \frac{1}{1-\gamma_n} \le \frac{1}{1-b}.$$

Therefore, we have

$$\lim_{n\to\infty} \| \alpha_n x_n - 1 + \beta_n T x_n - 1 + \gamma_n T x_n - T x_n\| = 0.$$

This implies $$\lim_{n\to\infty} \| x_n - T x_n \| = 0.$$

Theorem 4.2. Let E be a UCED Banach space which satisfies Opial's condition and K anonempty weakly compact convex subset of E. Let T: K → K be λ- firmly nonexpansive mappings with a nonempty fixed point set F. Let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ be three real sequencessatisfying $\alpha_n + \beta_n + \gamma_n = 1$, $0 < a \le \alpha_n, \beta_n, \gamma_n \le b < 1$. From arbitrary $x_0 \in K$, define the sequence $\{x_n\}$ by (2). For the sequence $\{x_n\}$ there is $z \in S_E$such that $x_n - T x_n \in \operatorname{span}(\{z\})$ for all n ∈ N. Then sequence $\{x_n\}$ converges weakly to a fixed point of T.

Proof. The existence of fixed point follows from Wei-Shih Du et al. Let q be afixed point of T. From the Lemma (4.1) $\lim_{n\to\infty} \| x_n - q\|$ exists. We prove that $\{x_n\}$ has a unique weak subsequential limit in F. Since $\{x_n\}$ is sequence in weakly compact subset of UCED Banach space E, there exists two weakly convergent subsequences $\{x_{ni}\}$ and $\{x_{nj}\}$ of $\{x_n\}$.Let w_1 and w_2 be two weak subsequential limits of the sequence $\{x_n\}$. We first shows that $w_1, w_2 \in F(T)$. In fact if $T w_1 \ne w_1$ then by Opial's condition, we have

$$\limsup_{i\to\infty}\| x_{ni} - w_1\| < \limsup_{i\to\infty} \| x_{ni} - T w_1\|.$$

Since T is λ-firmly nonexpansive for some $\lambda \in (0, 1)$ we obtain

$$\limsup_{i\to\infty} \|T x_{ni} - T w_1\| \limsup_{i\to\infty} \|(1-\lambda)(x_{ni} - w_1) + \lambda (T x_{ni} - T w_1)\|$$
$$\le (1-\lambda) \limsup_{i\to\infty}\| x_{ni} - w_1\| + \lambda \limsup_{i\to\infty}\|Tx_{ni} - T w_1\|$$

and thus

$$\limsup_{i\to\infty}\|T x_{ni} - Tw_1\| = \limsup_{i\to\infty}\|Tx_{ni} - Tw_1\| \le \limsup_{i\to\infty}\|x_{ni} - w_1\|.$$

Since we have $\limsup_{i\to\infty} \|x_{ni} - T w_1\| \le \limsup_{i\to\infty} \|x_n - w_1\|$.

This contradicts the Opial's condition. Hence we have $T w_1 = w_1$. Similarly we have $T w_2 = w_2$. Next, we have to show that $w_1 = w_2$. If not (i.e $w_1 = w_2$). Then by Opial's condition, we have

$$\begin{aligned}
\lim_{n\to\infty} \|x_n - w_1\| &= \lim_{i\to\infty}\|x_{ni} - w_1\| \\
&\le \lim_{n\to\infty}\|x_{ni} - w_2\| \\
&= \lim_{n\to\infty}\|x_n - w_2\| \\
&= \lim_{j\to\infty} \|x_{nj} - w_2\| \\
&< \lim_{j\to\infty} \|x_{nj} - w_1\| \\
&= \lim_{n\to\infty} \|x_n - w_1\|
\end{aligned}$$

which is contradiction. Then $\{x_n\}$ converges weakly to a fixed point of T.

Theorem 4.3. Let E be UCED Banach space and K be a nonempty weakly compact convex subset of E. Suppose T: K $\to$ K is λ-firmly nonexpansive for some $\lambda \in (0, 1)$ such that T (K) is contained in compact subset of K. Then the implicit iterative algorithm $\{x_n\}$ defined by

$$x_n = \alpha_n x_n - 1 + \beta_n T x_n - 1 + \gamma_n T x_n \text{f or all } n \in N.$$

with initial arbitrary $x_0 \in K$. Where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ be three real sequence satisfying $\alpha_n + \beta_n + \gamma_n = 1$, $0 < a \le \alpha_n$, β_n, $\gamma_n \le b < 1$. For the sequence $\{x_n\}$ there is $z \in S_E$ such that $x_n - T x_n \in \text{span}(\{z\})$ for all $n \in N$, converges strongly to a fixed point of T .

Proof. $\{x_n\}$ is well defined. The existence of a fixed point follows from Wei-Shih Du et al. By Mazur's theorem $\overline{co}$ (x $\cup$ T (K)) is compact subset of K containing $\{x_n\}$. There exists a subsequence $\{x_m\}$ of the sequence $\{x_n\}$ and a point $z \in K$ suchthat $x_m \to z$ as in proof of lemma (2.1) $\{x_n - T x_n\} \to 0$ as $n \to \infty$. Since T is λ-firmly nonexpansive for some $\lambda \in (0, 1)$ we obtain

$$\begin{aligned}
\|z - T z\| &= \| z - x_m + x_m - T x_m + T x_m - Tz\| \\
&\le \|z - x_m\| + \| x_m - Tx_m\| + \|T x_m - T z\| \\
&\le \|z - x_m\| + \|x_m - Tx_m\| + \|x_m - z\| \\
&= 2\| z - x_m\| + \| x_m - T x_m\| \to 0 \text{ as } n \to \infty
\end{aligned}$$

Hence

$$T z = z$$

$$\begin{aligned}
\|x_n - z\| &= \| \alpha_n x_{n-1} + \beta_n T x_{n-1} + \gamma_n T x_n - z\| \\
&= \|\alpha_n (x_n - z) + \beta_n (T x_{n-1} - z) + \gamma_n (T x_n - z)\| \\
&\le \alpha_n \|x_{n-1} - z \| + \beta_n \|T x_{n-1} - z \| + \gamma_n \|T x_n - z\| \\
&\le \alpha_n \|x_{n-1} - z \| + \beta_n \|x_{n-1} - z \| + \gamma_n \|x_n - z\|
\end{aligned}$$

This implies

$$(1 - \gamma_n) \| x_n - z\| \le (\alpha_n + \beta_n) \| x_{n-1} - z \| = (1 - \gamma_n) \| x_{n-1} - z\|$$

$$\|xn - z \| \le \| xn - 1 - z\| \quad \text{f or all } n \ge 1.$$

So $\{x_n\}$ is bounded and $\lim_{n\to\infty} \| x_n - z \|$ exists. Hence we have $\lim_{n\to\infty} \| x_n - z \| = 0$.

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