International Journal of Mathematical Archive-3(10), 2012, 3853-3863
IJMA Available online through www.ijma.info ISSN 2229 – 5046

FROM DYNAMICAL TREE GRAPH INTO DYNAMICAL MANIFOLD

M. El-GHOUL* & A. A. SAAD
**Mathematics Department, Faculty of science, Tanta University, Tanta, Egypt*
***Mathematics Department, Faculty of science, Alexandria University, Alexandria, Egypt*

(Received on: 20-06-12; Revised & Accepted on: 26-09-12)

ABSTRACT

In this paper, we shall discuss the changes which will occur on the dynamical tree graph to transform into dynamical manifold. We will deduce the changes on the vertices, edges, matrices and polynomial of dynamic graph under the change of the time. Finally, we will discuss some dynamical manifold analytically.

Keywords: *dynamic graphs, dynamical manifold, polynomial of graphs.*

2000 Mathematics Subject Classification: *51H20, 57N10, 57M05, 14F35, 20F34.*

1. INTRODUCTION

A graph is an abstract representation of a set of objects where some pair of the objects is connected by links. The interconnected objects are represented by mathematical abstractions called vertices, and the links that connect some pairs of vertices are called edges. A dynamic graph is a graph which defined on dynamical system changed by time. The deformations of graphs discussed in many papers, these papers explained how graphs are deformed by transitive of vertex and edges [8]. More studies of polynomials of graphs were studied in [6, 7]. In this article, we will discuss the changes of graphs by time to transform into dynamical manifold. Also, we will discuss some dynamical manifolds analytically.

2. DEFINITIONS

(1) An abstract graph is a pair (V, E) where V is a finite set and E a set of unordered pairs of distinct elements of V. Thus an element of E is of the form {v, w} where v and w belong to V and $\neq$ w. The elements of V are called vertices (or nodes) and the element {v, w} of E is called the edge joining V and W (or W and V) [1-4]. A graph is called directed (or digraph) if the edges between vertices have an implied direction, and undirected otherwise.

(2) The Tutte polynomial which was introduced by Tutte (1954, [7]) is defined by a two-variable polynomial as follow:

$$T(G;x,y)=\begin{cases} xT(G-e;x,y), & e{:}\,bridge \\ yT(G\backslash e;x,y), & e{:}\,loop \\ T(G-e;x,y)+T(G\backslash e;x,y), & otherwise. \end{cases}$$

(3) Let v and w be two vertices of a graph G. If v and w are joined by edges then v and w are said to be adjacent. Also, v and w are said to be incident with e then e is said to be incident with v and [5].

(4) Let G be a graph without loops, with n vertices labeled 1,2,3,…,n. and m edges labeled 1,2,3,…,m. The adjacency matrix A(G) is $the\ n\times m$ matrix in which the entry in row i and column j is the number of edges joining the vertices i and j [5].

(5) Let G be a graph without loops, with n vertices labeled 1, 2, 3… n. and m edges labeled 1, 2, 3… m. The incidence matrix I(G) is $the\ n\times m$ matrix in which the entry in row i and Column j is 1 if the vertex i is incident with edge j and 0 otherwise [5].

(6) Loop-graph is an edge that joins a single endpoint (vertex) to itself [5].

(7) K-coloring of a graph G is a function $\sigma{:}\,V(G)\to\{1,2,\dots,k\}$ which satisfy $\sigma(i)\neq\sigma(j)$ for an edge $e=ij$, where i, j are adjacent vertices. Chromatic polynomial of a graph is polynomial of degree $n=|V(G)|$ which is defined by [6]

$$Chr(G,k)=Chr(G+ij,k)+Chr(G/ij,k).$$

Corresponding author: *M. El-GHOUL**
**Mathematics Department, Faculty of science, Tanta University, Tanta, Egypt*

- For graph has an n vertex only: $Chr(G,k) = k^n$.
- For a tree of n vertices: $Chr(T_n,k)$=$k(k-1)^n$.
- For cycle graph: $Chr(G,k) = (k-1)^n + (-1)^n(k-1)$[6].

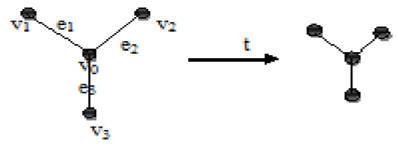
3- MAIN RESULT

Let G be a tree graph with four vertices $\{v_0, v_1, v_2, v_3\}$ and three edges $\{e_1, e_2, e_3\}$. We will discuss the changes which will occur by time on graph to change the matrices and polynomial to give dynamical manifold. Now, we will discuss the following cases:

(A) No change occurs on polynomial or matrices where our graphs defined in R^2:

Case (A): The change in volume by time induces no change in the matrices and the polynomial. But in the Limit, they are changed.

Proof: Transform (1):

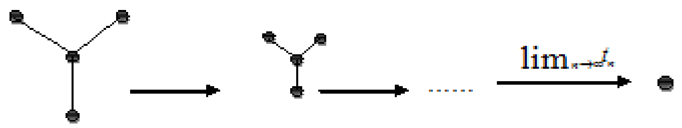


Under the action of time, the tree graph transforms into isomorphic graph which changes its volume (decrease) but no changes occur in matrices and Tutte polynomial. They are given as follow:

$$A_{af}(G) = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} = A_b(G), \quad I_{af}(G) = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} = I_b(G) \quad \text{and}$$

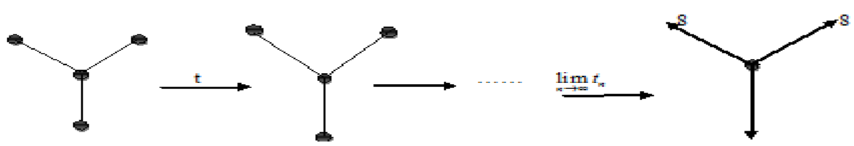
$$T_{af}(G) = x^3(t) = T_b(G) \quad where\ \ af = after \quad \text{b=before.}$$

But in the limit case the matrices and Tutte polynomial will be changed into $I(p) = [0]$, $T(p) = 1$, see the following figure as follow:



In this case, the graph transform into a vertex (point) which represents manifold of dimension zero.

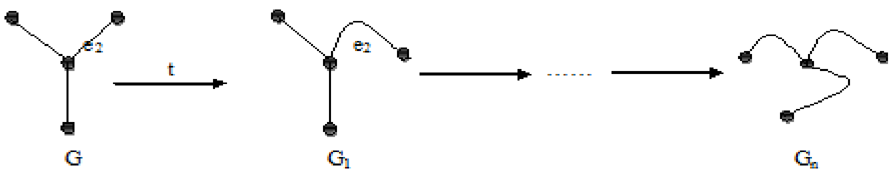
Transform (2):



Under the action of time, the tree graph transforms into graph at infinity which changes its volume (increase) which not represents a graph.

Case (B): Let G be a tree graph, the local change in the graph G by the time not changes in matrices and polynomials.

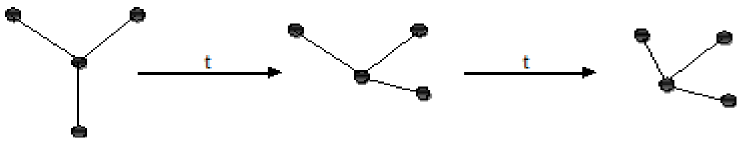
Proof: Let $G \xrightarrow{t} G_1$, G_1 is variation of the curvature of any edge and in the limit we get the graph G_n see the following figure:



Then, under the action of time, the tree graph changes into another graph with the same vertices and edges. We notice that the resultant matrices and polynomial are not changed.

Case (C): The change of the graph by the time into isomorphic graph not change the matrices and polynomial.

Proof:

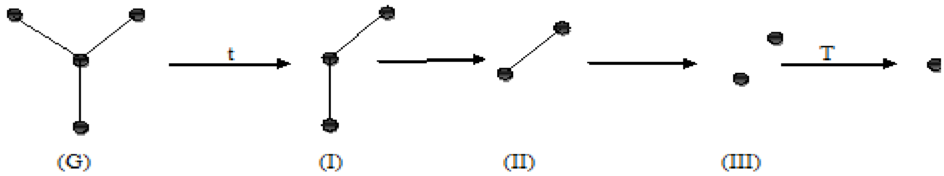


Under the action of time, the tree graph changes into isomorphic graph such that no changes in matrices and polynomial occur. But the shape of graph changes.

(B) Change occurs in polynomials and matrices where:

Define tree graph in R^2 (one dimension graph) and graph of two dimensions in R^3. Now, we discuss the change which will occur on this graphs by time:

Transform (1):



Under the action of time, tree graph transforms into a vertex by collapsing the edges pass through three cases which describe their matrices and polynomials as:

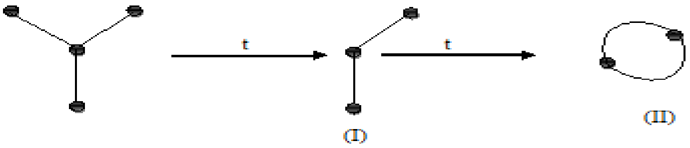
$$I(I) = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \; A(I) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \; and \; T(I) = x^2(t).$$

$$I(II) = (1 \quad 1), \; A(II) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \; and \; T(II) = x(t), chr(II) = k(k-1).$$

This graph described by chromatic polynomial which gives

$$T(III) = 1^2, \; chr(III) = k^2$$

Transform (2):



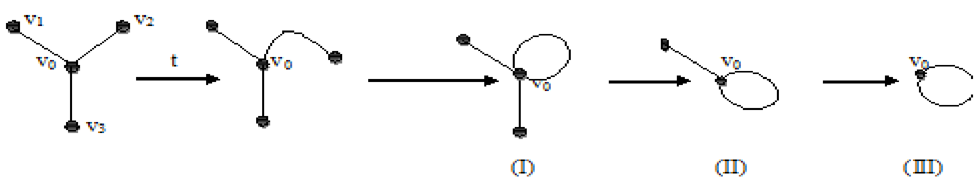
Under the action of time, one dimension graph G transformed into graph I by deleting the edge and transform into graph II by contraction of the edge which gives dynamical manifold (circle) of one dimension. We can describe the resultant graphs as follows:

$$I(II) = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, A(II) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \; and \; T(II) = (x+y)(t).$$

We notice that the graph II is a manifold also and can represented by Tutte polynomial.

Now, we will discuss the change of graph by the time with fixed point v_0 in the following cases:

Case (I):

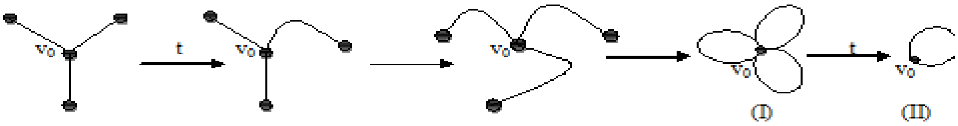


By changing the time, the tree graph is transformed into graph I, graph II and graph III which different in number of vertices and edge. Notice that the resultant graph may represent manifold with boundary. We can describe these graphs by matrices and polynomials as follow:

$$I(I)=\begin{pmatrix}1&1&0\\0&1&0\\0&1&1\end{pmatrix},\ A(I)=\begin{pmatrix}0&1&0\\1&0&1\\0&1&0\end{pmatrix}\ and\ T(I)=x^2(t)+y(t).$$

$$I(II)=\begin{pmatrix}1&1\\0&1\end{pmatrix},\ A(II)=\begin{pmatrix}0&1\\1&1\end{pmatrix}\ and\ T(II)=(xy)(t).$$

Case (II):

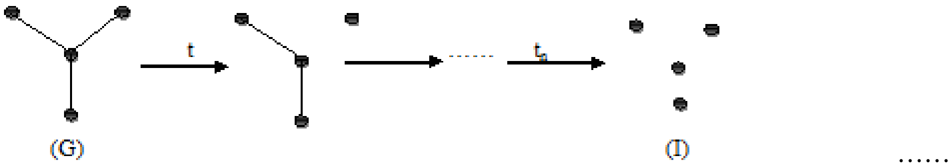


By changing of time, the dynamic graph G transformed into graph I $\longrightarrow$ graph II such that we get dynamic graph and also manifold without boundary. We can describe the resultant graph I and II as follows:

$$I(I)=\begin{pmatrix}1\\1\\1\end{pmatrix}, A(I)=(1), T(I)=y^3(t).$$

$$T(II)=y(t).$$

If the change in the time delete the interior of the edges as follows:

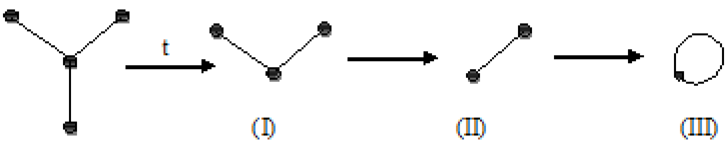


By changing of time, a dynamic graph changes into a set of points by deleting the interior of the edges. We notice that this set of points may be transformed into one point by changing its positions (a manifold of zero dimensions). We can describe this transformation of graph by the following polynomial:

$$Chr(G)=k(k-1)^3, Chr(I)=k^4\ and$$

$$T(G)=x^3(t),\ \ T(I)=1^4.$$

When the limit of changes by the time is aloop like in the following figure:



Under the action of time, tree graph transformed into manifold with boundary (line determent by two points) which describe by the following polynomial:

$$T(I)=x^2(t), T(II)=x(t).$$

Also, the graph II transformed by contraction into one loop which represent a manifold without boundary.

(C) Dynamical graph of higher dimension:

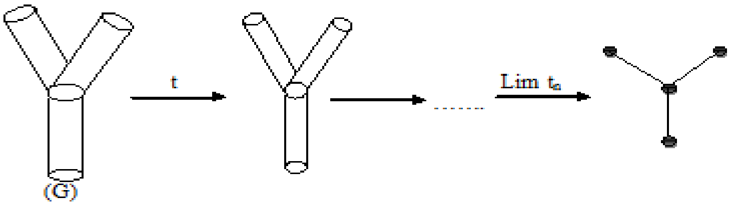
We discuss the graph of two dimensions with vertices defined in R^2(circles of one dimension) and edges defined in R^3(cylinders of two dimensions). We will make a modification of Tutte polynomial for graph of higher dimension R^n (such that we considered that $x, y\ and\ z$ as a coordinates) as follow:

$$\mathrm{M}T(G;x,y,z_1,\dots,z_{n-1})=\begin{cases}(x+z_1+\cdots+z_{n-1})T(G-e;x,y), & e: bridge\\ (y+z_1+\cdots+z_{n-1})T(G\backslash e;x,y), & e: loop\\ T(G-e)+T(G\backslash e), & otherwise.\end{cases}$$

Now, we deduce how we obtain the dynamical manifolds from the dynamical graphs by changing the time as follow:

Case (I): Let G be tree graph of two dimensions with edges (cylinders) and vertices (circles). The limit change of the time changes in dimension and polynomials.

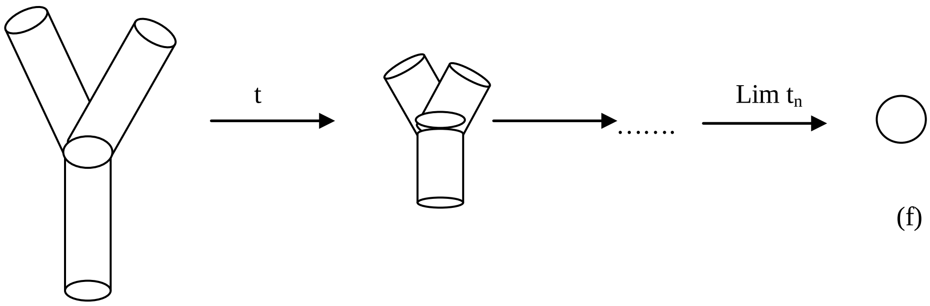
Proof: Transform (1)



Under the change of time, tree graph of two dimensions transform into dynamic graph of one dimension by changing the volume of edges (cylinder to line) and the volume of vertices (circles to points) in such case we notice that Tutte polynomial transform from modified Tutte into ordinary Tutte as follow:

$$MT(G) = (x+z)^3(t) \rightarrow T(g) = x^3(t) \; as \; z = 0.$$

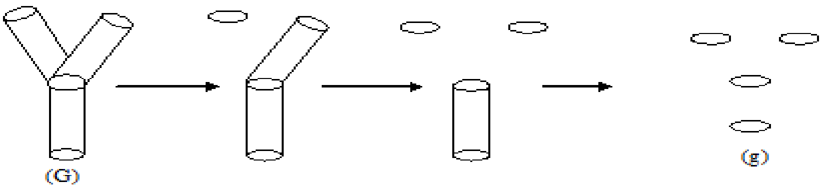
Transform (2)



Under the action of time, there is a map from $R^3 \rightarrow R^2$ such that the tree graph G transforms into one dimension manifold (circle) in such case the Tutte polynomial changes as follows:

$$MT(G) = (x+z)^3(t) \rightarrow MT(f) = 1, there \; is \; no \; edges.$$

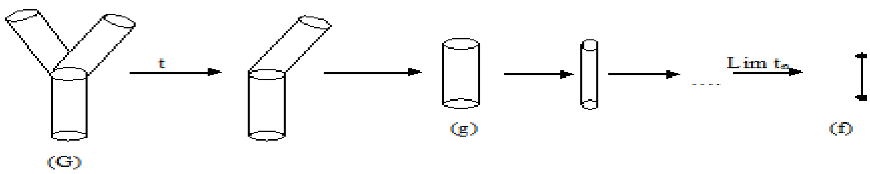
Transform (3):



Under the change of time, tree graph transformed into vertices only by deleting the edges (set of circles of one dimension). We can describe this transform by Chromatic polynomial as follow:

$$Chr(G) = k(k-1)^3 \; and \; Chr(g) = k^4$$

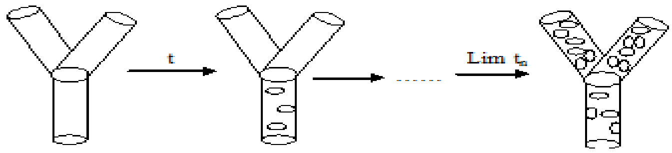
Transform (4):



Under the change of the time, the tree graph of two dimensions transformed by deleting the edges into one edge in R^3. in the limit changes, the graph g transformed into one edge in R^2. we notice that the polynomial changes as follows:

$$MT(G) = (x+z)^3(t) \rightarrow MT(g) = (x+z)(t) \rightarrow T(f) = x(t).$$

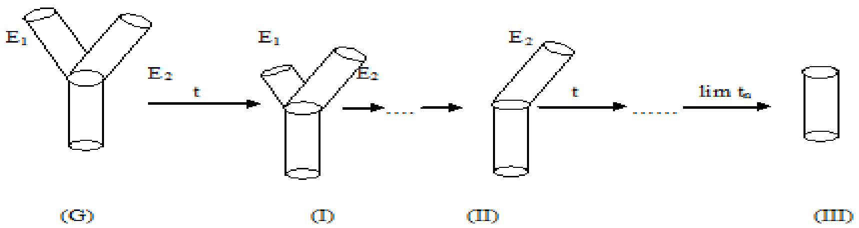
Also, we can obtain on one dimension filter edges which describe as follow transform:



In this transformation, the tree graph of two dimensions transformed into graph with hole by the time. In the limit changes, this graph transformed into graph with filter edges (graph of one dimension).

Case (II): The tree graph of two dimensions will be changed by the time into dynamical manifold with changes in polynomials.

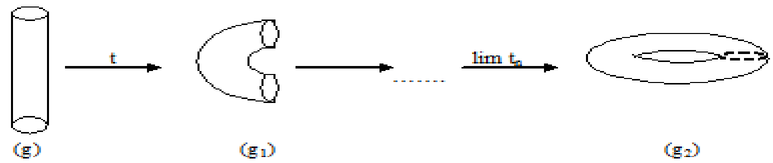
Proof:



Under the change of the time, the tree graph of two dimensions transformed into dynamical manifold without boundary (cylinder of two dimensions) such that $E_1 \to 0$ and $E_2 \to 0$ which can be described by the following polynomial:

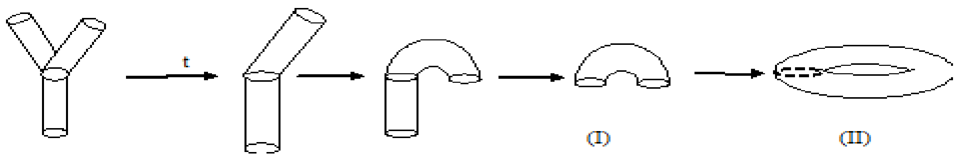
$$MT(G) = (x+z)^3(t) = MT(I),\ MT(\text{II}) = (x+z)^2(t)\ and\ MT(III) = (x+z)(t)\ .$$

The resultant manifold without boundary can be changed by the time into dynamical manifold of two dimensions and the polynomial describe as follows:



$$MT(g) = (x+z)(t) = MT(g_1) \to MT(g_2) = (y+z)(t).$$

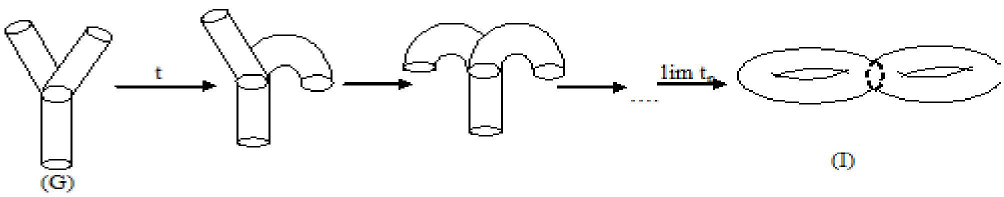
Or



By changing of time, the tree graph of two dimensions transformed into half torus (manifold with boundary) and get on torus of two dimensions at the end of the changing of time such that we get on manifolds from dynamical graph which described by the following polynomial:

$$MT(I) = (x+z)(t) \to MT(II) = (y+z)(t).$$

Or



Under the change of time, tree graph transformed into graph with two loops which is a connected sum of tori. Describe this transform by polynomial as follows:

$$MT(I) = (y+z)^2(t).$$

The same transformations occur on tree graph of three dimensions which transform from dynamical graph of three dimensions into dynamical manifold of three dimensions or two dimensions.

Finally, we shall discuss the dynamical manifold analytically by solving some dynamical system and classify the resultant solution (dynamical manifolds) as follows:

Proposition (1): Consider the dynamical system

$$\frac{dx}{dt} = A, \qquad \frac{dy}{dt} = B$$

with initial condition

$$x(0) = 0, \; y(0) = 1$$

$where\ A\ and\ B\ are\ constants.$

Define

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{B}{A} = \alpha$$

By separation of variable and integrating both sides, we get

$$y = \alpha x + \beta, \; where\ \beta\ is\ constant\ of\ integration \tag{1}$$

This equation represented by set of straight lines which are defined in R^2. This straight lines cutting y-axis in 1 and with slope may be positive or negative according to sign of constant α as follow:

Notice that in Fig (1), when α > 0 the slope of the straight line changes if $\alpha \to 0$, then the straight lines become y = 1 which is line parallel to x-axis.

If initial condition changes be $x(0) = 0, y(0) = 0$, then the straight lines take the form $y = \alpha x$ which represent lines passing through the origin (see Fig (2)).

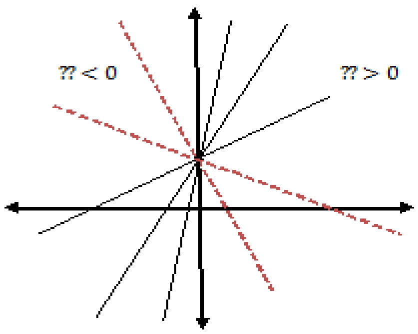


Fig (1)

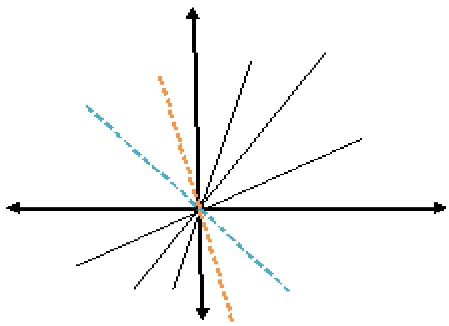


Fig (2)

The change of time makes equations (1) on the following form:

$$y(t) = \alpha x(t) + \beta \tag{2}$$

Or

$$y = \alpha x^2 + \beta \tag{3}$$

Or may be changed into the following form:

$$y = \alpha x^3 + \beta \tag{4}$$

These equations (3) and (4) will be described in the following figures:

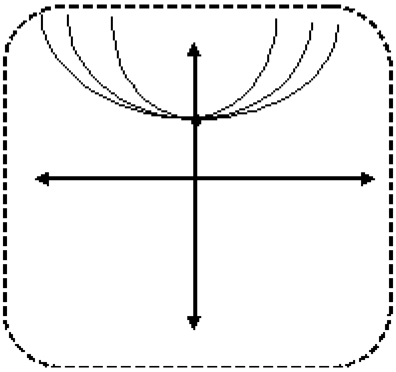


Fig (3)

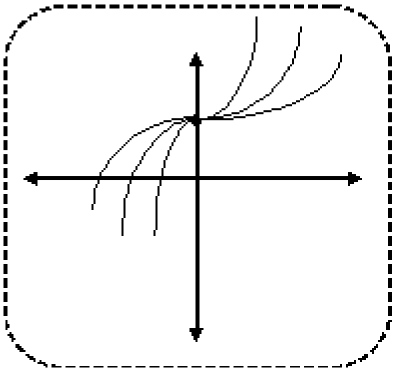


Fig (4)

by changing the time, the equations transformed into the following forms:

$$y = \alpha x^4 + \beta, \ldots\ldots, y = \alpha x^n + \beta, \quad for\, n \geq 1.$$

All dynamical changes are changed in the curvature.

Proposition (2): Consider the dynamical system with initial condition

$$\frac{dx}{dt} = x, \qquad \frac{dy}{dt} = -y$$

$$x(0) = x_o, \quad y(0) = y_o$$

Where x_0, y_0 are constants

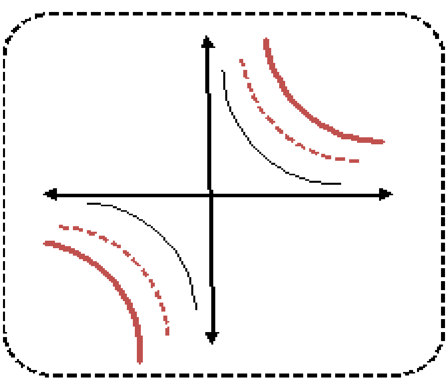
$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{-y}{x}$$

By separation of variable and integrating both sides, we get

$$\ln y = -\ln x + C, \quad C\ is\ constant\ of\ integration$$

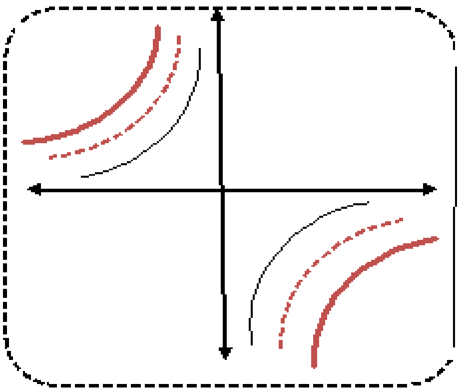
$$y = \frac{y_0 x_0}{x} = \frac{\alpha}{x}, \quad \alpha = y_0 x_0 \ .$$

These equations represent set of curves which represent manifolds of one dimension which changed by the time into line ($y = 0\ as\ \alpha = 0$). Curves change with change the value of constant α (see Fig (5), Fig (6)).



$\alpha > 0$

Fig (5)



$\alpha < 0$

Fig (6)

By changing the time, the curves transformed into the following form:

$$y(t) = \frac{\alpha}{x(t)}$$

Such that the time acting on the function $x(t)$ to be as follows:

$$y = \frac{\alpha}{x^2}, \dots \dots, y = \frac{\alpha}{x^n}, \quad for\, n \geq 1$$

We can describe the cases $for\, n = 2\ \ and\ \ n = 7$ as follows:

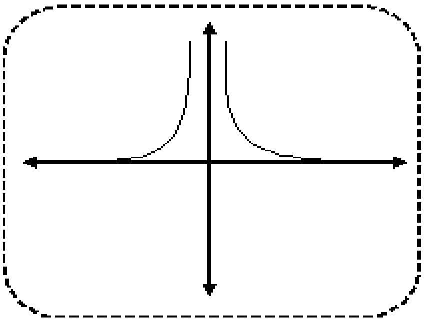


Fig (7)
$n = 2$

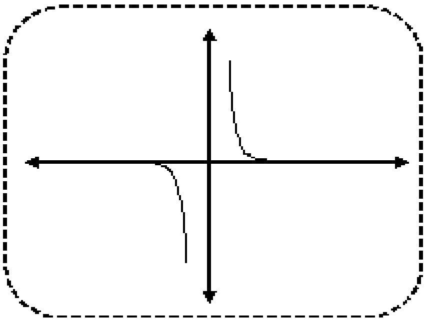


Fig (8)
$n = 9$

All dynamical changes like as curvature and positions of curves by changing the time such that for different values of n (odd or even) the position for curve changes for odd number and the curvature decreases for increasing n. All these curves represented manifolds of one dimension.

Proposition (3): Consider the dynamical system with initial conditions as follow

$$\frac{dx}{dt} = y, \qquad \frac{dy}{dt} = -x$$

$$x(0) = x_0, \quad y(0) = y_0$$

where x_0, y_0 are constants

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{-x}{y}$$

By separation of variable and integrating both sides, we get

$$x^2 + y^2 = A \to x^2 + y^2 = x_0^2 + y_0^2 \quad \Longrightarrow \quad x^2 + y^2 = \alpha^2 \qquad (5)$$

These equations (5) represent a set of circles centered at the origin and with radius α which are changed increases or decreases (see Fig (9))

$i.e.\ \ as\ \alpha \to 0,\ \ the\ limit\ of\ the\ circles\ is\ the\ point\ (origin)\ also\ as\ K \to \infty,\ \ then\ the\ circles\ are\ lines.$

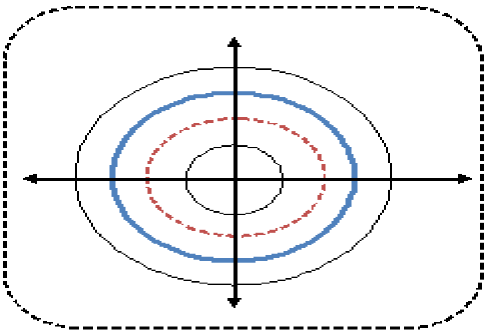


Fig (9)

These circles represent one dimension manifolds under the change of the time these equations changed into the form:

$$x^2(t) + y^2(t) = \alpha^2$$

By the time, the equations changed from circles into another curves as follows:

$$x^4 + y^4 = A, x^6 + y^6 = A, \dots\dots, x^m + y^m = A, \quad for\ m = 2,4, \dots..$$

We shall describe the cases for $m = 6$ as follows:

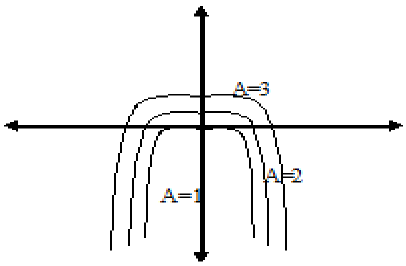


Fig (10)

Proposition (4): Consider dynamical system with initial condition

$$\begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \\ \frac{dz}{dt} \end{pmatrix} = \begin{pmatrix} -x+y \\ -x-y \\ -2z \end{pmatrix}$$

$$Since\ \frac{dz}{dt} = -2z$$
$$\therefore \frac{dz}{z} = -2dt$$

Integrating both sides of last equation, we get

$$z(t) = K\,e^{-2t}, \quad K\ is\ constant. \quad \rightarrow (1)$$

$$Since\ \frac{dx}{dt} = -x + y\ and\ \frac{dy}{dt} = -x - y$$

$$\Longrightarrow \frac{dx}{dt} + x = y \Longrightarrow \frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 2x = 0$$

Which is homogenous equation has a homogenous solution as follows:

$$x(t) = e^{-t}(a\cos t + b\ sint), a, b\ \ constants \rightarrow (2)$$

Similarly,

$$y(t) = e^{-t}(-a\sin t + b\ cost) \rightarrow (3)$$

The three equations (1), (2) and (3) satisfy the following equation $z = A(x^2 + y^2)$ which represented equations of parabolids centered at origin changes with change the constant (manifolds of two dimension) (see Fig (11)).

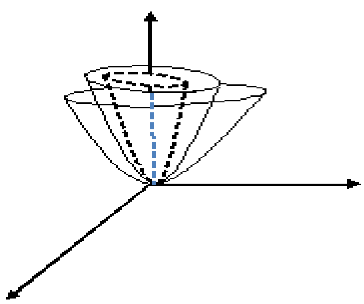


Fig (11)

We notice that when $A \to 0, then\ the\ parabolid\ approch\ to\ z = 0$.

4-REFERENCES

[1] P. J .Giblin: Graphs, Surfaces and Homology, an introduction to algebraic topology. Chapman and Hall Ltd, London, 1977.

[2] R. J. Wilson: Introduction to graph theory. Oliver & Boyed, Edinburgh, 1972.

[3] R. Diestel: Graph Theory. Electronic Edition 2000. Springer-Verlag New York 1997, 2000.

[4] A. T. White: Graphs, Groups and surfaces, Amsterdam, North-Holland, Publishing Company, 1973.

[5] R. J. Wilson, J. J. Watkins: Graphs, an introductory approach, a first course in discrete mathematics. Jon Wiley & Sons Inc, Canada, 1990.

[6] H. Tamás, László Lovász. The Chromatic Polynomial, Master Thesis, Eötvös Loránd Univeristy,

[7] W. T. Tutte. A contribution to theory of chromatic polynomials. Canadian Journal of Mathematics, 6:80-91, 1954.

[8] Chris Godsil and Gordon Royle, Algebraic Graph Theory, pages 1, 33-40, 1949.

Source of support: Nil, Conflict of interest: None Declared