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GELFAND-SHILOV TYPE SPACES FOR DISTRIBUTIONAL FOURIER-MELLIN TRANSFORM AND THEIR TOPOLOGICAL STRUCTURE

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ABSTRACT

The Fourier-Mellin transform is a useful mathematical tool for image recognition because its resulting spectrum is invariant in rotation, translation and scale. This paper generalizes the Fourier-Mellin transform to the Gelfand-shilov type spaces of generalized function and obtained their topological structure.

Keywords: *Fourier-Mellin Transforms, Testing Function Space, Generalized Function, Signal Processing, Optics.*

1. Introduction

The magnitude spectrum of a time domain signal has the property of delay-invariance. Similar to the delay-invariance property of the Fourier transform, the Mellin transform has the property of scale-invariance. By combining these two transforms together one can form the Fourier-Melllin transform that yields a signal representation which is independent of both delay and scale change.

Time and frequency represents two fundamental physical variables of signal analysis and processing. The Fourier transform (FT), which provides a mapping between the time domain and frequency domain representation of signal, has been used extensively in signal processing applications [4]. It is itself translation invariant and its conversion to log polar co-ordinates converts the scale and rotation differences to vertical and horizontal offsets that can be measured. A second FFT called the Mellin transform (MT) gives a transform space image that is invariant to translation, rotation and scale.

The application of the Fourier-Mellin transform has been studies in digital signal and image processing, pattern recognition, speech processing, ship target recognition by sonar system [1] and radar signal analysis. Also, Fourier-Mellin transform is used in detecting watermarks in images regardless of scale or rotation [5].

In the present work Fourier-Mellin transform is generalized in distributional sense. Gelfand-Shilove type testing function spaces are proved. Also their topological structures are given.

2. S-type spaces

2.1. The space $FM_{a,b,\propto}$

A function $\varphi(t,x)$ defined on $0<x<\infty,\ 0<t<\infty$ is said to be a member of $FM_{a,b,\propto}$ if $\varphi(t,x)$ is smooth.

The space $FM_{a,b,\propto}$ is given by

$$FM_{a,b,\propto} = \{\varphi:\varphi\in E_+ \mid \gamma_{a,b,k,q,l}\,\varphi(t,x) = \sup_{I_1}\left|t^k\,\xi_{a,b}(x)\,x^{q+1}D_t^l\,D_x^q\,\varphi(t,x)\right| \leq C_{lq}A^k k^{k\alpha}\} \tag{2.1}$$

Where the constant A and C_{lq} depend on the testing function φ.

The Topology of the space $FM_{a,b,\propto}$ is generated by the countable multinorm $S=\{\gamma_{a,b,k,q,l}\}_{k,q,l=0}^{\infty}$. With this topology $FM_{a,b,\propto}$ is a countable multinormed space.

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A sequence $\{\varphi_\mu\}_{\mu=1}^{\infty}$ is said to converge in $FM_{a,b,\propto}$ to φ if for each triplet of non negative integer $k, q, l,$ $\gamma_{a,b,k,q,l}(\varphi_\mu - \varphi \to 0$ as $\mu\to\infty$.

2.2. The space $FM_{a,b}^{\beta}$

This space is given by

$$FM_{a,b}^{\beta} = \{\varphi : \varphi \in E_+ | \ \sigma_{\ a,b,k,q,l}\ \varphi(t,x) = \sup_{I_1} \left| t^k\ \xi_{a,b}(x)\ x^{q+1} D_t^l\ D_x^q \varphi(t,x) \right| \leq C_{kq} B^l l^{l\beta} \} \quad (2.2)$$

The constants C_{kq} and B depends on φ.

2.3. The space $FM_{a,b,\alpha}^{\beta}$

This space is formed by combining the conditions of 2.1 and 2.2

$$FM_{a,b,\alpha}^{\beta} = \{\varphi : \varphi \in E_+ | \ \rho_{\ a,b,k,q,l}\ \varphi(t,x) = \sup_{I_1} \left| t^k\ \xi_{a,b}(x)\ x^{q+1} D_t^l\ D_x^q \varphi(t,x) \right| \leq C A^k k^{k\alpha} B^l l^{l\beta} \} \quad (2.3)$$

2.4. The space $FM_{a,b,\gamma}$

It is given by

$$FM_{a,b,\gamma} = \{\varphi : \varphi \in E_+ | \ \xi_{\ a,b,k,q,l}\ \varphi(t,x) = \sup_{I_1} \left| t^k\ \xi_{a,b}(x)\ x^{q+1} D_t^l\ D_x^q \varphi(t,x) \right| \leq C_{lk} A^q q^{q\gamma} \} \quad (2.4)$$

The right hand side of equation (2.1) depends on $A, k, \propto$ while the right hand side of equation (2.2) depends on B, l, β. Thus it is clear that the spaces $FM_{a,b}^{\beta}$ and $FM_{a,b,\propto}$ are different from each other. From equation (2.1), (2.2) and (2.3) it is easy to see that the space $FM_{a,b,\alpha}^{\beta}$ is contained in the intersection of the spaces $FM_{a,b,\alpha}$ and $FM_{a,b}^{\beta}$.

The topology of each of the spaces $FM_{a,b,\propto}$, $FM_{a,b}^{\beta}$ and $FM_{a,b,\alpha}^{\beta}$ is $T_{a,b,\propto}$, $T_{a,b}^{\beta}$ and $T_{a,b,\alpha}^{\beta}$ respectively. is generated by the seminorms $\{\gamma_{a,b,k,q,l}\}_{k,q,l=0}^{\infty}$, $\{\rho_{a,b,k,q,l}\}_{k,q,l=0}^{\infty}$ and $\{\xi_{a,b,k,q,}\}_{k,q,l=0}^{\infty}$. On assigning the topology generated by the countable multinorms $S = \{\gamma_{a,b,k,q,l}\}_{k,q,l=0}^{\infty}$ etc. , $FM_{a,b,\propto}$, $FM_{a,b}^{\beta}$ and $FM_{a,b,\alpha}^{\beta}$ becomes countably multinormed spaces. It can be easily proved that these spaces are Frechet spaces.

3. Subspaces

In this section subspaces of each of the above spaces are introduced, which are used in defining the inductive limits of these spaces.

3.1 The space $FM_{a,b,\propto,A}$

Let $FM_{a,b,\propto,A}$ be the space of testing function φ is $FM_{a,b,\propto}$ such that

$$\gamma_{a,b,k,q,l}\ \varphi(t,x) = \sup_{\substack{0<t<\infty \\ 0<x<\infty}} \left| t^k\ \xi_{a,b}(x)\ x^{q+1} D_t^l\ D_x^q \varphi(t,x) \right| \leq C_{lq\delta} (A+\delta)^k k^{k\alpha}, \qquad k, q = 0,1,2 \ldots \ldots$$

for any $\delta > 0$, where A is the constant depending on the function φ.

3.2 The space $FM_{a,b}^{\beta B}$

Let $FM_{a,b}^{\beta B}$ be the space of testing function φ is $FM_{a,b}^{\beta}$ such that

$$\sigma_{a,b,k,q,l}\ \varphi(t,x) = \sup_{\substack{0<t<\infty \\ 0<x<\infty}} \left| t^k\ \xi_{a,b}(x)\ x^{q+1} D_t^l\ D_x^q \varphi(t,x) \right| \leq C_{k,q\mu} (B+\mu)^l l^{l\beta}, \qquad k, q = 0,1,2 \ldots \ldots$$

for any $\mu > 0$, where B is the constant depending on the function φ.

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3.3 The space $FM_{a,b,\alpha,A}^{\beta,B}$

The space is defined by combining the conditions 3.1 and 3.2 as

$$\rho_{a,b,k,q,l}\ \varphi(t,x) = \sup_{\substack{0<t<\infty \\ 0<x<\infty}} \left|t^k\ \xi_{a,b}(x)\ x^{q+1} D_t^l\ D_x^q \varphi(t,x)\right| \\ \leq C_{\delta\mu}\,(A+\delta)^k\ (B+\mu)^l\ k^{k\alpha}\ l^{l\beta}, \qquad k,q = 0,1,2 \ldots ....$$

for any $\delta > 0$, $\mu > 0$ and for given $m > 0$ and $n > 0$.

3.4 The space $FM_{a,b,\gamma,p}$

Let $FM_{a,b,\gamma,p}$ be the space of testing function φ in $FM_{a,b,\gamma}$ such that

$$\xi_{a,b,k,q,l}\ \varphi(t,x) = \sup_{\substack{0<t<\infty \\ 0<x<\infty}} \left|t^k\ \xi_{a,b}(x)\ x^{q+1} D_t^l\ D_x^q \varphi(t,x)\right| \\ \leq C_{lkr}\,(p+r)^q\ \ q^{qr}\ , \qquad k,q = 0,1,2 \ldots ....$$

For any $r > 0$, where p is the constant depending on the function φ.

4. Space of type $F^\mu M_{a,b}$

In this section spaces of the functions on the domain $R_- \times R_+$ is defined.

4.1. The space $F^\mu M_{a,b,\propto}$

A smooth function $\varphi(t,x)$ defined on $-\infty < t < 0,\ 0 < x < \infty$ is in $F^\mu M_{a,b,\propto}$, if $\varphi^\mu(t,x) = \varphi(-t,x)$ is in $FM_{a,b,\propto}$.

$$i_{a,b,k,q,l}\ \varphi(t,x) = \sup_{\substack{-\infty<t<0 \\ 0<x<\infty}} \left|(-t)^k\ \xi_{a,b}(x)\ x^{q+1} D_t^l\ D_x^q \varphi(t,x)\right|$$

$$\leq C_{lq}\ A^k\ k^{k\propto}\ , \qquad k,q = 0,1,2, \ldots \ldots$$

It can be easily proved that $F^\mu M_{a,b,\propto}$ is a Frechet space, Also if $\leq p$, $F^\mu M_{P,b,\propto} \subset F^\mu M_{a,b,\propto}$. We define that countable union space

$$F^\mu M(w,b,\propto) = \bigcup_{v=1}^{\infty} F^\mu M_{a_\mu,b_\mu,\propto}\ .$$

A sequence $\{\varphi_\mu\}_{\mu=1}^{\infty}$ converges in $F^\mu M(w,b,\propto)$ to φ iff it converges to φ in some $F^\mu M_{a_\mu,b_\mu,\propto}$

4.2. The space $F^\mu M_{a,b}^{\beta}$

A smooth function $\varphi(t,x)$ defined on $-\infty < t < 0,\ 0 < x < \infty$ is in $M^\beta{}_{a,b}$, if $\varphi^\mu(t,x) = \varphi(-t,x)$ is in $FM_{a,b}^{\beta}$,

$$j_{a,b,k,q,l}\ \varphi(t,x) = \sup_{\substack{-\infty<t<0 \\ 0<x<\infty}} \left|(-t)^k\ \xi_{a,b}(x)\ x^{q+1} D_t^l\ D_x^q \varphi(t,x)\right| \\ \leq C_{kq}\ B^l\ l^{l\beta}\ , \qquad k,q = 0,1,2, \ldots \ldots$$

4.3. The space $F^\mu M_{a,b,\alpha}^{\beta}$

Combining the conditions of (3.1) and (3.2) we get the space.

A smooth function $\varphi(t,x)$ defined on $-\infty < t < 0,\ 0 < x < \infty$ is in $FM_{a,b,\alpha}^{\beta}$, if $\varphi^\mu(t,x) = \varphi(-t,x)$ is in $FM_{a,b,\alpha}^{\beta}$.

$$\mu_{a,b,k,q,l}\ \varphi(t,x) = \sup_{\substack{-\infty<t<0 \\ 0<x<\infty}} \left|(-t)^k\ \xi_{a,b}(x)\ x^{q+1} D_t^l\ D_x^q \varphi(t,x)\right| \\ \leq C\ A^k\ k^{k\propto} B^l\ l^{l\beta}\ , \qquad k,q = 0,1,2, \ldots \ldots$$

where the constant A, B, C depend on the testing function φ.

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4.4. The space $FM^{\mu}_{a,b,\gamma}$

A smooth function $\varphi(t,x)$ defined on $-\infty < x < 0, 0 < t < \infty$ is in $FM_{a,b,\gamma}$ if $\varphi^{\mu}(t,x) = \varphi(t,-x)$ is in $FM_{a,b,\gamma}$

$$\lambda_{a,b,k,q,l}\, \varphi(t,x) = \sup_{\substack{0<t<\infty \\ -\infty<x<0}} \left| t^k\, \xi_{a,b}(x)\, (-x)^{q+1} D_t^l\, D_x^q \varphi(t,x) \right| \le C_{l,q}\, A^q\, q^{q\gamma}$$

5. The space of the type $\tilde{F}M_{a,b}$

5.1 The space $\tilde{F}M^{\beta}_{a,b,\alpha}$

Let $\tilde{F}M^{\beta}_{a,b,\alpha}$ denote the set of all Fourier-Mellin transforms of $\varphi \in FM^{\beta}_{a,b,\alpha}$. It is easily seen that FM_b is a one to one mapping from $FM^{\beta}_{a,b,\alpha}$ into $\tilde{F}M^{\beta}_{a,b,\alpha}$. Since FM_b and FM_b^{-1} are inverse of each other, FM_b^{-1} is a one to one mapping from $\tilde{F}M^{\beta}_{a,b,\alpha}$ onto $FM^{\beta}_{a,b,\alpha}$.

The topology $\tilde{F}M^{\beta}_{a,b,\alpha}$ is generated by the multinorms $S = \{\rho_{a,b,k,q,l}\}_{k,q,l=0}^{\infty}$, where $\rho_{a,b,k,q,l}(\varphi) = r_{a,b,k,q,l}(\varphi)$ and $\varphi = FM_b^{-1}(\psi)$.

Note that the space $\tilde{F}M_{a,b}$ can be considered to be limiting case of the space $\tilde{F}M^{\beta}_{a,b,\alpha}$.

$\tilde{F}M_{a,b} = \tilde{F}M^{\infty}_{a,b,\infty}$ where the right hand side is understood to be the countable union space such that

$$\tilde{F}M^{\infty}_{a,b,\infty} = \bigcup_{\alpha_i,\beta_i=1}^{\infty} \tilde{F}M^{\beta_i}_{a,b,\alpha_i}$$

5.2 The space $\widetilde{F^{\mu}}M^{\beta}_{a,b,\alpha}$

If $\varphi(t,x)$ is a suitably restricted function on $-\infty < t < 0,\ 0 < x < \infty$. We define the Fourier Mellin transform by

$$F(s,p) = FM\{f(t,x)\} = \int_{-\infty}^{0}\int_{0}^{\infty} f(t,x)\, e^{-ist}\, x^{p-1}\, dt\, dx\,,$$

where $a < p < b$ and $s > 0$.

Let $\widetilde{F^{\mu}}M^{\beta}_{a,b,\alpha}$ denote the set of all Fourier-Mellin transform of $\varphi \epsilon F^{\mu} M^{\beta}_{a,b,\alpha}$. The space $\widetilde{F^{\mu}}M^{\beta}_{a,b,\alpha}$ and $\widetilde{F^{\mu}}M_{a,b}$ can now be defined as we have defined the spaces $\widetilde{FM}^{\beta}_{a,b,\alpha}$ and $\widetilde{FM_{a,b}}$. The countable union spaces $\overline{FM}(W,b)$ and $F^{\mu}M(W,b)$ can be defined as $\widetilde{FM}(W,b) = \cup_{v=1}^{\infty} \widetilde{FM_{a_{\mu},b}}$ and $\widetilde{F^{\mu}}M(W,b) = \cup_{\mu=1}^{\infty} \widetilde{F^{\mu}}M_{a_{\mu},b}$.

6. The Dual spaces $FM^{*}_{a,b}$, $F^{\mu}M^{*}_{a,b}$, $\tilde{F}M^{*}_{a,b}$ and $\widetilde{F^{\mu}}M^{*}_{a,b}$

The space $FM^{*}_{a,b}$ is the dual space of the space $FM_{a,b}$ and contains set of all continuous linear functions $f(t,x)$ defined on $FM_{a,b}$. Similarly the spaces $F^{\mu}M^{*}_{a,b}$, $\tilde{F}M^{*}_{a,b}$ and $F^{\tilde{\mu}}M^{*}_{a,b}$ are dual spaces of the spaces $F^{\mu}M_{a,b}$, $\tilde{F}M_{a,b}$ and $F\widetilde{^{\mu}M_{a,b}}$ respectively.

6.1 Distributional Fourier-Mellin transforms

It can be easily sean that $e^{-ist}x^{p-1} \in FM_{a,b,\alpha}$ for $a < p < b$ and $s > 0$. Now we defined distributional Fourier-Mellin transform for $f(t,x) \in FM^{*}_{a,b,\alpha}$ where $FM^{*}_{a,b,\alpha}$ is the dual space of $FM_{a,b,\alpha}$. We define distributional Fourier-Mellin transform as

$$FM\{f(t,x)\} = F(s,p) = \langle f(t,x), e^{-ist}x^{p-1}\rangle$$

The right hand side has a sense for $f \in FM^{*}_{a,b,\alpha}$ and $e^{-ist}x^{p-1} \in FM_{a,b,\alpha}$.

7. CONCLUSION

In this paper Gelfand shilov type space and their topological study for distributional generalized Fourier Mellin transform is proved.

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