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ON FELICITOUS LABELINGS OF CYCLE RELATED GRAPHS

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ABSTRACT

A *simple graph G is called felicitous if there exists a 1 – 1 function* $f : V(G) \rightarrow \{0,1,2, . . . , q\}$ *such that the set of induced edge labels* $f^*(uv) = (f(u) + f(v)) \pmod q$ *are all distinct. In this paper, cycle related graphs are shown to be felicitous.*

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1. INTRODUCTION

The graphs we consider are simple. For notation and terminology, we refer to [1]. Lee, Schmeichel and Shee [2] introduced the concept of a felicitous graph as a generalization of a harmonious graph. A graph G with q edges is called *harmonious* if there is an injection $f : V(G) \rightarrow Z_q$, the additive group of integers modulo q such that when each edge xy of G is assigned the label $(f(x) + f(y)) \pmod q$, the resulting edge labels are all distinct. A *felicitous* labeling of a graph G, with q edges is an injection $f : V(G) \rightarrow \{0,1,2, . . . ,q\}$ so that the induced edge labels $f^*(xy) = (f(x) + f(y)) \pmod q$ are distinct.

2. DEFINITIONS & RESULTS

Definition 2.1: $\langle C_3, K_{1,m}\rangle$ $(m \geq 1)$ be the graph obtained by attaching $K_{1,m}$ to one vertex of the cycle C_3.

Definition 2.2: $\langle C_3 * K_{1,m}\rangle$ $(m \geq 1)$ be the graph by attaching $K_{1,m}$ to any one vertex of C_3.

Definition 2.3: $(C_4 \square C_4)_n$ is a graph obtained by joining C_4's by an edge.

Remark 2.3: Let G be a (p, q) graph. Let f be a felicitous labeling. Define $f_1(uv) = f(u) + f(v)$ for every $uv \in E(G)$. Then $f^*(uv) = f_1(uv) \pmod q$.

Remark 2.4: [4] Let G be a graph with an odd number of edges and let $f: V(G) \rightarrow \{0, 1, 2 . . . q\}$ be an odd edge labeling of G. Then, f is a felicitous labeling for G.

3. MAIN RESULTS

Theorem 3.1: $<C_n, K_{1,m}>$ is a felicitous graph for any m.

Proof: Case (i) when n = 3.

Let $V(<C_3, K_{1,m}>) = \{u_1, u_2, u_3, v_1, v_2, . . . ,v_m\}$ and $E(<C_3, K_{1,m}>) = \{(u_1 u_2), (u_2 u_3), (u_3 u_1)\} \cup \{(u_2 v_i) : 1 \leq i \leq m\}$.

Define $f: V(<C_3, K_{1,m}>) \rightarrow \{0,1,2, . . . , q = m+3\}$ by

$$f(u_1) = 0, f(u_2) = 1, f(u_3) = 2$$

$$f(v_i) = 2 + i, 1 \leq i \leq m$$

The induced edge labels are:

$$f^*(u_1 u_2) = 1, f^*(u_3 u_1) = 2, f^*(u_2 u_3) = 3, f^*(u_2 v_i) = 3 + i, 1 \leq i \leq m$$

Clearly, $f_1(E(G)) = \{1, 2, 3, . . . , m + 3\}$.

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After taking (mod q), $f^*(E(G)) = f_1(E(G)) \pmod q = \{0, 1, 2, \ldots, m + 2\}$. Hence, $<C_3, K_{1,m}>$ is a felicitous graph for any m.

For example, a felicitous labeling of $<C_3, K_{1,5}>$ is shown in Figure 3.1.

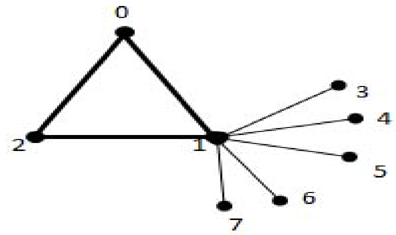

Fig. 3.1

Case (ii): $<C_n, K_{1,m}>$ is a felicitous graph when $n \geq 5$ and $n \equiv 1 \pmod 2$.

Let $V(<C_n, K_{1,m}>) = \{u_i : 1 \leq i \leq n \text{ and } v_j : 1 \leq j \leq m\}$ and $E(<C_n, K_{1,m}>) = \{(u_i u_{i+1}): 1 \leq i \leq n-1\} \cup \{u_n u_1\} \cup \{u_{n-2} v_j : 1 \leq j \leq m\}$.

Define $f : V(<C_n, K_{1,m}>) \to \{0,1,2, \ldots, q = m + n\}$ by

$$f(u_{2i-1}) = i - 1,\ 1 \leq i \leq \frac{n+1}{2}$$

$$f(u_{2i}) = f(u_n) + i,\ 1 \leq i \leq \frac{n-1}{2}$$

$$f(v_j) = f(u_{n-1}) + j + 1,\ 1 \leq j \leq m$$

The induced edge labels are as follows:

$$f_1(u_i u_{i+1}) = \frac{n-1}{2} + i,\ 1 \leq i \leq n-1$$

$$f_1(u_n u_1) = \frac{n-1}{2}$$

$$f_1(u_{n-2} v_j) = 3\frac{n-1}{2} + j,\ 1 \leq j \leq m$$

Clearly, $f_1(E(G)) = \{\frac{n-1}{2}, \frac{n-1}{2} + 1, \frac{n-1}{2} + 2, \ldots, \frac{n-1}{2} + n - 1, \frac{n-1}{2} + n, \ldots, \frac{n-1}{2} + m + n - 1\}$. After taking (mod q), $f^*(E(G)) = f_1(E(G)) \pmod q = \{\frac{n-1}{2}, \frac{n-1}{2} + 1, \frac{n-1}{2} + 2, \ldots, \frac{n-1}{2} + n - 1, \frac{n-1}{2} + n, \ldots, \frac{n-1}{2} - 1\}$.

Hence, $<C_n, K_{1,m}>$ is a felicitous graph when $n \geq 5$ and $n \equiv 1 \pmod 2$.

For example, a felicitous labeling of $<C_7, K_{1,4}>$ is shown in Figure 3.2.

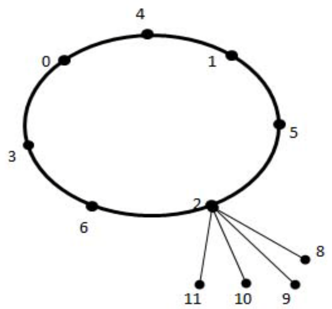

Fig. 3.2

Case (iii): when n = 4

Let $V(\langle C_4, K_{1,m}\rangle) = \{u_1, u_2, u_3\ u_4, v_1, v_2, \ldots, v_m\}$ and $E(\langle C_4, K_{1,m}\rangle) = \{(u_1 u_2), (u_2 u_3), (u_3 u_4), (u_4 u_1)\} \cup \{(u_3 v_i) : 1 \le i \le m\}$.

Define $f : V(\langle C_4, K_{1,m}\rangle) \to \{0,1,2, \ldots, q = m+4\}$ by

$$f(u_1) = 0,\ f(u_2) = 2,\ f(u_3) = 1,\ f(u_4) = 4$$
$$f(v_i) = i + 4,\ 1 \le i \le m$$

The induced edge labels are:

$$f_1(u_1 u_2) = 2,\ f_1(u_2 u_3) = 3,\ f_1(u_3 u_4) = 5,\ f_1(u_4 u_1) = 4$$
$$f_1(u_3 v_i) = i + 5,\ 1 \le i \le m$$

Clearly, $f_1(E(G)) = \{2, 3, 4, \ldots, m + 5\}$. After taking (mod q), $f^*(E(G)) = f_1(E(G)) \pmod q = \{0, 1, 2, \ldots, m + 3\}$.

Hence, $\langle C_4, K_{1,m}\rangle$ is a felicitous graph for any m.

For example, a felicitous labeling of $\langle C_4, K_{1,4}\rangle$ is shown in Figure 3.3.

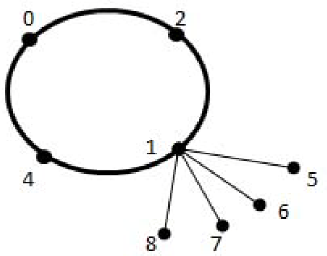

Fig. 3.3

Case (iv) : $\langle C_{2n+4}, K_{1,m}\rangle$ is a felicitous graph when $n \ge 1$.

Let $V(\langle C_{2n+4}, K_{1,m}\rangle) = \{u_i : 1 \le i \le n$ and $v_j : 1 \le j \le m\}$ and $E(\langle C_{2n+4}, K_{1,m}\rangle) = \{(u_i u_{i+1}) : 1 \le i \le n - 1\} \cup \{u_n u_1\} \cup \{u_{2n+3} v_j : 1 \le j \le m\}$

Define $f : V(\langle C_{2n+4}, K_{1,m}\rangle) \to \{0,1,2, \ldots, q = m+2n+4\}$ by

$$f(u_{2i-1}) = i - 1,\ 1 \le i \le n + 2$$

$$f(u_{2i}) = \begin{cases} n + 1 + i, & 1 \le i \le n + 1 \\ 3n + 4 & i = n + 2 \end{cases}$$

$$f(v_j) = \begin{cases} 2n + 3 + j, & 1 \le j \le n \\ 2n + 4 + j, & n + 1 \le j \le m \end{cases}$$

The induced edge labels are as follows :

$$f_1(u_i u_{i+1}) = \begin{cases} n + 1 + i, & 1 \le i \le 2n + 2 \\ 4n + 5, & i = 2n + 3 \\ 3n + 4 & \end{cases}$$

$$f_1(u_{2n+3} v_j) = \begin{cases} 3n + 4 + j, & 1 \le j \le n \\ 3n + 5 + j, & n + 1 \le j \le m \end{cases}$$

Clearly, $f_1(E(G)) = \{n + 2, n + 3, n + 4, \ldots, 3n + 3, 3n + 4, 3n + 5, \ldots, 4n + 4, 4n + 5, 4n + 6, \ldots, 3n + m + 5\}$.

After taking (mod q), $f^*(E(G)) = f_1(E(G)) \pmod q = \{n + 2, n + 3, \ldots, 3n + 3, 3n + 4, \ldots, 4n + 4, 4n + 5, \ldots, n + 1\}$.

Hence, $\langle C_{2n+4}, K_{1,m}\rangle$ is a felicitous graph, when $n \ge 1$.

For example, a felicitous labeling of $<C_8, K_{1,4}>$ is shown in Figure 3.4.

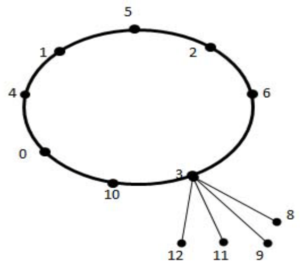

Fig. 3.4

In all cases, the edge values are distinct and so $<C_n, K_{1,m}>$ admits felicitous labeling.

Theorem 3.2: $<C_n * K_{1,m}>$ is a felicitous graph for any m.

Proof: Case (i): when n = 3

Let $V(<C_3 * K_{1,m}>) = \{u_1, u_2, u_3\} \cup \{v_i : 0 \leq i \leq m\}$, where $u_3 = v_1$ and let $E(<C_3 * K_{1,m}>) = \{(u_1 u_2), (u_2 u_3), (u_3 u_1)\} \cup \{(v_0 v_i) : 1 \leq i \leq m\}$.

Define $f : V(<C_3 * K_{1,m}>) \rightarrow \{0,1,2, . . . , q = m + 3\}$ by

$$f(u_1) = 0,\ f(u_2) = 2,\ f(u_3) = 3,\ f(v_o) = 1$$
$$f(v_i) = 3 + i,\ 2 \leq i \leq m$$

The induced edge labels are

$$f_1 (u_1 u_2) = 2,\ f_1 (u_3 u_1) = 3,\ f_1 (u_2 u_3) = 5,\ f_1 (v_o v_1) = 4,$$
$$f_1 (v_o v_i) = i + 4,\ 2 \leq i \leq m$$

Clearly, $f_1(E(G)) = \{2, 3, 4, . . . , m + 4\}$.

After taking (mod q), $f^*(E(G)) = f_1(E(G)) \pmod q = \{0, 1, 2, . . . , m + 2\}$.

Hence, $<C_3 * K_{1,m}>$ is a felicitous graph for any m.

For example, a felicitous labeling of $<C_3 * K_{1,5}>$ is shown in Figure 3.5.

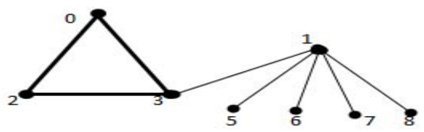

Fig. 3.5

Case (ii) : when n = 4

Let $V(<C_4 * K_{1,m}>) = \{u_1, u_2, u_3, u_4\} \cup \{v_i : 0 \leq i \leq m\}$, where $u_3 = v_1$ and let $E(<C_4 * K_{1,m}>) = \{(u_1 u_2), (u_2 u_3), (u_3 u_4), (u_4 u_1)\} \cup \{(v_0 v_i) : 1 \leq i \leq m\}$.

Define $f : V(<C_4 * K_{1,m}>) \rightarrow \{0,1,2, . . . , q = m + 4\}$ by

$$f(u_1)= 0,\ f(u_2)= 3f(u_3)= 1f(u_4) = 5,\ f(v_o) = 2$$
$$f(v_i) = 4 + i,\ 2 \leq i \leq m$$

The induced edge labels are

$$f_1 (u_1 u_2) = 3,\ f_1 (u_2 u_3) = 4,\ f_1 (u_3 u_4) = 6,\ f_1 (u_4 u_1) = 5,$$
$$f_1 (v_o v_i) = i + 4,\ 2 \leq i \leq m$$

Clearly, $f_1(E(G)) = \{3, 4, 5, \ldots, m + 6\}$.

After taking (mod q), $f^*(E(G)) = f_1(E(G)) \pmod q = \{1, 2, 3, \ldots, m + 4\}$.

Hence, $<C_3 * K_{1,m}>$ is a felicitous graph for any m.

For example, a felicitous labeling of $<C_3 * K_{1,5}>$ is shown in Figure 3.6.

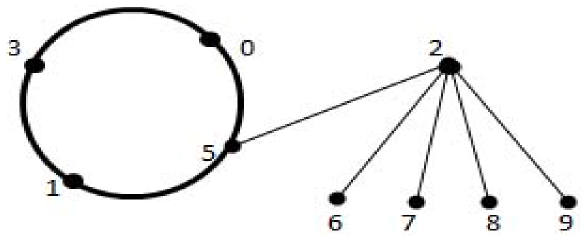

Fig. 3.6

Case (iii): when $n \geq 5$ and $n \equiv 1 \pmod 2$

Let $V(<C_n * K_{1,m}>) = \{u_i : 1 \leq i \leq n\} \cup \{v_j : 0 \leq i \leq m\}$ where $f(u_{n-1}) = f(v_1)$ let $E(<C_n * K_{1,m}>) = \{(u_i u_{i+1} : 1 \leq i \leq n - 1\} \cup \{u_n u_1\} \cup \{v_o v_j : 1 \leq j \leq m\}$

Define $f : V(<C_n * K_{1,m}>) \rightarrow \{0,1,2, \ldots, q = m + n\}$ by

$$f(u_{2i-1}) = i - 1,\ 1 \leq i \leq \frac{n-1}{2}$$

$$f(u_{2i})) = \frac{n+1}{2} + i,\ 1 \leq i \leq \frac{n-1}{2}$$

$$f(u_n) = \frac{n+1}{2}$$

$$f(v_o) = \frac{n-1}{2}$$

$$f(v_j) = n + j,\ 2 \leq j \leq m$$

The induced edge labels are as follows:

$$f_1(u_i u_{i+1}) = \begin{cases} \frac{n+1}{2} + i, & 1 \leq i \leq n-2 \\ 3\frac{n+1}{2} - 1 & i = n - 1 \end{cases}$$

$$f_1(u_n u_1) = \frac{n+1}{2}$$

$$f_1(v_o v_1) = 3\frac{n+1}{2} - 2,$$

$$f_1(v_o v_j) = 3\frac{n+1}{2} - 2 + j,\ 2 \leq j \leq m.$$

Clearly, $f_1(E(G)) = \{\frac{n+1}{2}, \frac{n+1}{2} + 1, \ldots, \frac{n+1}{2} + n - 2, \frac{n+1}{2} + n - 1, \frac{n+1}{2} + n, \ldots, \frac{n+1}{2} + n + m - 1\}$.

After taking (mod q), $f^*(E(G)) = f_1(E(G)) \pmod q = \{\frac{n+1}{2}, \frac{n+1}{2} +1, \ldots, \frac{n+1}{2} + n - 2, \frac{n+1}{2} + n - 1, \ldots, \frac{n+1}{2} - 1\}$.

Hence, $<C_n * K_{1,m}>$ where $n \geq 5$ and $n \equiv 1 \pmod 2$ is a felicitous graph for any m.

For example, a felicitous labeling of $<C_7 * K_{1,4}>$ is shown in Figure 3.7.

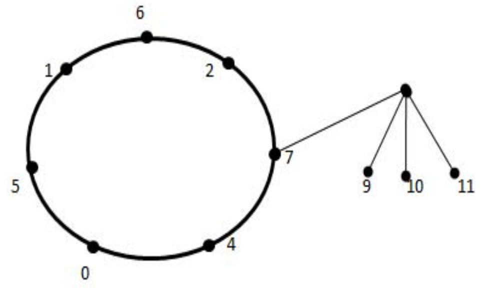

Fig.3.7

Case (iv): when $n \geq 6$ and $n \equiv 0 \pmod 2$

Let $V(<C_n * K_{1,m}>) = \{u_i : 1 \leq i \leq n\} \cup \{v_j : 0 \leq i \leq m\}$ where $f(u_n) = f(v_1)$ and $E(<C_n * K_{1,m}>) = \{(u_i u_{i+1} : 1 \leq i \leq n-1\} \cup \{u_n u_1\} \cup \{v_o v_j : 1 \leq j \leq m\}$

Define an injective function $f : V(<C_n * K_{1,m}>) \rightarrow \{0,1,2, \dots, q = m + n\}$ by

$$f(u_{2i-1}) = i - 1, \quad 1 \leq i \leq \frac{n}{2}$$

$$f(u_{2i}) = \begin{cases} \frac{n}{2} + i, & 1 \leq i \leq \frac{n}{2} - 1 \\ \frac{3n}{2} - 1, & i = \frac{n}{2} \end{cases}$$

$$f(v_o) = \frac{n}{2}$$

$$f(v_j) = \begin{cases} n + j - 2, 2 \leq j \leq \frac{n}{2} - 1 \\ n + j, & \frac{n}{2} \leq j \leq m \end{cases}$$

The edge labels are

$$f_1(u_i u_{i+1}) = \begin{cases} \frac{n}{2} + i, & 1 \leq i \leq n-2 \\ 2(n-1), i = n-1 \end{cases}$$

$$f_1(u_n u_1) = \frac{3n}{2} - 1$$

$$f_1(v_o v_j) = \begin{cases} 2n - 1, & j = 1, \\ \frac{3n}{2} + j - 2, & 2 \leq j \leq \frac{n}{2} - 1 \\ \frac{3n}{2} + j, \ \frac{n}{2} \leq j \leq m \end{cases}$$

Clearly, $f_1(E(G))=\{\frac{n}{2} + 1, \frac{n}{2} + 2, \dots, \frac{n}{2} + n - 2, 2n - 2, \frac{n}{2} + n - 1, 2n - 1, \frac{n}{2} + n, \dots, 2n - 3, 2n, \dots \frac{n}{2} + n + m\}$.

After taking (mod q), $f^*(E(G)) = f_1(E(G))$ (mod q) = $\{\frac{n}{2}+1, \frac{n}{2}+2, \ldots, \frac{n}{2}+n-1, \frac{n}{2}+n, \ldots, 2n-3, 2n-2, 2n-1, 2n, \ldots \frac{n}{2}\}$.

Hence, $\langle C_n * K_{1,m}\rangle$ where $n \geq 6$ and $n \equiv 0 \pmod 2$ is a felicitous graph for any m.

For example, a felicitous labeling of $\langle C_8 * K_{1,4}\rangle$ is shown in Figure 3.8.

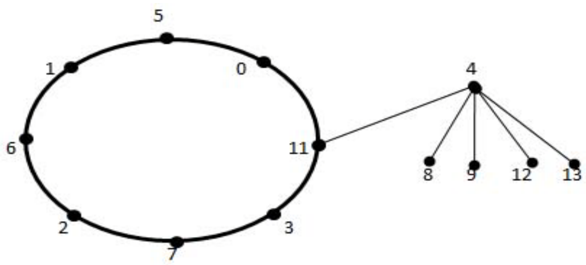

Fig.3.8

In all the cases, the edge values are distinct and so $\langle C_n * K_{1,m}\rangle$ admits felicitous labeling.

Theorem 3.3: $C_n \odot K_1$, $n \equiv 1 \pmod 2$ is a felicitous graph for any $n \geq 3$.

Proof: Let $V(C_n \odot K_1) = \{u_i, v_i : 1 \leq i \leq n\}$ and $E(C_n \odot K_1) = \{(u_i u_{i+1}) : 1 \leq i \leq n-1\} \cup \{u_n u_1\} \cup \{u_i v_i : 1 \leq i \leq n\}$.

Define $f: V(C_n \odot K_1) \rightarrow \{0, 1, 2 \ldots q = 2n\}$ by

$$f(u_i) = i-1, \quad 1 \leq i \leq n$$

$$f(v_i) = \begin{cases} n+i, & 1 \leq i \leq n-1 \\ n, & i = n \end{cases}$$

The labels of the edges are as follows:

$$f^*(u_i u_{i+1}) = 2i-1, \quad 1 \leq i \leq n-1$$

$$f^*(u_n u_1) = n-1$$

$$f^*(u_i v_i) = \begin{cases} n+1+2(i-1), & 1 \leq i \leq \frac{n+1}{2} \\ 2i-n-1, & \frac{n+1}{2}+1 \leq i \leq n-1 \\ 2n-1 & i = n \end{cases}$$

$f^*(E(G)) = \{1, 3, 5, \ldots, n-2, n, n+2, \ldots, 2(n-1)-1\} \cup \{n-1\} \cup \{n+1, n+3, \ldots, n+1+2(\frac{n+1}{2}-2), n+1+2(\frac{n+1}{2}-1)\} \cup \{2(\frac{n+1}{2}+1)-n-1, 2(\frac{n+1}{2}+2)-n-1, \ldots, 2(n-1)-n-1\} \cup \{2n-1\}$.

$= \{1, 3, 5, \ldots, n-2, n, n+2, \ldots, 2n-3\} \cup \{n-1\} \cup \{n+1, n+3, \ldots, 2n-2, 2n\} \cup \{2, 4, \ldots, n-3\} \cup \{2n-1\}$.

$= \{1, 2, 3, 4, 5, \ldots, n-3, n-2, n-1, n, n+1, \ldots, 2n-3, 2n-2, 2n-1, 2n\}$.

Clearly, the above edge values are distinct and hence, $C_n \odot K_1$, $n \equiv 1 \pmod 2$ is a felicitous graph for any $n \geq 3$.

For example, the felicitous labeling of $C_7 \odot K_1$ is shown in the Figure 3.9.

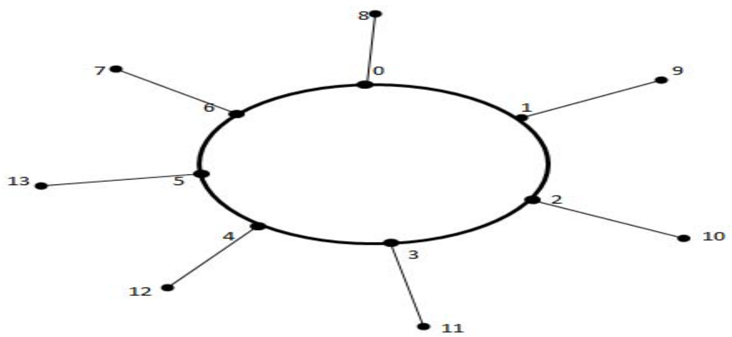

Fig. 3.9

Theorem 3.4: $(C_4 \square C_4)_{2n}$ is a felicitous graph for any $n \geq 1$.

Proof : Let $V(C_4 \square C_4)_{2n} = \{ u_i^j : 1 \leq j \leq 2n$ and $1 \leq i \leq 4\}$ and $E(C_4 \square C_4)_{2n} = \{ u_i^j \, u_{i+1}^j : 1 \leq j \leq 2n$ and $1 \leq i \leq 3\}$ $\cup \{(u_4^j \, u_1^j) : 1 \leq j \leq 2n\} \cup \{(u_3^j \, u_1^{j+1}) : 1 \leq j \leq 2n - 1\}$.

It is enough to show that $(C_4 \square C_4)_{2n}$ admits odd edge labeling.

Define f: $V((C_4 \square C_4)_{2n}) \rightarrow \{0, 1, 2. . . q = 10n - 1\}$ by
]

$f(u_1^1) = 0$ $\quad$ $f(u_2^1) = 1$, $\quad$ $f(u_3^1) = 4$, $\quad$ $f(u_4^1) = 3$

For $2 \leq j \leq 2n$,

$$f(u_i^j) = f(u_i^1) + 5(j - 1), \quad 1 \leq i \leq 4$$

The labels of the edges are as follows:

$f(u_1^j u_2^j) = 10j - 9, \quad 1 \leq j \leq 2n$
$f(u_2^j u_3^j) = 10j - 5, \quad 1 \leq j \leq 2n$
$f(u_3^j u_4^j) = 10j - 3, \quad 1 \leq j \leq 2n$
$f(u_4^j u_1^j) = 10j - 7, \quad 1 \leq j \leq 2n$
$f(u_3^j \, u_1^{j+1}) = 10\,j - 1, \quad 1 \leq j \leq 2n - 1$

Clearly, $f(E(G)) = \{1, 11, \ldots, 20n - 9\} \cup \{5, 15, \ldots, 20n - 5\} \cup \{7, 17, \ldots, 20n - 3\} \cup \{3, 13, \ldots, 20n - 7\} \cup \{9, 19, \ldots, 10(2n - 1) - 1\}$.

$= \{1, 11, \ldots, 20n - 9\} \cup \{5, 15, \ldots, 20n - 5\} \cup \{7, 17, \ldots, 20n - 3\} \cup \{3, 13, \ldots, 20n - 7\} \cup \{9, 19, \ldots, 20n - 11\}$.

$= \{1, 3, 5, 7. . . 20n - 11, 20n - 9, 20n - 7, 20n - 5, 20n - 3\} = \{1, 3, 5, \ldots, 2q - 1\}$.

Clearly, the above edge values are distinct and odd and hence G admits odd edge labeling. Therefore by 2.4, $(C_4 \square C_4)_{2n}$ is a felicitous graph for any $n \geq 1$.

For example, a felicitous labeling of $(C_4 \square C_4)_4$ is shown in Figure 3.10.

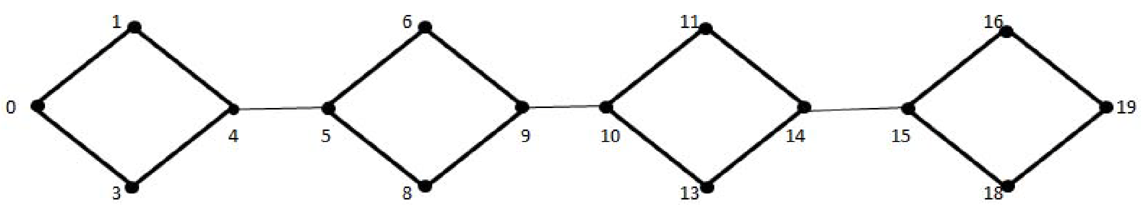

Fig. 3.10

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