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HOMOTOPY ANALYSIS METHOD AND FOKKER-PLANCK EQUATION

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ABSTRACT

Here, an analytical technique, namely the homotopy analysis method (HAM), is applied to solve various linear and nonlinear Fokker-Planck equations. HAM is strong and easy-to-use analytical tool for nonlinear problem and does not need small parameters in the equations. Comparison of the results with those of Adomain decomposition method (ADM) and Homotopy perturbation method (HPM), has led just us to significant consequences. The homotopy analysis method contains the auxiliary parameter $\hbar$ *, which provides us with a simple way to adjust and control the convergence region of solution series.*

Keywords: *Homotopy analysis method, Fokker-Planck equation, Kolmogorov equation, approximate solution, exact solution.*

1. INTRODUCTION

The investigations of the exact solutions to nonlinear equations play an important role in the study of nonlinear physical phenomena. To date, various nonlinear equations were presented, which described, for example, the motion of the isolated waves, and in many fields such as hydrodynamics, plasma physics, nonlinear optics etc. In most cases it is difficult to solve nonlinear problems, especially analytically. Perturbation techniques [1, 2] were among the popular ones and are based on the existence of small or large parameters, namely the perturbation quantities. Unfortunately many nonlinear problems in science and engineering do not contain such kind of perturbation quantities at all. Hence some non-perturbative techniques [3, 4] have been developed, in which these techniques are independent upon small parameters. However, both perturbative and non-perturbative cannot provide a simple way to adjust or control the convergence region and the rate of given approximate series [5].

To overcome such problems, the homotopy analysis method (HAM) is developed and proposed by Liao [6] in 1992. The method is powerful analytical method for nonlinear problems and has been applied to solve many types of nonlinear problems in science and engineering by many authors [7-25] and the references therein. HAM is different from the perturbation and non-perturbation methods mentioned above as it's provide a simple way to adjust or control the convergence region and the rate of given approximate series. It is also aimed to confirm that the HAM is efficient in handling scientific and engineering problems. In this paper we apply HAM to various Fokker-Planck equations (discussed in next section). Various techniques have been used to solve these equations as [27- 29] Comparisons of the present method with these techniques are also discussed in this paper.

2. FOKKER-PLANCK EQUATION

The Fokker-Planck equation was first introduced by Fokker and Planck to describe the Brownian motion of particles [26]. This equation has been used in different fields in natural sciences such as quantum optics, solid state physics, chemical physics, theoretical biology and circuit theory. Fokker-Planck equations describe the erratic motions of small particles that are immersed in fluids, fluctuations of the intensity of laser light, velocity distributions of fluid particles in turbulent flows and the stochastic behavior of exchange rates. In general, Fokker-Planck equations can be applied to equilibrium and nonequilibrium systems [30-33].

Fokker-Plank equation in general form can be expressed as follows [26].

$$\frac{\partial u}{\partial t} = \left[-\frac{\partial}{\partial x} A(x) + \frac{\partial^2}{\partial x^2} B(x) \right] u, \tag{1}$$

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with the initial condition:

$$u(x,0) = f(x), \quad x \in R$$

where $u(x,t)$ is an unknown function, $A(x)$ and $B(x)$ are called Diffusion and Drift coefficients, such that $B(x) > 0$. Diffusion and Drift coefficient in Eq. (1) can be function of x and t as well as

$$\frac{\partial u}{\partial t} = \left[-\frac{\partial}{\partial x} A(x,t) + \frac{\partial^2}{\partial x^2} B(x,t) \right] u, \tag{2}$$

Eq. (1) is also well known as a forward Kolmogorov equation. There exists another type of this equation is called a backward one as [26].

$$\frac{\partial u}{\partial t} = \left[-A(x,t)\frac{\partial}{\partial x} + B(x,t)\frac{\partial^2}{\partial x^2} \right] u, \tag{3}$$

A generalization of Eq. (1) to N-variables of $x_1, x_2, \ldots x_N$, yields to

$$\frac{\partial u}{\partial t} = \left[-\sum_{i=1}^{N} \frac{\partial}{\partial x_i} A_i(x) + \sum_{i,j=1}^{N} \frac{\partial^2}{\partial x_i \partial x_j} B_{i,j}(x) \right] u, \tag{4}$$

with the following initial condition

$$u(x,0) = f(x), \quad x = (x_1, x_2, \ldots x_N) \in R^N$$

The nonlinear Fokker-Planck equation is a more general form of linear one which has also been applied in vast areas such as plasma physics, surface physics, astrophysics, the physics of polymer fluids and particle beams, nonlinear hydrodynamics, theory of electronic-circuitry and laser arrays, engineering, biophysics, population dynamics, human movement sciences, neurophysics, psychology and marketing [34].

The nonlinear form of Fokker-Planck equation can be expressed in the following form:

$$\frac{\partial u}{\partial t} = \left[-\frac{\partial}{\partial x} A(x,t,u) + \frac{\partial^2}{\partial x^2} B(x,t,u) \right] u, \tag{5}$$

A generalization of Eq. (5) with N-variables of $x_1, x_2, \ldots x_N$ yields to

$$\frac{\partial u}{\partial t} = \left[-\sum_{i=1}^{N} \frac{\partial}{\partial x_i} A_i(x,t,u) + \sum_{i,j=1}^{N} \frac{\partial^2}{\partial x_i \partial x_j} B_{i,j}(x,t,u) \right] u. \tag{6}$$

3. BASIC IDEA OF HOMOTOPY ANALYSIS METHOD (HAM)

In this paper, we apply the HAM to the four problems to be discussed. In order to show the basic idea of HAM, consider the following differential equation:

$$N[u(x,t)] = 0, \tag{7}$$

where *N* is a nonlinear operator, *x* and *t* denote the independent variables and *u* is an unknown function. For simplicity, we ignore all boundary or initial conditions, which can be treated in the similar way. By means of the HAM, we first construct the so-called zeroth-order deformation equation.

$$(1-q)\, L[\phi(x,t;q) - u_0(x,t)] = q\, \hbar\, H(x,t)\, N[\phi(x,t;q)] \tag{8}$$

where $q \in [0,1]$ is the embedding parameter, $\hbar \neq 0$ is an auxiliary parameter, L is an auxiliary linear operator, $\phi(x,t;q)$ is an unknown function, $u_0(x,t)$ is an initial guess of $u(x,t)$ and $H(x,t)$ denotes a nonzero auxiliary function. It is obvious that when the embedding parameter $q = 0$ and $q = 1$, equation (8) becomes

$\varphi(x,t;0) = u_0(x,t), \varphi(x,t;1) = u(x,t),$ respectively.

Thus as q increases form 0 to1, the solution $\phi(x,t;q)$ varies from the initial guess $u_0(x,t)$ to the solution $u(x,t)$. Expanding $\phi(x,t;q)$ in Taylor series with respect to q, one has

$$\phi(x,t;q) = u_0(x,t) + \sum_{m=1}^{\infty} u_m(x,t)\, q^m, \tag{9}$$

where

$$u_m(x,t) = \frac{1}{m!}\frac{\partial^m \phi(x,t;q)}{\partial q^m}\Big| q = 0. \tag{10}$$

The convergence of the series (9) depends upon the auxiliary parameter $\hbar$. If it is convergent at $q = 1$, one has

$$u(x,t) = u_0(x,t) + \sum_{m=1}^{\infty} u_m(x,t), \tag{11}$$

which one of the solutions of the original nonlinear equation, as proven by Liao [6, 7]. Define the vectors

$$\vec{u}_n = \{u_0(x,t),\ u_1(x,t), \ldots,\ u_n(x,t)\}. \tag{12}$$

Differentiating the zeroth-order deformation equation (8) *m*-times with respect to *q* and then dividing them by $m!$ and finally setting $q = 0$, we get the following mth-order deformation equation:

$$L[u_m(x,t) - \chi_m u_{m-1}(x,t)] = \hbar \Re_m(\vec{u}_{m-1}), \tag{13}$$

where

$$\Re_m(\vec{u}_{m-1}) = \frac{1}{(m-1)!}\frac{\partial^{m-1} N[\phi(x,t;q)]}{\partial q^{m-1}}\Big| q = 0, \tag{14}$$

and

$$\chi_m = \begin{cases} 0, & m \le 1, \\ 1, & m > 1. \end{cases} \tag{15}$$

It should be emphasized that $u_m(x,t)$ for $m \ge 1$ is governed by the linear equation (13) with linear boundary conditions that comes from the original problem, which can be easily solved by the symbolic computation softwares such as Maple, Mathematica and Matlab.

4. APPLICATIONS

Example 4.1 Consider the following linear Fokker-Planck equation

$$u_t = u_x + u_{xx}$$

with the initial condition $u(x,0) = x$ (16)

According to the HAM, we take the initial guess as

$$u_0(x,t) = x \tag{17}$$

The rule of solution expression also suggests that we define the linear operator L by

$$L[\phi(x,t;q)] = \frac{\partial \phi(x,t;q)}{\partial t} \tag{18}$$

with the property

$$L\left[C\right] = 0 \tag{19}$$

The nonlinear operator is

$$N[\phi(x,t;q)] = \frac{\partial \phi(x,t;q)}{\partial t} - \frac{\partial \phi(x,t;q)}{\partial x} - \frac{\partial^2 \phi(x,t;q)}{\partial x^2} \tag{20}$$

and thus

$$\Re_m(\vec{u}_{m-1}) = \frac{\partial u_{m-1}}{\partial t} - \frac{\partial u_{m-1}}{\partial x} - \frac{\partial^2 u_{m-1}}{\partial x} \tag{21}$$

The m^{th} -order deformation equation is given by

$$L[u_m(x,t) - \chi_m u_{m-1}(x,t)] = \hbar \Re_m(\vec{u}_{m-1}) \tag{22}$$

solving above equation (22) under the initial conditions $u_m(x,0) = 0,\ m = 1,2,3\ldots$ we get

$$\begin{aligned} u_1(x,t) &= -\hbar t \\ u_2(x,t) &= -\hbar(1+\hbar)t \\ u_3(x,t) &= \hbar(1+\hbar)^2 t \\ &\vdots \end{aligned} \tag{23}$$

and so on

Taking $\hbar = -1$, the approximate solution is given by

$$u(x,t) = \sum_{r=0}^{m-1} u_r(x,t) = x + t \tag{24}$$

Which is the exact solution and is same as obtained by ADM [27], VIM [28] and HPM[29]

Example 4.2 Consider the following linear Fokker-Planck equation (2) such that

$$A(x,t) = e^t \coth x \cosh x + e^t \sinh x - \coth x$$
$$B(x,t) = e^t \cosh x$$

i.e

$$u_t = -\frac{\partial}{\partial x} A(x,t)u + \frac{\partial^2}{\partial x^2} B(x,t)\, u$$

with the initial condition $u(x,0) = \sinh x, \quad x \in R$ (25)

According to the HAM, we take the initial guess as

$$u_0(x,t) = \sinh x \tag{26}$$

The nonlinear operator is

$$N[\phi(x,t;q)] = \frac{\partial \phi(x,t;q)}{\partial t} + \frac{\partial\ A(x,t)\ \phi(x,t;q)}{\partial x} - \frac{\partial^2\ B(x,t)\ \phi(x,t;q)}{\partial x^2} \tag{27}$$

and thus

$$\Re_m(\vec{u}_{m-1}) = \frac{\partial u_{m-1}}{\partial t} + \frac{\partial A(x,t)\, u_{m-1}}{\partial x} - \frac{\partial^2 B(x,t)\, u_{m-1}}{\partial x^2} \tag{28}$$

The m^{th} -order deformation equation is given by

$$L[u_m(x,t) - \chi_m u_{m-1}(x,t)] = \hbar \Re_m(\vec{u}_{m-1}) \tag{29}$$

solving above equation (29) under the initial conditions $u_m(x,0) = 0,\ m = 1,2,3...$ we get

$$\begin{aligned}
u_1(x,t) &= -\hbar\, t\, \sinh x \\
u_2(x,t) &= -\hbar(1+\hbar) t \sinh x + \frac{\hbar^2 t^2 \sinh x}{2} \\
u_3(x,t) &= -\hbar(1+\hbar)^2 t \sinh x + \hbar^2(1+\hbar) t^2 \sinh x - \frac{\hbar^3 t^3 \sinh x}{6} \\
&\vdots
\end{aligned} \tag{30}$$

and so on

Taking $\hbar = -1$, the approximate solution is given by

$$u(x,t) = e^t \sinh x \tag{31}$$

Which is the exact solution and is same as obtained by ADM [27], VIM [28] and HPM [29]

Example 4.3 Consider the Backward Kolmogorov Eq. (3) such that

$$A(x,t) = -(x+1), \quad B(x,t) = x^2 e^t \tag{32}$$

i.e.

$$u_t = (x+1)\, u_x + x^2 e^t u_{xx}$$

with the initial condition

$$u(x,0) = x+1, \quad x \in R \tag{33}$$

According to the HAM, we take the initial guess as

$$u_0(x,t) = (x+1) \tag{34}$$

The nonlinear operator is

$$N[\phi(x,t;q)] = \frac{\partial \phi(x,t;q)}{\partial t} - (x+1)\frac{\partial \phi(x,t;q)}{\partial x} - x^2 \frac{\partial^2 \phi(x,t;q)}{\partial x^2} \tag{35}$$

and thus

$$\Re_m(\vec{u}_{m-1}) = \frac{\partial u_{m-1}}{\partial t} - (x+1)\frac{\partial u_{m-1}}{\partial x} - x^2 \frac{\partial^2 u_{m-1}}{\partial x} \tag{36}$$

The m^{th} -order deformation equation is given by

$$L[u_m(x,t) - \chi_m u_{m-1}(x,t)] = \hbar \Re_m(\vec{u}_{m-1}) \tag{37}$$

solving above equation (37) under the initial conditions $u_m(x,0) = 0,\ m = 1,2,3...$ we get

$$u_1(x,t) = -\hbar(x+1) t$$

$$u_2(x,t) = -\hbar(1+\hbar)(x+1)t + \frac{\hbar^2(x+1)t^2}{2} \tag{38}$$
$$u_3(x,t) = -\hbar(1+\hbar)^2(x+1)t + \frac{\hbar^2(1+\hbar)(x+1)t^2}{2} + \frac{\hbar^3(x+1)t^3}{6}$$
$$\vdots$$

and so on

Taking $\hbar = -1$, the approximate solution is given by

$$u(x,t) = e^t(x+1) \tag{39}$$

Which is the exact solution and is same as obtained by ADM [27], VIM [28] and HPM [29]

Example 4.4 Consider the following Nonlinear Fokker-Planck eq. (5) such that

$$A(x,t,u) = \frac{4}{x}u - \frac{x}{3}$$
$$B(x,t,u) = u \tag{40}$$

subject to the initial condition:

$$u(x,0) = x^2, \quad x \in R$$

i.e.

$$u_t = \frac{\partial}{\partial x}\left(\frac{xu}{3} - \frac{4}{x}u^2\right) + \frac{\partial^2}{\partial x^2}\left(u^2\right) \tag{41}$$

According to the HAM, we take the initial guess as

$$u_0(x,t) = x^2 \tag{42}$$

The nonlinear operator is

$$N[\phi(x,t;q)] = \frac{\partial\phi(x,t;q)}{\partial t} - \frac{\partial}{\partial x}\left(\frac{x\phi(x,t;q)}{3} - \frac{4\phi^2(x,t;q)}{x}\right) + \frac{\partial^2\phi^2(x,t;q)}{\partial x^2} \tag{43}$$

and thus

$$\Re_m(\vec{u}_{m-1}) = \frac{\partial u_{m-1}}{\partial t} - \frac{\partial}{\partial x}\left(\frac{x\,u_{m-1}}{3} - \frac{4}{x}\left(\sum_{r=0}^{m-1} u_r\, u_{m-1-r}\right)\right) + \frac{\partial^2}{\partial x^2}\left(\sum_{r=0}^{m-1} u_r\, u_{m-1-r}\right) \tag{44}$$

The m^{th} -order deformation equation is given by

$$L[u_m(x,t) - \chi_m u_{m-1}(x,t)] = \hbar\Re_m(\vec{u}_{m-1}) \tag{45}$$

solving above equation (45) under the initial conditions $u_m(x,0) = 0,\ m = 1,2,3\ldots$ we get

$$u_1(x,t) = -\hbar x^2 t$$
$$u_2(x,t) = -\hbar(1+\hbar)x^2 t + \frac{\hbar^2 x^2 t^2}{2} \tag{46}$$
$$u_3(x,t) = -\hbar(1+\hbar)^2 x^2 t + \frac{\hbar^2(1+\hbar)x^2t^2}{2} - \frac{\hbar^3 x^2 t^3}{6}$$
$$\vdots$$

and so on

Taking $\hbar = -1$, the approximate solution is given by

$$u(x,t) = x^2 e^t \tag{47}$$

Which is the exact solution and is same as obtained by ADM [27], VIM [28] and HPM [29]

If we denote the approximation of k^{th} terms by ψ_k, then 4-terms approximation is denoted by $\psi_4 = \sum_{i=0}^{3} u_i(x,t)$.

The error between exact and approximate solution is given in Table 1.

Table 1
Comparison of the exact solution with 4-term HAM taking $\hbar$ = -1 solution of Ex. 4.4

(x_i, t_i)	Exact Solution $u(x,t)$	Approximate Solution $\psi_4(x,t)$	Error $\lvert u(x,t) - \psi_4(x,t) \rvert$
(0.1, 0.1)	0.011052	0.011052	1.40898E-11
(0.2, 0.2)	0.048856	0.048856	3.65974E-09
(0.3, 0.3)	0.121487	0.121487	9.51818E-08
(0.4, 0.4)	0.238692	0.238692	9.64965E-08
(0.5, 0.5)	0.412184	0.412184	5.83851E-08

5 CONCLUSIONS

In this paper, the homotopy analysis method (HAM) is applied to obtain the solution of Fokker-Planck equation. HAM provides us with a convenient way to control the convergence of approximation series by adapting $\hbar$, which is a fundamental qualitative difference in analysis between HAM and other methods. The error between the exact solution and approximate solution are very small and tends to zero. Fig 1-6 shows that the exact solution and approximate solution have the same values for larger values of *t*. Also it has been shown that the HPM and ADM are the particular case of this method. This technique is algorithmic and it is easy to implementation by symbolic computation software, such a Maple and Mathematica. The numerical results of the above example display a fast convergence, with minimal calculations. It shows that the HAM is a very effective method and might find wide applications. Mathematica 7.0 has been used for numerical computation in this paper.

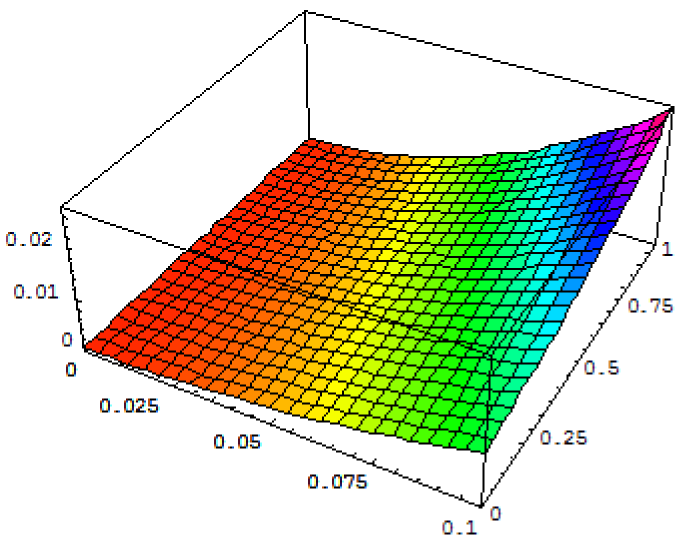

Fig.1: Exact solution graph of Fokker-Planck ex. 4.4 for t= 0 to t =1

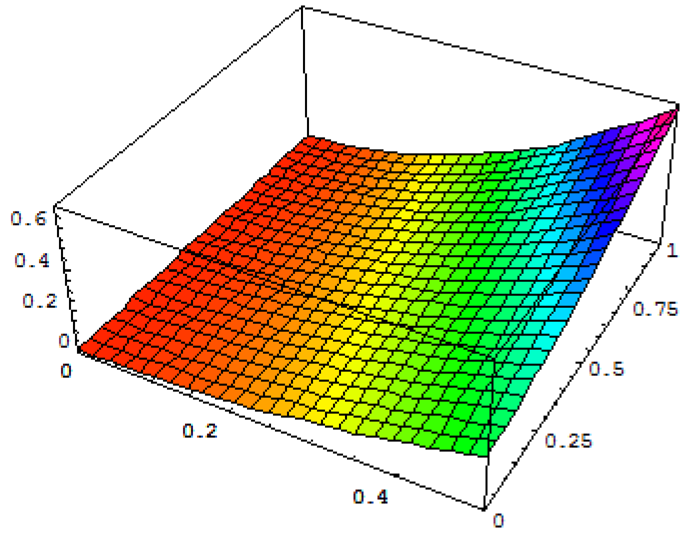

Fig.2: Approximate solution graph of Fokker-Planck ex. 4.4 for t= 0 to t =1 by HAM for fourth-order approximation

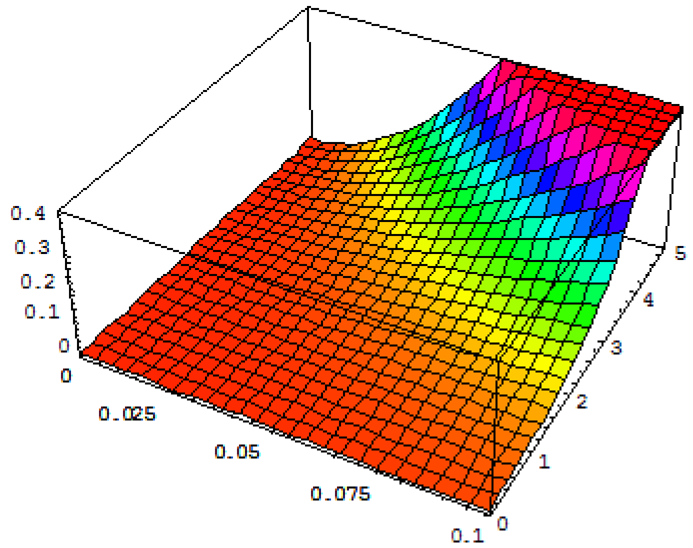

Fig.3: Exact solution graph of Fokker-Planck ex. 4.4 for t= 0 to t =5

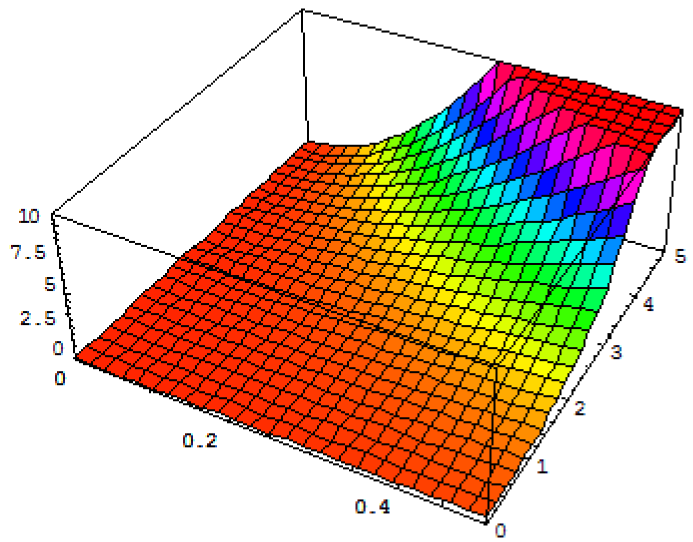

Fig.4: Approximate solution graph of Fokker-Planck ex. 4.4 for t= 0 to t =5 by HAM for fourth-order approximation

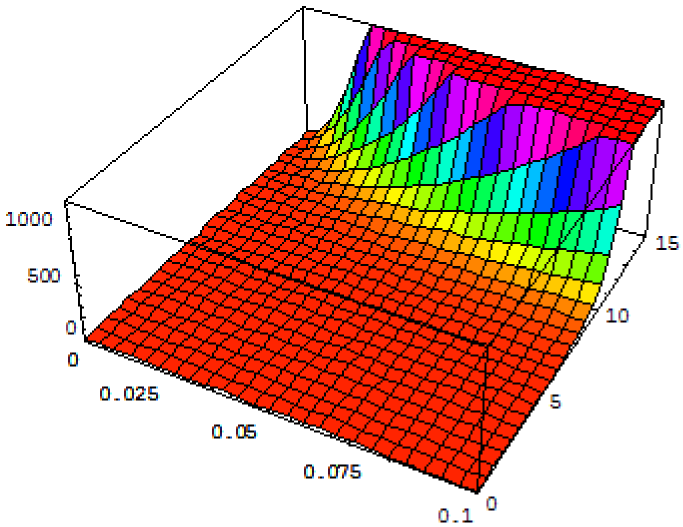

Fig.5: Exact solution graph of Fokker-Planck ex. 4.4 for t= 0 to t =10

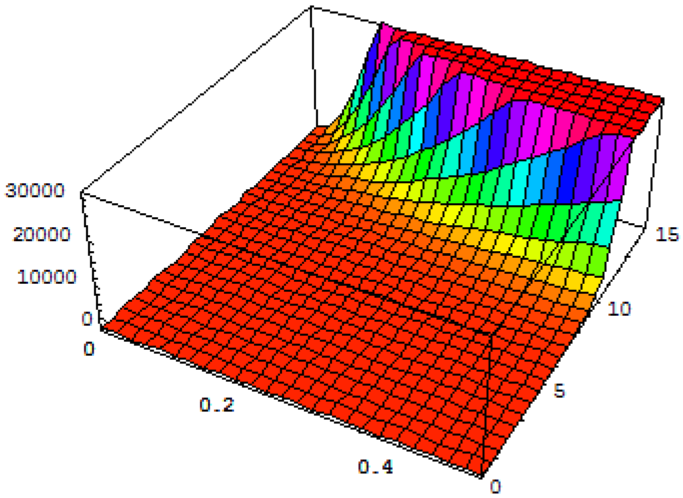

Fig 6: Approximate solution graph of Fokker-Planck ex. 4.4 for t= 0 to t =10 by HAM for fourth-order approximation

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