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A FIXED POINT THEOREM FOR FOUR SELF MAPS ON A FUZZY METRIC SPACE SATISFYING A CERTAIN CONTROL CONDITION

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ABSTRACT

The purpose of this paper is to show that four self maps on a complete fuzzy metric space which satisfy a certain control condition turn out to be equal and constant. From this result,we show that the theorem of Saluja and Mukesh Kumar Jain [8] follows as a corollary, even under a weaker condition.

Key words: *Fuzzy metric space, compatible maps, weak compatible maps.*

Mathematical subject classification (2010): *47H10, 54H25.*

1. INTRODUCTION:

The concept of fuzzy sets introduced by Zadeh [12], was the foundation of fuzzy metric space. Fuzzy metric spaces have been introduced by Kramosil and Michalek [5], and George and Veeramani [3] modified the notion of fuzzy metric spaces with the help of continuous t-norms. Recently many authors such as [1], [2], [6] and [11] have proved fixed point theorems involving fuzzy sets. Recently Singh and Jain [10] have introduced semi compatibility of maps in fuzzy metric spaces.

Using this concept Saluja and Mukesh Kumar Jain [8] proved a common fixed point theorem for six maps in fuzzy metric spaces. In this paper, we prove a fixed point theorem and obtain the result of Saluja and Mukesh Kumar Jain [8] as a corollary.

We start with

Definition: 1.1 (Schweizer. B and Sklar. A [9]) $A\ binary\ operation\ *: [0,1] \times [0,1] \rightarrow [0,1]\ is\ a\ continuous\ t-norm\ \ if * satisfies\ the\ following\ conditions$:

(i) $*$ is commutative and associative
(ii) $*$ is continuous
(iii) $a * 1 = a\ for\ all\ a\ \in [0,1]$
(iv) $a * b\ \leq c * d\ whenever\ a\ \leq c\ and\ b\ \leq d\ and\ a, b, c, d\ \in [0,1]$

Definition: 1.2 (Kramosil. I and Michelek. J [5]) A - triple $(X, M,*)$ is said to be a fuzzy metric space (FM sapce, briefly) if X is a nonempty set, $*$ is a continuous t-norm and M is a fuzzy set on $X^2 \times [0, \infty)$ $i.e\ \ M: X^2 \times [0, \infty) \longrightarrow [0.1]\ \ satisfying\ the\ following\ conditions: for\ all\ x, y, z \in X\ and\ s, t > 0.$

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(i) $M(x,y,t) > 0$
(ii) $M(x,y,t) = 1$ $if\ and\ only\ if\ x = y$
(iii) $M(x,y,t) = M(y,x,t)$
(iv) $M(x,y,t) * M(y,z,s) \le M(x,z,t+s)$
(v) $M(x,y,t): [0,\infty) \longrightarrow [0,1]$ $it\ left\ continuous$
(vi) $\lim_{t\to\infty} M(x,y,t) = 1$

Then M is called a fuzzy metric space on X.

The function $M(x,y,t)$ denotes the degree of nearness between x and y with respect to t.

Definition: 1.3 (**George. A and Veeramani. P [3]**) Let $(X, M, *)$ be a fuzzy metric space.Then,

(i) A sequence $\{x_n\}$ in X is said to be convergent to a point $x \in X$ if $\lim_{n\to\infty} M(x_n, x, t) = 1\ \forall\ t > 0$.

(ii) A sequence $\{x_n\}$ in X is called a Cauchy sequence if $\lim_{n\to\infty} M(x_{n+p}, x_n, t) = 1\ \forall\ t > 0$ and $p = 1,2,\ldots$

(iii) An F.M –space in which every Cauchy sequence is convergent is said to be complete.

Definition: 1.4 (G. Jungck [4]) Two maps F and G of a fuzzy metric space $(X, M, *)$ into itself are said to be compatible if $\lim_{n\to\infty} M(FGx_n, GFx_n, t) = 1\ \forall\ t > 0$ whenever $\{x_n\}$ is a sequence in X such that

$\lim_{n\to\infty} Fx_n = \lim_{n\to\infty} Gx_n = x$ for some $x \in X$.

Definition: 1.5 (G. Jungck [4]) Two self maps F and G of a fuzzy metric space $(X, M, *)$ are said to be weak compatible if $F(x) = G(x) \Rightarrow FG(x) = GF(x)$.

The following lemma is due to Mishra. S. N, Sharma. N and Singh. S.L [7**]**.

Lemma: 1.6 (Mishra. S. N, Sharma. N and Singh. S.L [7]) Let $(X, M, *)$ be a fuzzy metric space. If there exists $k > 1$ such that $M(x,y,kt) \le M(x,y,t)$ then $x = y$.

Saluja and Mukesh Kumar Jain **[8]** proved the following result.

Theorem: 1.7 (Saluja and Mukesh Kumar Jain [8], Theorem 3.1): Let $A, B, S, T, I\ and\ J$ be self maps of a complete fuzzy metric space $(X, M, *)$ with $*$ = min (so that $*$ in continuous) and

(i) $AB(X) \subset J(X),\ ST(X) \subset I(X)$
(ii) AB, I are continuous
(iii) pair (AB, I) is compatible
(iv) pair (ST, J) is weak compatible
(v) There exists $k > 1$ and $\alpha \epsilon (1,2)$ such that for all $x, y \in X$ and $t > 0$

$$M(ABx, STy, kt) \le M(Ix, ABx, t) * M(Jy, STy, t) * M(ABx, STy, t) * M(Ix, Jy, t) * M(ABx, Jy, (\alpha - 1)t) * M(Ix, STy, \tfrac{\alpha t}{2})$$

for all $x, y \in X$, $\lim_{t\to\infty} M(x,y,t) = 1\ \forall\ t > 0$

Then $AB, ST, I\ and\ J$ have a unique common fixed point in X.

2. MAIN RESULTS:

Now we state our main result and obtain Theorem 1.7 as a corollary.

Theorem: 2.1 Let A, B, C, D be self mpas of a complete fuzzy metric sapce $(X, M, *)$ with $* = min$. Suppose
(i) $A(X) \subset C(X) and\ B(X) \subset D(X)$,
(ii) pairs $(A, D), (B, C)$ are weakly compatible,
(iii) for some $k > 1, \alpha \in (0,1)$ and for all $x, y \in X$, and $t > 0$

$$M(Ax, By, kt) \le M(Dx, Ax, t) * M(Cy, By, t) * M(Ax, By, t) * M(Dx, Cy, t) * M(Ax, Cy, \alpha t) * M(Dx, By, \alpha t).$$

Then A is a constant function, $A = B = C = D$ is constant and hence A, B, C, D have unique common fixed point.

Proof: Let $x_0 \in X$. From (i) there exist $x_1, x_2, x_3, \ldots.\ x_n \ldots$ in X such that

$Ax_0 = Cx_1 = y_1 \quad and \quad Bx_1 = Dx_2 = y_2,$

$Ax_2 = Cx_3 = y_3 \ and \ Bx_3 = Dx_4 = y_4$

In general, $Ax_{2n} = Cx_{2n+1} = y_{2n+1}\ and \quad Bx_{2n+1} = Dx_{2n+2} = y_{2n+2}, n = 0,1,2 \ldots$

Now for $x = x_{2n},\ y = x_{2n+1}$ in (iii) we have, for $t > 0$,

$$M(Ax_{2n}, Bx_{2n+1}, kt) \le M(Dx_{2n}, Ax_{2n}, t) * M(Cx_{2n+1}, Bx_{2n+1}, t) * M(Ax_{2n}, Bx_{2n+1}, t) * M(Dx_{2n}, Cx_{2n+1}, t) * M(Ax_{2n}, Cx_{2n+1}, \alpha t) * M(Dx_{2n}, Bx_{2n+1}, \alpha t).$$

$$M(y_{2n+1}, y_{2n+2}, kt) \le M(y_{2n}, y_{2n+1}, t) * M(y_{2n+1}, y_{2n+2}, t) * M(y_{2n+1}, y_{2n+2}, t) * M(y_{2n}, y_{2n+1}, t) * M(y_{2n+1}, y_{2n+1}, \alpha t) * M(y_{2n}, y_{2n+2}, \alpha t).$$

$$\le M(y_{2n}, y_{2n+1}, t) * M(y_{2n+1}, y_{2n+2}, t) * M(y_{2n}, y_{2n+2}, \alpha t)$$

$$\le M(y_{2n+1}, y_{2n+2}, t) * M(y_{2n+1}, y_{2n+2}, (1+\alpha)t) \quad (\because 1 + \alpha > 1)$$

$$\le M(y_{2n+1}, y_{2n+2}, t)$$

$\therefore M(y_{2n+1}, y_{2n+2}, kt) \le M(y_{2n+1}, y_{2n+2}, t)$, for $t > 0$ (2.1.1)

Again for $x = x_{2n+2},\ y = x_{2n+1}$ in (iii) we have, for $t > 0$,

$$M(Ax_{2n+2}, Bx_{2n+1}, kt) \le M(Dx_{2n+2}, Ax_{2n+2}, t) * M(Cx_{2n+1}, Bx_{2n+1}, t) * M(Ax_{2n+2}, Bx_{2n+1}, t) * M(Dx_{2n+2}, Cx_{2n+1}, t) * M(Ax_{2n+2}, Cx_{2n+1}, \alpha t) * M(Dx_{2n+2}, Bx_{2n+1}, \alpha t)$$

i.e. $M(y_{2n+3}, y_{2n+2}, kt) \le M(y_{2n+2}, y_{2n+3}, t) * M(y_{2n+1}, y_{2n+2}, t) * M(y_{2n+3}, y_{2n+2}, t) * M(y_{2n+2}, y_{2n+1}, t) * M(y_{2n+3}, y_{2n+1}, \alpha t) * M(y_{2n+2}, y_{2n+2}, \alpha t)$

$$\le M(y_{2n+2}, y_{2n+3}, t) * M(y_{2n+1}, y_{2n+2}, t) * M(y_{2n+3}, y_{2n+1}, \alpha t)$$

$$\le M(y_{2n+2}, y_{2n+3}, t) * M(y_{2n+2}, y_{2n+3}, (1+\alpha)t) \ (\because 1 + \alpha > 1)$$

$$\le M(y_{2n+2}, y_{2n+3}, t).$$

i.e $M(y_{2n+3}, y_{2n+2}, kt) \le M(y_{2n+2}, y_{2n+3}, t)$, for $t > 0$ (2.1.2)

Thus from (2.1.1) and (2.1.2), we have $M(y_n, y_{n+1}, kt) \le M(y_n, y_{n+1}, t)\ \forall n \in \mathbb{N}$ and $t > 0$.

Hence $y_n = y_{n+1}\ \forall n \in \mathbb{N}$ (by Lemma 1.6) (2.1.3)

Hence $\{y_n\}$ is a constant sequence, say, z in X. So it is Cauchy in X and converges to z in X.

Thus the sequences $\{Ax_{2n}\}, \{Cx_{2n+1}\}, \{Dx_{2n}\}\ and\ \{Bx_{2n+1}\}$ also converge to z.

Now we have $Ax_0 = Cx_1 = y_1 \quad and \quad Bx_1 = Dx_2 = y_2,$

$\Rightarrow Ax_0 = Cx_1 = y_1 = y_2 = Bx_1 = Dx_2$

Therefore $Ax_2 = Cx_1 = y_1 \Rightarrow Bx_1 = Cx_1 = y_1 \Rightarrow BCx_1 = CBx_1$

$\Rightarrow B(Bx_1) = C(Bx_1) \Rightarrow By_1 = Cy_1$ [$by\ (ii)\ B, C\ are\ weak\ compatible$] and $Ay_1 = Dy_1$

[$by\ (ii)\ A, D\ are\ weak\ compatible$]

By taking $x = x_2, y = y_1$ in (iii), we get

$$M(Ax_2, By_1, kt) \leq M(Dx_2, Ax_2, t) * M(Cy_1, By_1, t) * M(Ax_2, By_1, t) * M(Dx_2, Cy_1, t) * M(Ax_2, Cy_1, \alpha t) * M(Dx_2, By_1, \alpha t)$$

$$\Rightarrow M(y_1, By_1, kt) \leq M(y_1, y_1, t) * M(By_1, By_1, t) * M(y_1, By_1, t) * M(y_1, By_1, t) * M(y_1, By_1, \alpha t) * M(y_1, By_1, \alpha t)$$

$$\Rightarrow M(y_1, By_1, kt) \leq M(y_1, By_1, t) * M(y_1, By_1, \alpha t)$$

$$\leq M(y_1, By_1, \alpha t) \qquad [\because \alpha < 1]$$

$$\leq M(y_1, By_1, t) \ \forall\ t > 0$$

Therefore by Lemma 1.6, we have $By_1 = y_1 = Cy_1$

By taking $x = y_1, y = y_1$ in (iii), we get

$$M(Ay_1, By_1, kt) \leq M(Dy_1, Ay_1, t) * M(Cy_1, By_1, t) * M(Ay_1, By_1, t) * M(Dy_1, Cy_1, t) * M(Ay_1, Cy_1, \alpha t) * M(Dy_1, By_1, \alpha t)$$

$$\Rightarrow M(Ay_1, y_1, kt) \leq M(Ay_1, Ay_1, t) * M(y_1, y_1, t) * M(Ay_1, y_1, t) * M(Ay_1, y_1, t) * M(Ay_1, y_1, \alpha t) * M(Ay_1, y_1, \alpha t)$$

$$\leq 1 * 1 * M(Ay_1, y_1, t) * M(Ay_1, y_1, \alpha t)$$

$$\leq M(Ay_1, y_1, \alpha t) \qquad [\because \alpha < 1]$$

$$\leq M(Ay_1, y_1, t)$$

$$\Rightarrow M(Ay_1, y_1, kt) \leq M(Ay_1, y_1, t) \ \forall\ t > 0$$

Therefore by Lemma 1.6, we have $Ay_1 = y_1 = Dy_1$

$\therefore y_1$ is a fixed point of $A, B, C \text{ and } D$.

By (iii), $A, B, C \text{ and } D$ cannot have more than one fixed point.

Hence A, B, C, D have a unique common fixed point in X and also $Ax_0 = y_1$ every$x_0 \in X$.

Thus we have shown that for any $x_0 \in X, Ax_0 = y_1$ is the unique fixed point of $A, B, C \text{ and } D$.

Hence A is a constant function.

From (i) of Theorem 2.1, we have $A(X) \subset C(X) \text{and } B(X) \subset D(X)$

For $x_0 \in X$, construct the sequences $\{x_n\}$ and $\{y_n\}$ as follows

$Bx_0 = Dx_1 = y_1 \quad and \quad Ax_1 = Cx_2 = y_2,$

$Bx_2 = Dx_3 = y_3 \ and \ Ax_3 = Cx_4 = y_4$

In general, $Bx_{2n} = Dx_{2n+1} = y_{2n+1} \ and \quad Ax_{2n+1} = Cx_{2n+2} = y_{2n+2}$

Then, it can be shown, as we did in the case of A, that B is constant, say, $Bx = w$ for every $x \in X$ and w is the common fixed point for $A, B, C \text{ and } D$.

Hence, $z = w$ by the uniqueness of the common fixed point $A, B, C \text{ and } D$.

Hence $A = B$

Let $x \in X$. By condition (iii) taking $x = y$, we get

$M(Ax, Bx, kt) \leq M(Dx, Ax, t) * M(Cx, Bx, t) * M(Ax, Bx, t) * M(Dx, Cx, t) * M(Ax, Cx, \alpha t) * M(Dx, Bx, \alpha t)$

$1 \leq M(Dx, Ax, t) * M(Cx, Bx, t) * 1 * M(Dx, Cx, t) * M(Ax, Cx, \alpha t) * M(Dx, Bx, \alpha t)$

$\therefore 1 \leq M(Ax, Dx, t)$ for every $t > 0 \Rightarrow Ax = Dx \forall x \in X \Rightarrow A = D$

$\therefore 1 \leq M(Cx, Bx, t)$ for every $t > 0 \Rightarrow Cx = Bx \forall x \in X \Rightarrow C = B$.

Therefore $D = A = B = C$

Therefore A, B, C, D are constants.

Corollary: 2.2 Let A, B, S, T, I *and* J be self maps of a FM-space $(X, M, *)$ with $* = min$, satisfying (i), (iii), (iv), (v), (vi) in the Theorem 1.7. Then AB, ST, I *and* J have a unique common fixed point.

Proof: Let $\beta = max\left\{\alpha - 1, \frac{\alpha}{2}\right\}$, *so that* $\beta \in (0,1)$. Then

$$M(ABx, STy, kt) \leq M(Ix, ABx, t) * M(Jy, STy, t) * M(ABx, STy, t) * M(Ix, Jy, t) * M(ABx, Jy, (\alpha - 1)t) * M(Ix, STy, \tfrac{\alpha t}{2})$$

$$\leq M(Ix, ABx, t) * M(Jy, STy, t) * M(ABx, STy, t) * M(Ix, Jy, t) * M(ABx, Jy, \beta t) * M(Ix, STy, \beta t)$$

$\therefore AB = I = J = ST = constant$ by our main result.

Hence AB, I, J, ST have a unique common fixed point in X.

Note 1: Thus the Theorem 1.7 is a corollary to our main result Theorem 2.1.

Note 2: However under the hypothesis of Theorem 3.1, of Saluja and Mukehs Kumar Jain [8], A, B, S *and* T need not be constant functions, even though $AB = ST$ = constant function.

This is a evident by taking $B = T =$ identity map and $A = S = I$.

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