

A FIXED POINT THEOREM FOR FOUR SELF MAPS
ON A FUZZY METRIC SPACE SATISFYING A CERTAIN CONTROL CONDITION

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ABSTRACT

The purpose of this paper is to show that four self maps on a complete fuzzy metric space which satisfy a certain control condition turn out to be equal and constant. From this result, we show that the theorem of Saluja and Mukesh Kumar Jain [8] follows as a corollary, even under a weaker condition.

Key words: Fuzzy metric space, compatible maps, weak compatible maps.

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1. INTRODUCTION:

The concept of fuzzy sets introduced by Zadeh [12], was the foundation of fuzzy metric space. Fuzzy metric spaces have been introduced by Kramosil and Michalek [5], and George and Veeramani [3] modified the notion of fuzzy metric spaces with the help of continuous t-norms. Recently many authors such as [1], [2], [6] and [11] have proved fixed point theorems involving fuzzy sets. Recently Singh and Jain [10] have introduced semi compatibility of maps in fuzzy metric spaces.

Using this concept Saluja and Mukesh Kumar Jain [8] proved a common fixed point theorem for six maps in fuzzy metric spaces. In this paper, we prove a fixed point theorem and obtain the result of Saluja and Mukesh Kumar Jain [8] as a corollary.

We start with

Definition: 1.1 (Schweizer. B and Sklar. A [9]) A binary operation $*$: $[0,1] \times [0,1] \rightarrow [0,1]$ is a continuous t - norm if $*$ satisfies the following conditions:

- (i) $*$ is commutative and associative
- (ii) $*$ is continuous
- (iii) $a * 1 = a$ for all $a \in [0,1]$
- (iv) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$ and $a, b, c, d \in [0,1]$

Definition: 1.2 (Kramosil. I and Michelek. J [5]) A - triple $(X, M, *)$ is said to be a fuzzy metric space (FM space, briefly) if X is a nonempty set, $*$ is a continuous t-norm and M is a fuzzy set on $X^2 \times [0, \infty)$ i.e $M: X^2 \times [0, \infty) \rightarrow [0,1]$ satisfying the following conditions: for all $x, y, z \in X$ and $s, t > 0$.

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- (i) $M(x, y, t) > 0$
- (ii) $M(x, y, t) = 1$ if and only if $x = y$
- (iii) $M(x, y, t) = M(y, x, t)$
- (iv) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$
- (v) $M(x, y, t): [0, \infty) \rightarrow [0, 1]$ it left continuous
- (vi) $\lim_{t \rightarrow \infty} M(x, y, t) = 1$

Then M is called a fuzzy metric space on X .

The function $M(x, y, t)$ denotes the degree of nearness between x and y with respect to t .

Definition: 1.3 (George. A and Veeramani. P [3]) Let $(X, M, *)$ be a fuzzy metric space. Then,

- (i) A sequence $\{x_n\}$ in X is said to be convergent to a point $x \in X$
if $\lim_{n \rightarrow \infty} M(x_n, x, t) = 1 \forall t > 0$.

- (ii) A sequence $\{x_n\}$ in X is called a Cauchy sequence if
 $\lim_{n \rightarrow \infty} M(x_{n+p}, x_n, t) = 1 \forall t > 0$ and $p = 1, 2, \dots$

- (iii) An F.M –space in which every Cauchy sequence is convergent is said to be complete.

Definition: 1.4 (G. Jungck [4]) Two maps F and G of a fuzzy metric space $(X, M, *)$ into itself are said to be compatible if $\lim_{n \rightarrow \infty} M(FGx_n, GFx_n, t) = 1 \forall t > 0$ whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n \rightarrow \infty} Fx_n = \lim_{n \rightarrow \infty} Gx_n = x \text{ for some } x \in X.$$

Definition: 1.5 (G. Jungck [4]) Two self maps F and G of a fuzzy metric space $(X, M, *)$ are said to be weak compatible if $F(x) = G(x) \Rightarrow FG(x) = GF(x)$.

The following lemma is due to Mishra. S. N, Sharma. N and Singh. S.L [7].

Lemma: 1.6 (Mishra. S. N, Sharma. N and Singh. S.L [7]) Let $(X, M, *)$ be a fuzzy metric space. If there exists $k > 1$ such that $M(x, y, kt) \leq M(x, y, t)$ then $x = y$.

Saluja and Mukesh Kumar Jain [8] proved the following result.

Theorem: 1.7 (Saluja and Mukesh Kumar Jain [8], Theorem 3.1): Let A, B, S, T, I and J be self maps of a complete fuzzy metric space $(X, M, *)$ with $*$ = min (so that $*$ in continuous) and

- (i) $AB(X) \subset J(X), ST(X) \subset I(X)$
- (ii) AB, I are continuous
- (iii) pair (AB, I) is compatible
- (iv) pair (ST, J) is weak compatible
- (v) There exists $k > 1$ and $\alpha \in (1, 2)$ such that for all $x, y \in X$ and $t > 0$

$$M(ABx, STy, kt) \leq M(Ix, ABx, t) * M(Jy, STy, t) * M(ABx, STy, t) * M(Ix, Jy, t) * M(ABx, Jy, (\alpha - 1)t) * M(Ix, STy, \frac{at}{2})$$

$$\text{for all } x, y \in X, \lim_{t \rightarrow \infty} M(x, y, t) = 1 \forall t > 0$$

Then AB, ST, I and J have a unique common fixed point in X .

2. MAIN RESULTS:

Now we state our main result and obtain Theorem 1.7 as a corollary.

Theorem: 2.1 Let A, B, C, D be self mpas of a complete fuzzy metric sapce $(X, M, *)$ with $*$ = min . Suppose

- (i) $A(X) \subset C(X)$ and $B(X) \subset D(X)$,
- (ii) pairs $(A, D), (B, C)$ are weakly compatible,
- (iii) for some $k > 1, \alpha \in (0, 1)$ and for all $x, y \in X$, and $t > 0$

$$M(Ax, By, kt) \leq M(Dx, Ax, t) * M(Cy, By, t) * M(Ax, By, t) * M(Dx, Cy, t) * M(Ax, Cy, \alpha t) * M(Dx, By, \alpha t).$$

Then A is a constant function, $A = B = C = D$ is constant and hence A, B, C, D have unique common fixed point.

Proof: Let $x_0 \in X$. From (i) there exist $x_1, x_2, x_3, \dots, x_n \dots$ in X such that

$$Ax_0 = Cx_1 = y_1 \quad \text{and} \quad Bx_1 = Dx_2 = y_2,$$

$$Ax_2 = Cx_3 = y_3 \quad \text{and} \quad Bx_3 = Dx_4 = y_4$$

In general, $Ax_{2n} = Cx_{2n+1} = y_{2n+1}$ and $Bx_{2n+1} = Dx_{2n+2} = y_{2n+2}, n = 0, 1, 2 \dots$

Now for $x = x_{2n}, y = x_{2n+1}$ in (iii) we have, for $t > 0$,

$$M(Ax_{2n}, Bx_{2n+1}, kt) \leq M(Dx_{2n}, Ax_{2n}, t) * M(Cx_{2n+1}, Bx_{2n+1}, t) * M(Ax_{2n}, Bx_{2n+1}, t) * M(Dx_{2n}, Cx_{2n+1}, t) \\ * M(Ax_{2n}, Cx_{2n+1}, \alpha t) * M(Dx_{2n}, Bx_{2n+1}, \alpha t).$$

$$M(y_{2n+1}, y_{2n+2}, kt) \leq M(y_{2n}, y_{2n+1}, t) * M(y_{2n+1}, y_{2n+2}, t) * M(y_{2n+1}, y_{2n+2}, t) * M(y_{2n}, y_{2n+1}, t) \\ * M(y_{2n+1}, y_{2n+1}, \alpha t) * M(y_{2n}, y_{2n+2}, \alpha t).$$

$$\leq M(y_{2n}, y_{2n+1}, t) * M(y_{2n+1}, y_{2n+2}, t) * M(y_{2n}, y_{2n+2}, \alpha t)$$

$$\leq M(y_{2n+1}, y_{2n+2}, t) * M(y_{2n+1}, y_{2n+2}, (1 + \alpha)t) \quad (\because 1 + \alpha > 1)$$

$$\leq M(y_{2n+1}, y_{2n+2}, t)$$

$$\therefore M(y_{2n+1}, y_{2n+2}, kt) \leq M(y_{2n+1}, y_{2n+2}, t), \text{ for } t > 0 \tag{2.1.1}$$

Again for $x = x_{2n+2}, y = x_{2n+1}$ in (iii) we have, for $t > 0$,

$$M(Ax_{2n+2}, Bx_{2n+1}, kt) \leq M(Dx_{2n+2}, Ax_{2n+2}, t) * M(Cx_{2n+1}, Bx_{2n+1}, t) * M(Ax_{2n+2}, Bx_{2n+1}, t) \\ * M(Dx_{2n+2}, Cx_{2n+1}, t) * M(Ax_{2n+2}, Cx_{2n+1}, \alpha t) * M(Dx_{2n+2}, Bx_{2n+1}, \alpha t)$$

$$\text{i.e. } M(y_{2n+3}, y_{2n+2}, kt) \leq M(y_{2n+2}, y_{2n+3}, t) * M(y_{2n+1}, y_{2n+2}, t) * M(y_{2n+3}, y_{2n+2}, t) * M(y_{2n+2}, y_{2n+1}, t) \\ * M(y_{2n+3}, y_{2n+1}, \alpha t) * M(y_{2n+2}, y_{2n+2}, \alpha t)$$

$$\leq M(y_{2n+2}, y_{2n+3}, t) * M(y_{2n+1}, y_{2n+2}, t) * M(y_{2n+3}, y_{2n+1}, \alpha t)$$

$$\leq M(y_{2n+2}, y_{2n+3}, t) * M(y_{2n+2}, y_{2n+3}, (1 + \alpha)t) \quad (\because 1 + \alpha > 1)$$

$$\leq M(y_{2n+2}, y_{2n+3}, t).$$

$$\text{i.e. } M(y_{2n+3}, y_{2n+2}, kt) \leq M(y_{2n+2}, y_{2n+3}, t), \text{ for } t > 0 \tag{2.1.2}$$

Thus from (2.1.1) and (2.1.2), we have $M(y_n, y_{n+1}, kt) \leq M(y_n, y_{n+1}, t) \quad \forall n \in \mathbb{N}$ and $t > 0$.

$$\text{Hence } y_n = y_{n+1} \quad \forall n \in \mathbb{N} \text{ (by Lemma 1.6)} \tag{2.1.3}$$

Hence $\{y_n\}$ is a constant sequence, say, z in X . So it is Cauchy in X and converges to z in X .

Thus the sequences $\{Ax_{2n}\}, \{Cx_{2n+1}\}, \{Dx_{2n}\}$ and $\{Bx_{2n+1}\}$ also converge to z .

Now we have $Ax_0 = Cx_1 = y_1$ and $Bx_1 = Dx_2 = y_2$,

$$\Rightarrow Ax_0 = Cx_1 = y_1 = y_2 = Bx_1 = Dx_2$$

Therefore $Ax_2 = Cx_1 = y_1 \Rightarrow Bx_1 = Cx_1 = y_1 \Rightarrow BCx_1 = CBx_1$

$\Rightarrow B(Bx_1) = C(Bx_1) \Rightarrow By_1 = Cy_1$ [by (ii) B, C are weak compatible] and $Ay_1 = Dy_1$

[by (ii) A, D are weak compatible]

By taking $x = x_2, y = y_1$ in (iii), we get

$$\begin{aligned}
 M(Ax_2, By_1, kt) &\leq M(Dx_2, Ax_2, t) * M(Cy_1, By_1, t) * M(Ax_2, By_1, t) * \\
 &\quad M(Dx_2, Cy_1, t) * M(Ax_2, Cy_1, \alpha t) * M(Dx_2, By_1, \alpha t) \\
 \Rightarrow M(y_1, By_1, kt) &\leq M(y_1, y_1, t) * M(By_1, By_1, t) * M(y_1, By_1, t) * M(y_1, By_1, t) * M(y_1, By_1, \alpha t) * M(y_1, By_1, \alpha t) \\
 \Rightarrow M(y_1, By_1, kt) &\leq M(y_1, By_1, t) * M(y_1, By_1, \alpha t) \\
 &\leq M(y_1, By_1, \alpha t) \quad [:\alpha < 1] \\
 &\leq M(y_1, By_1, t) \quad \forall t > 0
 \end{aligned}$$

Therefore by Lemma 1.6, we have $By_1 = y_1 = Cy_1$

By taking $x = y_1, y = y_1$ in (iii), we get

$$\begin{aligned}
 M(Ay_1, By_1, kt) &\leq M(Dy_1, Ay_1, t) * M(Cy_1, By_1, t) * M(Ay_1, By_1, t) * \\
 &\quad M(Dy_1, Cy_1, t) * M(Ay_1, Cy_1, \alpha t) * M(Dy_1, By_1, \alpha t) \\
 \Rightarrow M(Ay_1, y_1, kt) &\leq M(Ay_1, Ay_1, t) * M(y_1, y_1, t) * M(Ay_1, y_1, t) * M(Ay_1, y_1, t) \\
 &\quad * M(Ay_1, y_1, \alpha t) * M(Ay_1, y_1, \alpha t) \\
 &\leq 1 * 1 * M(Ay_1, y_1, t) * M(Ay_1, y_1, \alpha t) \\
 &\leq M(Ay_1, y_1, \alpha t) \quad [:\alpha < 1] \\
 &\leq M(Ay_1, y_1, t)
 \end{aligned}$$

$$\Rightarrow M(Ay_1, y_1, kt) \leq M(Ay_1, y_1, t) \quad \forall t > 0$$

Therefore by Lemma 1.6, we have $Ay_1 = y_1 = Dy_1$

$\therefore y_1$ is a fixed point of A, B, C and D .

By (iii), A, B, C and D cannot have more than one fixed point.

Hence A, B, C, D have a unique common fixed point in X and also $Ax_0 = y_1$ every $x_0 \in X$.

Thus we have shown that for any $x_0 \in X, Ax_0 = y_1$ is the unique fixed point of A, B, C and D .

Hence A is a constant function.

From (i) of Theorem 2.1, we have $A(X) \subset C(X)$ and $B(X) \subset D(X)$

For $x_0 \in X$, construct the sequences $\{x_n\}$ and $\{y_n\}$ as follows

$$Bx_0 = Dx_1 = y_1 \quad \text{and} \quad Ax_1 = Cx_2 = y_2,$$

$$Bx_2 = Dx_3 = y_3 \quad \text{and} \quad Ax_3 = Cx_4 = y_4$$

$$\text{In general, } Bx_{2n} = Dx_{2n+1} = y_{2n+1} \quad \text{and} \quad Ax_{2n+1} = Cx_{2n+2} = y_{2n+2}$$

Then, it can be shown, as we did in the case of A , that B is constant, say, $Bx = w$ for every $x \in X$ and w is the common fixed point for A, B, C and D .

Hence, $z = w$ by the uniqueness of the common fixed point A, B, C and D .

Hence $A = B$

Let $x \in X$. By condition (iii) taking $x = y$, we get

$$M(Ax, Bx, kt) \leq M(Dx, Ax, t) * M(Cx, Bx, t) * M(Ax, Bx, t) * M(Dx, Cx, t) * M(Ax, Cx, at) * M(Dx, Bx, at)$$

$$1 \leq M(Dx, Ax, t) * M(Cx, Bx, t) * 1 * M(Dx, Cx, t) * M(Ax, Cx, at) * M(Dx, Bx, at)$$

$$\therefore 1 \leq M(Ax, Dx, t) \text{ for every } t > 0 \Rightarrow Ax = Dx \forall x \in X \Rightarrow A = D$$

$$\therefore 1 \leq M(Cx, Bx, t) \text{ for every } t > 0 \Rightarrow Cx = Bx \forall x \in X \Rightarrow C = B.$$

Therefore $D = A = B = C$

Therefore A, B, C, D are constants.

Corollary: 2.2 Let A, B, S, T, I and J be self maps of a FM-space $(X, M, *)$ with $*$ = min, satisfying (i), (iii), (iv), (v), (vi) in the Theorem 1.7. Then AB, ST, I and J have a unique common fixed point.

Proof: Let $\beta = \max\{\alpha - 1, \frac{\alpha}{2}\}$, so that $\beta \in (0,1)$. Then

$$\begin{aligned} M(ABx, STy, kt) &\leq M(Ix, ABx, t) * M(Jy, STy, t) * M(ABx, STy, t) * M(Ix, Jy, t) \\ &\quad * M(ABx, Jy, (\alpha - 1)t) * M(Ix, STy, \frac{\alpha t}{2}) \\ &\leq M(Ix, ABx, t) * M(Jy, STy, t) * M(ABx, STy, t) * M(Ix, Jy, t) \\ &\quad * M(ABx, Jy, \beta t) * M(Ix, STy, \beta t) \end{aligned}$$

$\therefore AB = I = J = ST = \text{constant}$ by our main result.

Hence AB, I, J, ST have a unique common fixed point in X .

Note 1: Thus the Theorem 1.7 is a corollary to our main result Theorem 2.1.

Note 2: However under the hypothesis of Theorem 3.1, of Saluja and Mukehs Kumar Jain [8], A, B, S and T need not be constant functions, even though $AB = ST = \text{constant function}$.

This is a evident by taking $B = T = \text{identity map}$ and $A = S = I$.

REFERENCES:

- [1] Balasubramaniam. P, Muralisankar. S and Pant. R.P: Common fixed points of four mappings in fuzzy metric spaces, J. Fuzzy math. 10(2) (2002), 379-384.
- [2] Grabiec. M: Fixed points in fuzzy metric spaces, Fuzzy sets and systems 27(1988), 385-389.
- [3] George. A and Veeramani. P: On some results in Fuzzy metric spaces, Fuzzy sets and system, 64 (1994), 395-399.
- [4] G. Jungck: Compatible mappings and common fixed point, Internat J. Math. Mat. Sci 9 (1996), 771-779.
- [5] Kramosil. I and Michelek. J: Fuzzy metric and statistical metric, kybernetika, 11(1975), 336-344.
- [6] Kutukcu. S, Sharma. S. S, Tokgoz. H: A fixed point theorem in Fuzzy metric spaces, Int. Journal of Math, Analysis, Vol. 1, 2007, no. 18, 861-872.
- [7] Mishra. S. N, Sharma. N and Singh. S.L: Common fixed points of maps on fuzzy metric spaces, Internet, J. Math. & Math Sci. 17 (1994), 253-258.
- [8] Saluja and Mukehs Kumar Jain: Fixed point theorem for expansion mapping including six maps in fuzzy metric space, Volume 6, no. 1 (2011), 127-134.
- [9] Schweizer. B and Sklar. A: Probabilistic Metric spaces, North Holland (1983).
- [10] Singh. B and Jain. S: Semi compatibility, Compatibility and fixed point theorem in fuzzy metric space, Journal of the chuncheong mathematical society (2005), 1-22.
- [11] Vasuki. R: common fixed point theorem in a fuzzy metric space, Fuzzy sets and system, 97 (1998), 395-397.
- [12] Zadeh. L.A: Fuzzy sets, Inform and control, 189 (1965), 338-353.
