International Journal of Mathematical Archive-2(1), Jan. - 2011, Page:131-145
Available online through www.ijma.info ISSN 2229 – 5046

HANKEL TYPE TRANSFORMATION AND CONVOLUTION ON SPACES OF DISTRIBUTIONS WITH EXPONENTIAL GROWTH

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(Received on: 25-12-10; Accepted on: 02-01-11)

ABSTRACT

In this paper we study Hankel type transformation on the spaces $M_{\alpha,\beta}$ and $Q_{\alpha,\beta}$ defined. Further Hankel type Convolution on $M_{\alpha,\beta}$ and $M'_{\alpha,\beta}$ is studied. We have given representation and characterization theorem for the elements of $M'_{\alpha,\beta,m,\#}$ and $M'_{\alpha,\beta}$ on $M_{\alpha,\beta}$ and $M'_{\alpha,\beta,\#}$ respectively. Finally interchange formula and algebraic properties are discussed.

Keywords: Hankel type transformation, Hankel type Convolution, distribution, Bessel type operator.

2000 Mathematics Subject Classification: 46F12.

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1. INTRODUCTION:

In the last few decades many authors have studied Hankel transformation theory and it's applications in various areas of mathematics, engineering and technologies, especially by Betancor[1], Betancor and Marrero[2,3,4,5,6,7], Waphare [18] and many other amongst them. The Hankel type transformation defined in Waphare [18] is given by,

$$\left(h_{\alpha,\beta}\,\phi\right)(s)=\int_0^\infty (st)^{\alpha+\beta}J_{\alpha-\beta}(st)f(t)dt\ ,\quad s\in(0,\infty) \tag{1.1}$$

$\alpha-\beta>\dfrac{-1}{2}$, where J_λ is the Bessel function of the first kind and order λ.

We introduce the space $H_{\alpha,\beta}$ that consists of all complex valued functions $f=f(x),\ x\in(0,\infty)$, such that

$$\rho^{\alpha,\beta}_{k,m}(f)=\sup_{x\in(0,\infty)}\left|x^k\left(\frac{1}{x}D\right)^m\left(x^{2\beta-1}f(x)\right)\right|<\infty, \tag{1.2}$$

For every $k,m\in N$. We note that $H_{\alpha,\beta}$ is a Frechet space. The dual space of $H_{\alpha,\beta}$ is denoted by $H'_{\alpha,\beta}$.

Now for every $a\in(0,\infty)$, we define the subspace $B_{\alpha,\beta,a}$ of $H_{\alpha,\beta}$ consisting of $\phi\in H_{\alpha,\beta}$ such that $\phi(x)=0$ for $x\geq a$. The space $B_{\alpha,\beta}=\bigcup_{a>0}B_{\alpha,\beta,a}$ endowed with the inductive topology is a dense subspace of $H_{\alpha,\beta}$ and the Hankel type transform on $B_{\alpha,\beta}$.

Cholewinski [8], Hirschman [12] and Haimo [10] studied a convolution for a variant of the Hankel transformation that, after straight forword manipulations, allows defining convolution for the h_u transform. A measurable function ϕ on $(0,\infty)$ is said to be in $L_{\alpha,\beta}$ if and only if, $x^{2\alpha}\phi$ is absolutely integrable on $(0,\infty)$. For $\phi,\psi\in L_{\alpha,\beta}$, we define the Hankel type convolution $\phi\#\psi$ of ϕ and ψ by

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$$(\phi \# \psi)(x) = \int_0^\infty \phi(y)(\tau_x \psi)(y) dy\ , \quad x \in (0,\infty) \tag{1.3}$$

where the Hankel type translation is given by

$$(\tau_x \psi)(y) = \int_0^\infty D_{\alpha,\beta}(x,y,z)\psi(z)dz\ , \quad x, y \in (0,\infty) \tag{1.4}$$

and

$$D_{\alpha,\beta}(x,y,z) = \int_0^\infty t^{2\beta-1}(xt)^{\alpha+\beta} J_{\alpha-\beta}(xt)(yt)^{\alpha+\beta} J_{\alpha-\beta}(yt)(zt)^{\alpha+\beta} J_{\alpha-\beta}(zt)dt\ , \tag{1.5}$$

$x, y, z \in (0,\infty)$.

The Fourier transform of distribution of exponential growth had been investigated earlier by Hasumi [11], Zielezny [24] and among others. However the Hankel type transformations has not been defined on distribution of exponential growth. In this paper we investigate the Hankel type transformation and Hankel type convolution on distribution of exponential growth.
The following boundedness properties of Bessel functions will be very much useful in the sequel:

(i) There exists $C > 0$ such that

$$\left| z^{-(\alpha-\beta)} J_{\alpha-\beta}(z) \right| \le C e^{\left|\operatorname{Im} z\right|}, z \in C \tag{1.6}$$

(ii) If $H^{(1)}_{\alpha,\beta}$ denotes the Hankel type function of the first kind and order $(\alpha - \beta)$ then there exist $C > 0$ such that

$$\left| z^{\alpha+\beta} H^{(1)}_{\alpha,\beta}(z) \right| \le C e^{\operatorname{Im} z}, z \in C, |z| \ge 1 \tag{1.7}$$

2. Hankel type transformation and the spaces $Q_{\alpha,\beta}$ and $M_{\alpha,\beta}$:

In this section we introduce new function spaces that the Hankel type transformation maps isomorphically.

Definition 2.1: The space $M_{\alpha,\beta}$ is the space of all smooth complex valued functions $\phi(x), x \in (0,\infty)$ such that

$$\eta^{\alpha,\beta}_{k,m}(\phi) = \sup_{x \in (0,\infty)} \left| e^{kx} \left(\frac{1}{x} D\right)^m \left(x^{2\beta-1}\phi(x)\right) \right| < \infty, \tag{2.1}$$

for every $k, m \in N$.

The space $M_{\alpha,\beta}$ is a Frechet space. By using Lemma 2.2 of Betancor and Marrero [4] we can see that the seminorms

$$\eta^{\alpha,\beta}_{k,m}(\phi) = \operatorname*{Sup}_{x \in (0,\infty)} \left| e^{kx} x^{2\beta-1} \Delta^m_{\alpha,\beta} \phi(x) \right|, \phi \in M_{\alpha,\beta}\ , \quad k, m \in N \tag{2.2}$$

where

$$\Delta_{\alpha,\beta} = x^{2\beta-1} D x^{4\alpha} D x^{2\beta-1}.$$

is the Bessel type operator, induce on $M_{\alpha,\beta}$ the same topology as defined by $\left\{\eta^{\alpha,\beta}_{k,m}\right\}_{k,m \in N}$ of seminorms. Notice that $M_{\alpha,\beta}$ is continuously contained in $H_{\alpha,\beta}$. It is easy to see that the Frechet function spaces introduced in Koh and Li [13] and Koh and Zemanian [14] contain $M_{\alpha,\beta}$.

We denote by θ_M the space of multipliers of $M_{\alpha,\beta}$ that is a function f is in θ_M whenever $f\phi \in M_{\alpha,\beta}$ for every $\phi \in M_{\alpha,\beta}$. Following a procedure used in Betancor and Marrero [2] and Yoshinaga [20], we can easily prove that $f \in \theta_M$ if and only if

(i) f is smooth on $(0,\infty)$, and

(ii) for every $m \in N$ there exists $k \in N$ and $C > 0$ such that

$$\left|\left(\frac{1}{x}D\right)^m f(x)\right| \le Ce^{kx}, \quad x \in \mathrm{I}.$$

We denote by $M'_{\alpha,\beta}$, the dual space of $M_{\alpha,\beta}$. θ_M is also the space of multipliers in $M'_{\alpha,\beta}$. Following techniques used in Betancor[1], Betancor and Marrero[4] and Treves[17], is not very difficult to conclude that a functional T on $M_{\alpha,\beta}$ is in $M'_{\alpha,\beta}$ if and only if there exists $r \in N$ and essentially bounded functions f_k on $(0,\infty)$, $0 \le k \le r$ such that

$$T = \sum_{k=0}^{r} \Delta_{\alpha,\beta}^{k}\left(e^{rx} x^{2\beta-1} f_k\right).$$

Definition 2.2: The space $Q_{\alpha,\beta}$ consists of all complex valued functions Φ such that

(i) $z^{2\beta-1}\Phi(z)$ is an even entire function, and

(ii) for every $k, m \in N$,

$$\omega_{k,m}^{\alpha,\beta}(\Phi) = \sup_{|\operatorname{Im} z| \le k} \left|\left(1+|z|^2\right) z^{2\beta-1}\Phi(z)\right| < \infty.$$

When endowed with the topology generated by the family $\left\{\omega_{k,m}^{\alpha,\beta}\right\}_{k,m\in\mathbb{N}}$ of norms; $Q_{\alpha,\beta}$ is a nuclear Frechet space. $Q'_{\alpha,\beta}$ denotes the dual space $Q_{\alpha,\beta}$.

Definition 2.3: The space θ_Q of all complex valued, even and entire functions F such that for every $k \in N$ there exists $m \in N$ for which

$$\sup_{|\operatorname{Im} z| \le k} \left(1+|z|^2\right)^{-m} |F(z)| < \infty.$$

It is seen that F is a multiplier of $Q_{\alpha,\beta}$ whenever $F \in \theta_Q$.

Theorem 2.4: The Hankel type transformation $h_{\alpha,\beta}$ is an isomorphism from $M_{\alpha,\beta}$ onto $Q_{\alpha,\beta}$. Moreover, the inverse of $h_{\alpha,\beta}$ is also $h_{\alpha,\beta}$.

Proof: Let $\phi \in M_{\alpha,\beta}$ and define $\Phi = h_{\alpha,\beta}(\phi)$. By (1.6), for every $k \in N$,

$$\begin{aligned}\int_0^\infty \left|(xz)^{-(\alpha-\beta)} J_{\alpha-\beta}(xz)\right| x^{2\alpha} |\phi(x)|\,dx &\le C\int_0^\infty e^{x|\operatorname{Im} z|} x^{2\alpha} |\phi(x)|\,dx \\ &\le C\sup_{x\in(0,\infty)} \left|e^{(k+1)x} x^{2\beta-1}\phi(x)\right|, \quad if\ |\operatorname{Im} z| \le k.\end{aligned}$$

This shows that $x^{2\beta-1}\Phi(z)$ is an even entire function.

Also as $M_{\alpha,\beta} \subset H_{\alpha,\beta}$, by lemma 5.41 of Zemanian [23], we have for every $k, m \in N$,

$$\sup_{|\operatorname{Im} z| \le k} \left(1+|z|^2\right)^m \left|z^{2\beta-1}\Phi(z)\right| \le C\left\{\eta_{k+1,0}^{\alpha,\beta}(\phi) + \eta_{k+1,m}^{\alpha,\beta}(\phi)\right\}.$$

Thus $h_{\alpha,\beta}$ is a continuous mapping from $M_{\alpha,\beta}$ into $Q_{\alpha,\beta}$.

Let $\Phi \in Q_{\alpha,\beta}$. Since Φ is absolutely integrable in $(0,\infty)$, we can define

$$\phi(x)=\int_0^\infty (xy)^{\alpha+\beta} J_{\alpha-\beta}(xy)\Phi(y)dy, \quad x\in(0,\infty),$$

and the integral is absolutely convergent for every $x\in(0,\infty)$.

By Zemanian [23, 5.1(7)], for every $m\in N$ and $x\in(0,\infty)$, we have

$$\left(\frac{1}{x}D\right)^m \left(x^{2\beta-1}\phi(x)\right)=(-1)^m\int_0^\infty (xy)^{-(\alpha-\beta)-m} J_{\alpha-\beta+m}(xy)\, y^{2m+2\alpha}\Phi(y)\,dy. \tag{2.3}$$

The integral in (2.3) is again absolutely convergent for every $x\in(0,\infty)$.By Lemma 6.1 of Eijndhoven and Kerkhof [9], we have from (2.3) that

$$\left(\frac{1}{x}D\right)^m \left(x^{2\beta-1}\phi(x)\right)=\frac{(-1)^m}{2}\int_{-\infty}^{\infty}\left(x(\xi+i\eta)\right)^{-(\alpha-\beta)-m} H^{(1)}_{\alpha-\beta+m}\left(x(\xi+i\eta)\right)(\xi+i\eta)^{2m+2\alpha}\,\Phi(\xi+i\eta)\,d\xi \text{ for every}$$

$x>1,\eta>0$ and $m\in N$ and here $H^{(1)}_{\alpha,\beta}$ is the Hankel type function of the first kind and order $(\alpha-\beta)$. As $\left|x(\xi+i\eta)\,d\xi\right|>\eta$ for $x>1$, (2.2) gives

$$\left|\left(\frac{1}{x}D\right)^m \left(x^{2\beta-1}\phi(x)\right)\right|\le Ce^{-\eta x}\int_{-\infty}^{\infty}\left|\xi+i\eta\right|^m\left|\Phi(\xi+i\eta)\right|d\xi \tag{2.4}$$

for every $x>1,\eta>0$ and $m\in N$, where the positive constant C depends on η. Now we choose $l\in N$ such that $l>2(2\alpha+\beta)$. By (2.4) we can obtain that for every $k,m\in N$,

$$\begin{aligned}\left|e^{kx}\left(\frac{1}{x}D\right)^m \left(x^{2\beta-1}\phi(x)\right)\right| &\le Ce^{-x}\int_{-\infty}^{\infty}\left|\xi+i(k+1)\right|^m\left|\Phi(\xi+i(k+1))\right|d\xi\\ &\le C\int_{\infty}^{-\infty}\left|\xi+i(k+1)\right|^{2\alpha-l}\left|\xi+i(k+1)\right|^{m+1}\left|\xi+i(k+1)\right|^{2\beta-l}\left|\Phi(\xi+i(k+1))\right|d\xi\\ &\le C\omega^{\alpha,\beta}_{k+1,m+1}(\Phi), \quad \text{for every } x>1.\end{aligned} \tag{2.5}$$

Now for every $x\in(0,1)$ and $k,m\in N$ we have

$$\begin{aligned}\left|e^{kx}\left(\frac{1}{x}D\right)^m \left(x^{2\beta-1}\phi(x)\right)\right| &\le e^k\int_0^\infty\left|(xy)^{-(\alpha-\beta)-m} J_{\alpha-\beta+m}(xy)\right|\left|y^{2m+2\alpha}\Phi(y)\right|dy\\ &\le C\omega^{\alpha,\beta}_{1,m+n}(\Phi),\end{aligned} \tag{2.6}$$

where $n\in N$ and $n>(3\alpha+\beta)$. By (2.5) and (2.6) we can conclude that $h_{\alpha,\beta}$ is continuous mapping from $Q_{\alpha,\beta}$ ino $M_{\alpha,\beta}$. Finally as $M_{\alpha,\beta}\subset H_{\alpha,\beta}$, it follows that $h_{\alpha,\beta}=h^{-1}_{\alpha,\beta}$. This completes the proof.

Corollary 2.5: The space $Q_{\alpha,\beta}$ is continuously contained in $H_{\alpha,\beta}$.

Proof: Proof is easy and follows from Theorem 2.4.

Now we define the generalized Hankel type transformation between $M'_{\alpha,\beta}$ and $Q'_{\alpha,\beta}$ to be the transpose of the $h_{\alpha,\beta}$ transformation, that is, the Hankel type transform $h'_{\alpha,\beta}(T)$ of $T\in M'_{\alpha,\beta}$ $(resp.\ Q'_{\alpha,\beta})$ is the element of $Q'_{\alpha,\beta}$ $(resp.\ M'_{\alpha,\beta})$ defined by

$$\left\langle h'_{\alpha,\beta}(T)\ ,\ \phi\right\rangle = \left\langle T\ ,\ h_{\alpha,\beta}(\phi)\right\rangle, \quad \phi\in Q_{\alpha,\beta}(resp.\ M_{\alpha,\beta}).$$

Theorem 2.6: The generalized Hankel type transformation is an isomorphism from $M'_{\alpha,\beta}$ $(resp.\ Q'_{\alpha,\beta})$ onto $Q'_{\alpha,\beta}$ $(resp.\ M'_{\alpha,\beta})$ when we consider on $M'_{\alpha,\beta}$ and $Q'_{\alpha,\beta}$ the $weak^*$ or strong topology.

3. Hankel type convolution on $M_{\alpha,\beta}$ and $M'_{\alpha,\beta}$:

In this section first we investigate the behavior of the Hankel type transformation and Hankel type convolution in $M_{\alpha,\beta}$.

Lemma 3.1: For every $x\in(0,\infty)$, the Hankel type translation τ_x defines a continuous linear mapping from $M_{\alpha,\beta}$ into itself.

Proof: Let $x\in(0,\infty)$. As $M_{\alpha,\beta}\subset H_{\alpha,\beta}$, by Marrero and Betancor [15] we have for every $\phi\in M_{\alpha,\beta}$,

$$(\tau_x\phi)(y) = h_{\alpha,\beta}\left[t^{2\beta-1}(xt)^{\alpha+\beta}J_{\alpha-\beta}(xt)h_{\alpha,\beta}(\phi)(t)\right](y), \quad y\in \mathrm{I} \tag{3.1}$$

By theorem 2.4 to prove that τ_x is continuous from $M_{\alpha,\beta}$ into itself, it is enough to prove that the function $\Phi_x(t)=(xt)^{-(\alpha-\beta)}J_{\alpha-\beta}(xt)$, $t\in N$ is a multiplier in $Q_{\alpha,\beta}$. Note that Φ_x is an even entire function. Moreover, from (1.6), it follows that

$$\left|\Phi_x(t)\right| \le Ce^{x|\operatorname{Im} z|},\quad t\in N$$

Thus $\Phi_x\in\theta_Q$ and the proof is complete.

Lemma 3.2: The Hankel type convolution defines a continuous linear mapping from $M_{\alpha,\beta}\times M_{\alpha,\beta}$ into $M_{\alpha,\beta}$.

Proof: Note that for each $\phi,\psi\in M_{\alpha,\beta}$, from Theorem 2.d of Hirschman [12] it is not very difficult to show that the following interchange formula holds.

$$h_{\alpha,\beta}(\phi\#\psi) = y^{2\beta-1}h_{\alpha,\beta}(\phi)\,h_{\alpha,\beta}(\psi). \tag{3.2}$$

But the mapping $(\Phi,\Psi)\mapsto y^{2\beta-1}\Phi\Psi$ is continuous from $M_{\alpha,\beta}\times M_{\alpha,\beta}$ into $M_{\alpha,\beta}$. Thus the proof can be completed by using Theorem 2.4.

Lemma 3.1 allows us to define the Hankel type convolution $T\#\phi$ of $T\in M'_{\alpha,\beta}$ and $\phi\in M_{\alpha,\beta}$ as below:

$$(T\#\phi)(x) = \left\langle T\ ,\ \tau_x\phi\right\rangle, \quad x\in(0,\infty).$$

Note that we cannot insure that $T\#\phi\in M_{\alpha,\beta}$. Infact, define

$$\left\langle T,\phi\right\rangle = \int_0^\infty x^{2\alpha}\phi(x)dx, \qquad \phi\in M_{\alpha,\beta}.$$

Note that $T\in M'_{\alpha,\beta}$. Let $\phi\in M_{\alpha,\beta}$, By Hirschman [12, §2, (2)], we have

$$\langle T\ ,\ \tau_x\phi\rangle = \int_0^\infty z^{2\alpha}\int_0^\infty D_{\alpha,\beta}(x,y,z)\phi(y)\,dydz$$

$$= \int_0^\infty \phi(y)\int_0^\infty D_{\alpha,\beta}(x,y,z)\,z^{2\alpha}dzdy$$

$$= C_{\alpha,\beta}^{-1}x^{2\alpha}\int_0^\infty y^{2\alpha}\phi(y)\,dy, \qquad x\in(0,\infty)$$

where, $C_{\alpha,\beta} = 2^{\alpha-\beta}\overline{|(3\alpha+\beta)}$. Thus

$$x^{2\beta-1}\langle T\ ,\ \tau_x\phi\rangle = C_{\alpha,\beta}^{-1}\langle T\ ,\ \tau_x\phi\rangle = C_{\alpha,\beta}^{-1}\int_0^\infty y^{2\alpha}\phi(y)\,dy, \qquad x\in(0,\infty),$$ and

$$e^x x^{2\beta-1}\langle T\ ,\ \tau_x\phi\rangle \to \infty \qquad as \qquad x\to\infty.$$

Therefore $T\#\phi\in M_{\alpha,\beta}$.

In the next lemma we shall prove that for every $T\in M'_{\alpha,\beta}$ and $\phi\in M_{\alpha,\beta}$, $x^{2\beta-1}T\#\phi$ is a multiplier of $M_{\alpha,\beta}$.

Lemma 3.3: If $T\in M'_{\alpha,\beta}$ and $\phi\in M_{\alpha,\beta}$ then $x^{2\beta-1}T\#\phi\in\theta_M$.

Proof: From Section 2, there exists $r\in N$ and essentially bounded functions f_k on $(0,\infty)$, $0\le k\le r$, such that

$$T = \sum_{k=0}^{r}\Delta_{\alpha,\beta}^{k}\left(e^{rx}x^{2\beta-1}f_k\right).$$

Thus it is enough to prove the result for

$$T = \Delta_{\alpha,\beta}^{k}\left(e^{rx}x^{2\beta-1}f\right),$$

where f is an essentially bounded functions on $(0,\infty)$, and $r,k\in N$. Let $\phi\in M_{\alpha,\beta}$.

Now by Proposition 2.1(ii) of Marrero and Betancor [15] we have

$$(T\#\phi)(x) = \langle T\ ,\ \tau_x\phi\rangle = \int_0^\infty f(y)e^{ry}y^{2\beta-1}\tau_x\left(\Delta_{\alpha,\beta}^{k}\phi\right)(y)\,dy$$

$$= (-1)^k\ x^{2\alpha}\int_0^\infty f(y)e^{ry}y^{2\beta-1}h_{\alpha,\beta}\left[(xt)^{-(\alpha-\beta)}\ J_{\alpha-\beta}(xt)h_{\alpha,\beta}(\phi)(t)t^{2k}\right](y)\,dy,$$

$x\in(0,\infty)$. Now let $n\in N$. By using 5.1(7) of Zemanian [21], it is clear that

$$\left(\frac{1}{x}D\right)^n\left(x^{2\beta-1}(T\#\phi)(x)\right)$$

$$= (-1)^{n+k}\int_0^\infty f(y)e^{ry}y^{2\beta-1}h_{\alpha,\beta}\left[t^{2(n+k)}(xt)^{-(\alpha-\beta)-n}\ J_{\alpha-\beta+n}(xt)h_{\alpha,\beta}(\phi)(t)\right](y)dy,\ \ x\in(0,\infty).$$

Then for each $x\in(0,\infty)$,

$$\left|\left(\frac{1}{x}D\right)^n\left(x^{2\beta-1}(T\#\phi)(x)\right)\right|$$

$$\le\ C\sup_{y\in(0,\infty)}\left|e^{(r+1)y}y^{2\beta-1}h_{\alpha,\beta}\left[t^{2(n+k)}(xt)^{-(\alpha-\beta)-n}\ J_{\alpha-\beta+n}(xt)h_{\alpha,\beta}(\phi)(t)\right](y)\right|.$$

As $z^{-(\alpha-\beta)}J_{\alpha-\beta}(z)\in\theta_Q$ and using Theorem 2.4, there exists $l,m\in N$ such that

$$\left|\left(\frac{1}{x}D\right)^n\left(x^{2\beta-1}(T\#\phi)(x)\right)\right|$$

$$\leq C\omega_{l,m}^{\alpha,\beta}\left(t^{2(k+n)}(xt)^{-(\alpha-\beta)-n}J_{\alpha-\beta+n}(xt)h_{\alpha,\beta}(\phi)(t)\right),\ x\in(0,\infty). \tag{3.3}$$

Now from (1.6), we have

$$\omega_{l,m}^{\alpha,\beta}\left((xt)^{-\alpha-\beta-n}J_{\alpha-\beta+n}(xt)t^{2(k+n)}h_{\alpha,\beta}(\phi)(t)\right)$$

$$\leq C\omega_{l,m+k+n}^{\alpha,\beta}\left(h_{\alpha,\beta}(\phi)\right)e^{xl},\ x\in\mathrm{I}, \tag{3.4}$$

where C is independent of $x\in(0,\infty)$.

Lastly by Theorem 2.4 again and by (3.3) and (3.4) we can infer that

$$\left|\left(\frac{1}{x}D\right)^n\left(x^{2\beta-1}(T\#\phi)(x)\right)\right|\leq C\,e^{xl},\quad x\in(0,\infty).$$

Thus $x^{2\beta-1}(T\#\phi)(x)\in\theta_M$. This completes the proof.

Now by Lemma 3.3, if $T\in M'_{\alpha,\beta}$ and $\phi\in M_{\alpha,\beta}$ then $T\#\phi$ defines an element of $M'_{\alpha,\beta}$ by

$$\langle T\#\phi,\psi\rangle=\int_0^\infty(T\#\phi)(x)\psi(x)\,dx,\qquad \psi\in M_{\alpha,\beta}.$$

Lemma 3.4: If $T\in M'_{\alpha,\beta}$ and $\phi\in M_{\alpha,\beta}$ then

$$\langle T\#\phi,\psi\rangle=\langle T,\phi\#\psi\rangle,\qquad \psi\in M_{\alpha,\beta}, \tag{3.5}$$

and the interchange formula

$$h'_{\alpha,\beta}(T\#\phi)=x^{2\beta-1}h'_{\alpha,\beta}(T)\,h_{\alpha,\beta}(\phi)$$

holds. Moreover, for every $T\in M'_{\alpha,\beta}$ the mapping $\phi\mapsto T\#\phi$ is continuous from $M_{\alpha,\beta}$ into $M'_{\alpha,\beta}$ when we consider on $M'_{\alpha,\beta}$ the strong topology.

Proof: Let $\psi\in M_{\alpha,\beta}$. We have

$$\langle T\#\phi,\psi\rangle=\int_0^\infty(T\#\phi)(x)\psi(x)\,dx=\int_0^\infty\langle T,\tau_x\phi\rangle\psi(x)\,dx.$$

As $(\tau_x\phi)(y)=(\tau_y\phi)(x),\quad x,y\in(0,\infty)$, thus to complete the proof of (3.5), it is enough to prove that

$$\int_0^\infty\langle T,\tau_x\phi\rangle\psi(x)\,dx=\left\langle T(y),\int_0^\infty(\tau_x\phi)(y)\psi(x)\,dx\right\rangle \tag{3.7}$$

First we prove the following case.

Case I: $$\lim_{a\to\infty}\int_a^\infty(\tau_x\phi)(y)\psi(x)\,dx=0 \tag{3.8}$$

and

$$\lim_{a\to 0}\int_0^a (\tau_x\phi)(y)\psi(x)\,dx = 0 \tag{3.9}$$

in the sense of convergence in $M_{\alpha,\beta}$.

Proof of Case I: First we prove (3.8). Let $a > 0$. It is clear that

$$\int_a^\infty (\tau_x\phi)(y)\psi(x)\,dx = (\psi_a \# \phi)(y), \quad y \in (0,\infty),$$

where

$$\psi_a(x) = \begin{cases} \psi(x), & x > a \\ 0, & x \le a. \end{cases}$$

But by Theorem 2.d of Hirschman [12], we have

$$h_{\alpha,\beta}(\psi_a \# \phi) = x^{2\beta-1} h_{\alpha,\beta}(\psi_a) h_{\alpha,\beta}(\phi)$$

Therefore by Theorem 2.4, $\psi_a \# \phi \to 0$ in $M_{\alpha,\beta}$ as $a \to \infty$, if and only if ,

$$x^{2\beta-1} h_{\alpha,\beta}(\psi_a) h_{\alpha,\beta}(\phi) \to 0 \text{ in } Q_{\alpha,\beta} \text{ as } a \to \infty.$$

Now by (1.6), $x^{2\beta-1} h_{\alpha,\beta}(\psi_a)(x)$ is an even entire and also for every $k \in N$, we have

$$\begin{aligned} \left| x^{2\beta-1} h_{\alpha,\beta}(\psi_a)(x) \right| &\le \int_a^\infty \left| (xy)^{-(\alpha-\beta)} J_{\alpha-\beta}(xy)\, y^{2\alpha}\psi(y) \right| dy \\ &\le C\int_a^\infty e^{yk} \left| y^{2\alpha}\psi(y) \right| dy, \qquad \left| \operatorname{Im} x \right| \le k. \end{aligned}$$

Hence

$$\lim_{a\to\infty} x^{2\beta-1} h_{\alpha,\beta}(\psi_a) h_{\alpha,\beta}(\phi) = 0$$

in the sense of convergence in $Q_{\alpha,\beta}$. This proves (3.8). In the same manner (3.9) can be established. Thus proof of Case I is completed.

Case II: Let $0 < a < b < \infty$. Then

$$\int_a^b \langle T, \tau_x\phi \rangle \psi(x)\,dx = \left\langle T(y), \int_a^b (\tau_x\phi)(y)\psi(x)\,dx \right\rangle \tag{3.10}$$

Proof of Case II: We use the Riemann sums techniques. Let $m \in N - \{0\}$. Define

$$x_n = \frac{a + n(b-a)}{m}, \qquad n = 0,1,2,.................m.$$

Because of linearity of T we have

$$\int_a^b \langle T, \tau_x\phi \rangle \psi(x)\,dx = \lim_{m\to\infty} \left\langle T(y),\ \frac{b-a}{m} \sum_{n=1}^m (\tau_y\phi)(x_n)\psi(x_n) \right\rangle.$$

Hence to complete the proof of Case II, it is enough to show that

$$\lim_{m\to\infty}\frac{b-a}{m}\sum_{n=1}^{m}(\tau_y\phi)(x_n)\psi(x_n)=\int_a^b(\tau_y\phi)(x)\psi(x)\,dx \tag{3.11}$$

in the sense of convergence in $M_{\alpha,\beta}$. Moreover by Theorem 2.d of Hirschman [12], (3.1) and Theorem 2.4, (3.11) is equivalent to

$$\begin{aligned}&\lim_{m\to\infty}\frac{b-a}{m}\sum_{n=1}^{m}t^{2\beta-1}h_{\alpha,\beta}(\phi)(t)(tx_n)^{\alpha+\beta}J_{\alpha-\beta}(tx_n)\psi(x_n)\\&=t^{2\beta-1}h_{\alpha,\beta}(\phi)(t)h_{\alpha,\beta}(\psi_{a,b})(t)\end{aligned} \tag{3.12}$$

in the sense of convergence in $Q_{\alpha,\beta}$, where

$$\psi_{a,b}(t)=\begin{cases}\psi(t), & t\in(a,b)\\ 0, & t\notin(a,b).\end{cases}$$

Now we prove (3.12). Let $l,k\in N$. From (1.6), we have

$$\begin{aligned}\operatorname{Im}(t)&=\left(1+\left|t^2\right|\right)^l t^{2\beta-1}h_{\alpha,\beta}(\phi)(t)\left|\frac{b-a}{m}\sum_{n=1}^{m}(tx_n)^{-(\alpha-\beta)}J_{\alpha-\beta}(tx_n)x_n^{2\alpha}\psi(x_n)-\int_a^b(tx)^{-(\alpha-\beta)}J_{\alpha-\beta}(tx)x^{2\alpha}\psi(x)\,dx\right|\\&\le C\left(1+\left|t^2\right|\right)^l\left|h_{\alpha,\beta}(\phi)(t)t^{2\beta-1}\right|, \qquad \left|\operatorname{Im}t\right|\le k,\end{aligned}$$

where C is independent of $m\in N$.

Let $\varepsilon>0$. There exists $R>0$ such that if $\left|\operatorname{Re}t\right|>R$ and $\left|\operatorname{Im}t\right|\le k$ then

$$\operatorname{Im}(t)<\varepsilon, \text{ for every } m\in\mathbb{N}. \tag{3.13}$$

But then there exists $m_0\in N$ such that for every $m\in N$ with $m\ge m_0$,

$$\left|\frac{b-a}{m}\sum_{n=1}^{m}(tx_n)^{-(\alpha-\beta)}J_{\alpha-\beta}(tx_n)x_n^{2\alpha}\psi(x_n)-\int_a^b(tx)^{-(\alpha-\beta)}J_{\alpha-\beta}(tx)x^{2\alpha}\psi(x)\,dx\right|<\varepsilon, \tag{3.14}$$

for $\left|\operatorname{Re}t\right|\le R$ and $\left|\operatorname{Im}t\right|\le k$. Combining (3.13) and (3.14), we get (3.12).
This completes the proof of Case II.

Now we prove (3.7). For every $0<a<b<\infty$, by Case I and Case II,

$$\begin{aligned}&\int_0^\infty\langle T,\tau_x\phi\rangle\psi(x)\,dx-\left\langle T(y),\int_0^\infty(\tau_x\phi)(y)\psi(x)\,dx\right\rangle\\&=\int_0^a\langle T,\tau_x\phi\rangle\psi(x)\,dx+\int_a^b\langle T,\tau_x\phi\rangle\psi(x)\,dx+\int_b^\infty\langle T,\tau_x\phi\rangle\psi(x)\,dx\\&\quad-\left\langle T(y),\int_0^a(\tau_x\phi)(y)\psi(x)\,dx\right\rangle-\left\langle T(y),\int_a^b(\tau_x\phi)(y)\psi(x)\,dx\right\rangle\\&\quad-\left\langle T(y),\int_b^\infty(\tau_x\phi)(y)\psi(x)\,dx\right\rangle \;\to 0, \; as\ a\to 0\ and\ b\to\infty.\end{aligned}$$

Thus (3.7) is proved.

Now let $\psi\in Q_{\alpha,\beta}$. By using (3.2) and (3.5), we can obtain

$$\begin{aligned}\left\langle h'_{\alpha,\beta}(T\#\phi)\,,\,\psi\right\rangle &= \left\langle T\#\phi\,,\,h_{\alpha,\beta}(\psi)\right\rangle\\ &= \left\langle T\,,\,\phi\# h_{\alpha,\beta}(\psi)\right\rangle\\ &= \left\langle h'_{\alpha,\beta}(T)\,,\,h_{\alpha,\beta}\left(\phi\# h_{\alpha,\beta}(\psi)\right)\right\rangle\\ &= \left\langle x^{2\beta-1}h'_{\alpha,\beta}(T)\,h_{\alpha,\beta}(\phi)\,,\,\psi\right\rangle\end{aligned}$$

This proves (3.6).

Finally, by invoking Lemma 3.2 we can conclude that for each $T\in M'_{\alpha,\beta}$ the mapping $\phi\mapsto T\#\phi$ is continuous from $M_{\alpha,\beta}$ into $M'_{\alpha,\beta}$ when on $M'_{\alpha,\beta}$ we consider the strong topology. This completes the proof.

Now we introduce a subspace $M'_{\alpha,\beta,\#}$ of $M'_{\alpha,\beta}$ such that $S\#\phi\in M_{\alpha,\beta}$ for every $S\in M'_{\alpha,\beta,\#}$ and $\phi\in M_{\alpha,\beta}$. The new space $M_{\alpha,\beta,\#}$ contains $M_{\alpha,\beta}$ and $\varepsilon'_{\alpha,\beta}$. Also we will define the Hankel type convolution on $M'_{\alpha,\beta}\times M'_{\alpha,\beta,\#}$. Let $m\in N$. The space $X_{\alpha,\beta,m,\#}$ consists of all smooth complex valued functions $\phi=\phi(x),\ x\in(0,\infty)$, such that

$$\delta^k_{m,\alpha,\beta}=\sup_{x\in(0,\infty)}\left|e^{mx}x^{2\beta-1}\Delta^k_{\alpha,\beta}\phi(x)\right|<\infty$$

for every $k\in N$. When endowed with the topology generated by the system $\left\{\delta^k_{m,\alpha,\beta}\right\}_{k\in\mathbb{N}}$ of seminorms, $X_{\alpha,\beta,m,\#}$ is a Frechet space. Notice that $M_{\alpha,\beta}\subset X_{\alpha,\beta,m,\#}$.

Now we define $M_{\alpha,\beta,m,\#}$ as the closure of $M_{\alpha,\beta}$ in $X_{\alpha,\beta,m,\#}$. One can easily conclude that $M_{\alpha,\beta,m,\#}$ is a Frechet space. Moreover, $M_{\alpha,\beta,m+1,\#}$ is continuously contained in $M_{\alpha,\beta,m,\#}$.

4. REPRESENTATION AND CHARACTERIZATION:

In this section we give representation for elements of $M'_{\alpha,\beta,m,\#}$ on $M_{\alpha,\beta}$ and characterization of elements of $M'_{\alpha,\beta}$ on $M'_{\alpha,\beta,\#}$.

Theorem 4.1 (Representation): Let $m\in N$. If $T\in M'_{\alpha,\beta,m,\#}$ then there exists $r\in N$ and essentially bounded functions f_k on $(0,\infty)$, $k=0,1,2,\ldots\ldots\ldots\ldots,r$, such that

$$T=\sum_{k=0}^{r}\Delta^k_{\alpha,\beta}\left[x^{2\beta-1}e^{(m+2)x}f_k\right]\text{ on }M_{\alpha,\beta}.$$

Proof: Let T be in $M'_{\alpha,\beta,m,\#}$. Then there exist $n\in N$ and $C>0$ such that

$$\left|\left\langle T\,,\,\phi\right\rangle\right|\le C\max_{0\le k\le n}\delta^k_{m,\alpha,\beta}(\phi),\quad \phi\in M_{\alpha,\beta,m,\#}. \tag{4.1}$$

Let $\phi\in M_{\alpha,\beta}$ and $k\in N$. Assume first that $m\in N$. For every $x\in(0,\infty)$ we have

$$x^{2\beta-1}\Delta^k_{\alpha,\beta}\phi(x)=\int_{\infty}^{x}D_t\left[t^{2\beta-1}\Delta^k_{\alpha,\beta,t}\phi(t)\right]dt.$$

Then

$$\left|e^{mx}x^{2\beta-1}\Delta^k_{\alpha,\beta}\phi(x)\right| \le e^{mx}\int_\infty^x \left|D_t\left[t^{2\beta-1}\Delta^k_{\alpha,\beta,t}\phi(t)\right]\right|dt$$

$$\le \int_0^\infty e^{mt}\left|D_t\left[e^{2\beta-1}\Delta^k_{\alpha,\beta,t}\phi(t)\right]\right|dt$$

$$= \int_0^1 e^{mt}t^{4\beta-2}\left|\int_0^t u^{2\alpha}\Delta^{k+1}_{\alpha,\beta}\phi(u)\,du\right|dt + \int_1^\infty e^{mt}t^{4\beta-2}\left|\int_t^\infty u^{2\alpha}\Delta^{k+1}_{\alpha,\beta}\phi(u)\,du\right|dt$$

$$\le \int_0^1 e^{mt}\int_0^t u^{2\beta-1}\left|\Delta^{k+1}_{\alpha,\beta}\phi(u)\right|du\,dt + \int_1^\infty e^{-t}t^{4\beta-2}\int_t^\infty e^{u(m+1)}u^{2\alpha}\left|\Delta^{k+1}_{\alpha,\beta}\phi(u)\right|du\,dt$$

$$\le C\left(\int_0^\infty u^{2\beta-1}\left|\Delta^{k+1}_{\alpha,\beta}\phi(u)\right|du + \int_0^\infty e^{(m+2)u}u^{2\beta-1}\left|\Delta^{k+1}_{\alpha,\beta}\phi(u)\right|du\right), x\in(0,\infty).$$

Thus

$$\delta^k_{m,\alpha,\beta}(\phi) \le C\int_0^\infty e^{(m+2)u}u^{2\beta-1}\left|\Delta^{k+1}_{\alpha,\beta}\phi(u)\right|du\,. \tag{4.2}$$

Let $m\in N, m\le -1$. For every $x\in(0,\infty)$ we can write

$$\left|e^{mx}x^{2\beta-1}\Delta^k_{\alpha,\beta}\phi(x)\right| \le \int_0^\infty \left|D_t\left(e^{mt}t^{2\beta-1}\Delta^k_{\alpha,\beta}\phi(t)\right)\right|dt$$

$$\le |m|\int_0^\infty e^{mt}t^{2\beta-1}\left|\Delta^k_{\alpha,\beta}\phi(t)\right|dt + \int_0^\infty e^{mt}\left|D_t\left[t^{2\beta-1}\Delta^k_{\alpha,\beta}\phi(t)\right]\right|dt$$

$$\le |m|\int_0^\infty e^{mt}t^{2\beta-1}\left|\Delta^k_{\alpha,\beta}\phi(t)\right|dt + \int_0^\infty e^{mt}t^{4\beta-2}\int_0^t u^{2\alpha}\left|\Delta^{k+1}_{\alpha,\beta}\phi(u)\right|du\,dt$$

$$\le C\left(\int_0^\infty e^{mt}t^{2\beta-1}\left|\Delta^k_{\alpha,\beta}\phi(t)\right|dt + \int_0^\infty e^{(m+1)t}t^{2\beta-1}\left|\Delta^{k+1}_{\alpha,\beta}\phi(t)\right|dt\right),\ x\in(0,\infty).$$

Then

$$\delta^k_{m,\alpha,\beta}(\phi) \le C\left(\int_0^\infty e^{mt}t^{2\beta-1}\left|\Delta^k_{\alpha,\beta}\phi(t)\right|dt + \int_0^\infty e^{(m+1)t}t^{2\beta-1}\left|\Delta^k_{\alpha,\beta}\phi(t)\right|dt\right). \tag{4.3}$$

Now by combining (4.1), (4.2) and (4.3) we conclude that

$$\left|\langle T\ ,\ \phi\rangle\right| \le C\max_{0\le k\le r}\int_0^\infty e^{(m+2)t}t^{2\beta-1}\left|\Delta^k_{\alpha,\beta}\phi(t)\right|dt.,\quad \phi\in M_{\alpha,\beta}, \tag{4.4}$$

For some $r\in N$. The required result can be deduced from (4.4) by using a standard procedure (see Betancor [1] and Treves [17]). This completes the proof.

We denote by $M_{\alpha,\beta,\#}$ the space $\bigcup_{m\in\mathbb{Z}} M_{\alpha,\beta,m,\#}$ endowed with the inductive topology. We now characterize the elements of $M'_{\alpha,\beta}$ that belong to $M'_{\alpha,\beta,\#}$.

Theorem 4.2(Characterization): Let $T\in M'_{\alpha,\beta}$ The following statements are equivalent:

(i) $T\in M'_{\alpha,\beta,\#}$.

(ii) $x^{2\beta-1}h'_{\alpha,\beta}(T)\in\theta_Q$.

(iii) For every $m\in N$ there exists $r\in N$ and continuous functions f_k on $(0,\infty)$, $k=0,1,2,\ldots\ldots\ldots\ldots\ldots,r$, such that

$$T=\sum_{k=0}^r \Delta^k_{\alpha,\beta}f_k \tag{4.5}$$

and $e^{mx} f_k$ is bounded on $(0,\infty)$ for every $k = 0,1,2,................,r$.

(iv) For every $m \in N$ there exists $r \in N$ and bounded continuous functions f_k on $(0,\infty)$, $k = 0,1,2,................,r$, such that (4.5) holds and $e^{mx} f_k \to 0$ as $x \to \infty$ for each $k = 0,1,2,................,r$.

(v) For every $m \in N$ there exists $r \in N$ and bounded continuous functions f_k on $(0,\infty)$, $k = 0,1,2,................,r$, such that (4.5) holds and $e^{mx} f_k$ is absolutely integrable on $(0,\infty)$ for each $k = 0,1,2,................,r$.

Proof: (i) $\Rightarrow$ (ii). Let $T \in M'_{\alpha,\beta,\#}$. Then $T \in M'_{\alpha,\beta,\#}$ for every $m \in N$. Let $m < -2$. By Theorem 4.1 there exists $r \in N$ and essentially bounded functions f_n on $(0,\infty)$, $n = 0,1,2,................,r$, such that

$$T = \sum_{n=0}^{r} \Delta_{\alpha,\beta}^{n} \left[e^{(m+2)x} x^{2\beta-1} f_n \right] \text{ on } M_{\alpha,\beta}.$$

Set

$$g_n(x) = e^{(m+2)x} x^{2\beta-1} f, \quad n = 0,1,2,................,r.$$

Now by Fubini's theorem we have,

$$\left\langle h'_{\alpha,\beta}(T), \Phi \right\rangle = \left\langle T, h_{\alpha,\beta}(\Phi) \right\rangle$$

$$= \sum_{n=0}^{r} (-1)^n \int_0^\infty g_n(x) h_{\alpha,\beta}\left[y^{2n} \Phi(y) \right](x)\, dx$$

Thus

$$= \sum_{n=0}^{r} (-1)^n \int_0^\infty y^{2\alpha+2n} \Phi(y) \int_0^\infty g_n(x) x^{2\alpha} (xy)^{-(\alpha-\beta)} J_{\alpha-\beta}(xy)\, dx dy, \ \Phi \in Q_{\alpha,\beta}.$$

$$y^{2\beta-1} h'_{\alpha,\beta}(T)(y) = \sum_{n=0}^{r} (-1)^n y^{2n} \int_0^\infty g_n(x) x^{2\alpha} (xy)^{-(\alpha-\beta)} J_{\alpha-\beta}(xy)\, dx. \tag{4.6}$$

For every $k \in N$ by choosing the representation (4.6) associated with $m = -k-3$ and using (1.6) we have

$$\left| y^{2\beta-1} h'_{\alpha,\beta}(T)(y) \right| \le C \sum_{n=0}^{r} |y|^{2n}, \qquad |\operatorname{Im} y| \le k.$$

Therefore $y^{2\beta-1} h'_{\alpha,\beta}(T)(y) \in \theta_Q$.

(ii) $\Rightarrow$ (iii). Let $m \in N$. We define $\theta = h'_{\alpha,\beta} T$. Then for every $k \in N$ thery exist $C_k > 0$ and $n_k \in N$ such that

$$\left| y^{2\beta-1} \theta(y) \right| \le C_k \left(1 + |y|^2 \right)^{n_k}, \qquad |\operatorname{Im} y| \le k.$$

Now set $v(y) = \left(M^2 + y^2 \right)^{-l} \theta(y)$, $|\operatorname{Im} y| \le m+1$. Here $m \in N$ is such that $M > m+1$ and $l \in N$ satisfies $l > n_{m+1} + 2\alpha + \beta$. Thus v is absolutely integrable on $(0,\infty)$ and $h'_{\alpha,\beta}(v) = h_{\alpha,\beta}(v)$. Thus by Zemanian [23 , Lemma 5.4-1], we can write

$$T = h'_{\alpha,\beta}(\theta) = h'_{\alpha,\beta}\left(\left(M^2 + y^2 \right)^l v(y) \right)$$

$$= \sum_{j=0}^{l} \binom{l}{j} (-1)^j M^{2(l-j)} \Delta_{\alpha,\beta}^{j} h_{\alpha,\beta}(v)$$

$$= \sum_{j=0}^{l} \Delta_{\alpha,\beta}^{j} f_j ,$$

where $$f_j = \binom{l}{j}(-1)^j M^{2(l-j)} h_{\alpha,\beta}(v)\ ,\ j = 0,1,2,.............,l.$$

For every $j = 0,1,2,.............,l$, it is clear that f_j is a continuous functions on $(0,\infty)$.

To prove (iii) we have to show that $e^{mx} h_{\alpha,\beta}(v)$ is a bounded function on $(0,\infty)$. As v is absolutely integrable on $(0,\infty)$, $e^{mx} h_{\alpha,\beta}(v)(x)$ is bounded on $(0,1)$. On the other side, by using a procedure similar to the one employed in Lemma 6.1 of Eijndhoven and Kerkhof [9], we have

$$h_{\alpha,\beta}(v)(x) = \frac{1}{2} x^{2\alpha} \int_{-\infty}^{\infty} \left(x(\xi + i\eta)\right)^{-(\alpha-\beta)} H_{\alpha,\beta}^{(1)}\left(x(\xi+i\eta)\right)(\xi+i\eta)^{2\alpha} v(\xi+i\eta)\, d\xi \tag{4.7}$$

for every $x > 1$ and $0 < \eta < M$. Now by (1.7) and (4.7) we have

$$\left|e^{mx} h_{\alpha,\beta}(v)(x)\right| \le C e^{-x} \int_{-\infty}^{\infty} \frac{\left|\xi + i(m+1)\right|^{2\alpha}}{\left(1 + \left|\xi + i(m+1)\right|^2\right)^{l-n_{m+1}}} d\xi \sup_{|\operatorname{Im} z| \le m+1} \left|\left(1+|z|^2\right)^{l-n_{m+1}} z^{2\beta-1} v(z)\right|, \quad x > 1.$$ As

$l > n_{m+1} + 2\alpha + \beta$, it follows that $e^{mx} h_{\alpha,\beta}(v)(x)$ is bounded on $(1,\infty)$.

(iii) $\Rightarrow$ (iv) and (iv) $\Rightarrow$ (v) are clear.

(v) $\Rightarrow$ (i) : Enough to show that $T \in M'_{\alpha,\beta,m,\#}$ for each $m \in N$. Now choose $r \in N$ such that $r > -m+1$. There exist $k \in N$ and bounded continuous functions f_i on $(0,\infty)$, $i = 0,1,2,................,r$, for which

$$T = \sum_{i=0}^{k} \Delta_{\alpha,\beta}^{i} f_i$$

and $e^{rx} f_i$ is absolutely integrable on $(0,\infty)$ for every $i = 0,1,2,................,k$.

Then

$$\langle T, \phi \rangle = \sum_{i=0}^{k} \int_0^{\infty} f_i(x) \Delta_{\alpha,\beta}^{i} \phi(x)\, dx, \qquad \phi \in M_{\alpha,\beta}.$$

Therefore

$$\left|\langle T, \phi \rangle\right| \le \sum_{i=0}^{k} \int_0^{\infty} \left|e^{rx} f_i(x)\right| x^{2\alpha} e^{(-m-r)x} e^{mx} x^{2\beta-1} \left|\Delta_{\alpha,\beta}^{i} \phi(x)\right| dx$$

$$\le C \sum_{i=0}^{k} \delta_{m,\alpha,\beta}^{i}(\phi), \qquad \phi \in M_{\alpha,\beta}$$

As $M_{\alpha,\beta}$ is a dense subset of $M_{\alpha,\beta,m,\#}$, it follows that T can be extended to an element of $M'_{\alpha,\beta,m,\#}$ defined by the same formula, and the proof is complete.

One can immediately note from Theorem 2.4 and Theorem 4.2 that $M_{\alpha,\beta}$ is a subspace of $M'_{\alpha,\beta,\#}$. Now we introduce the space $\xi_{\alpha,\beta}$ consisting of an smooth complex-valued functions $\phi(x),\ x \in (0,\infty)$, such that the limit

$$\lim_{x \to 0^+} \left(\frac{1}{x} D\right)^k \left(x^{2\beta-1} \phi(x)\right)$$

exists for every $k \in N$. This space is equipped with the topology generated by the family $\left\{B_{m,k}^{\alpha,\beta}\right\}_{m \in \mathbb{N} - \{0\}}$, where for each $m \in N - \{0\}$ and $k \in N$,

$$B_{m,k}^{\alpha,\beta}(\phi) = \sup_{x \in (0,\infty)} \left|\left(\frac{1}{x} D\right)^k \left(x^{2\beta-1} \phi(x)\right)\right|, \qquad \phi \in \xi_{\alpha,\beta}.$$

We now prove that elements of $M'_{\alpha,\beta,\#}$ define convolution operators in $M_{\alpha,\beta}$.

Theorem 4.3: Let $S \in M'_{\alpha,\beta,\#}$. Then the mapping $\phi \mapsto S \# \phi$ is continuous from $M_{\alpha,\beta}$ into itself.

Proof: By Lemma 3.4, for every $\phi \in M_{\alpha,\beta}$, we have

$$h'_{\alpha,\beta}(S \# \phi) = x^{2\beta-1} h'_{\alpha,\beta}(S) h_{\alpha,\beta}(\phi).$$

Hence by Theorem 2.4 and Theorem 4.2, $h'_{\alpha,\beta}(S \# \phi) \in Q_{\alpha,\beta}$ for each $\phi \in M_{\alpha,\beta}$. Moreover the mapping $\phi \mapsto h'_{\alpha,\beta}(S \# \phi)$ is continuous from $M_{\alpha,\beta}$ onto $Q_{\alpha,\beta}$. Finally as $h'_{\alpha,\beta}$ reduces to $h_{\alpha,\beta}$ on $Q_{\alpha,\beta}$ we can infer from Theorem 2.4 that $\phi \mapsto S \# \phi$ defines a continuous mapping from $M_{\alpha,\beta}$ into itself. This completes the proof.

From Theorem 4.3 we have the following definition:
For $T \in M'_{\alpha,\beta}$ and $S \in M'_{\alpha,\beta,\#}$, the Hankel type convolution $T \# S$ is the element of $M'_{\alpha,\beta}$ defined by

$$\langle T \# S , \phi \rangle = \langle T , S \# \phi \rangle, \qquad \phi \in M_{\alpha,\beta}.$$

Note that by Lemma 3.4 the definition of Hankel type convolution on $M'_{\alpha,\beta} \times M'_{\alpha,\beta,\#}$ is a generalization of the above definition of Hankel type convolution on $M'_{\alpha,\beta} \times M_{\alpha,\beta}$.

5. Interchange formula and algebraic properties of the generalized Hankel type convolution:

In this section first we establish the interchange formula.

Theorem 5.1(Interchange formula): Let $T \in M'_{\alpha,\beta}$ and $S \in M'_{\alpha,\beta,\#}$. Then

$$h'_{\alpha,\beta}(T \# S) = x^{2\beta-1} h'_{\alpha,\beta}(T) h'_{\alpha,\beta}(S).$$

Proof: According to Lemma 3.4 and Theorem 4.3 we can write

$$\begin{aligned}\langle h'_{\alpha,\beta}(T \# S) , \Phi \rangle &= \langle T \# S , h_{\alpha,\beta}(\Phi) \rangle = \langle T , S \# h_{\alpha,\beta}(\Phi) \rangle \\ &= \langle h'_{\alpha,\beta}(T) , h_{\alpha,\beta}(S \# h_{\alpha,\beta}(\Phi)) \rangle \\ &= \langle h'_{\alpha,\beta}(T) , x^{2\beta-1} h'_{\alpha,\beta}(S) \Phi \rangle \\ &= \langle x^{2\beta-1} h'_{\alpha,\beta}(T) h'_{\alpha,\beta}(S) , \Phi \rangle, \qquad \Phi \in Q_{\alpha,\beta}.\end{aligned}$$

This completes the proof.

Corollary 5.2: If $R, S \in M'_{\alpha,\beta,\#}$ then $R \# S \in M'_{\alpha,\beta,\#}$.

Proof: It is immediate consequence of Theorem 4.2 and Theorem 5.1.
Now we show some algebraic properties of the generalized Hankel type convolution.

Theorem 5.3: Let $T \in M'_{\alpha,\beta}$ and $R, S \in M'_{\alpha,\beta,\#}$. Then

(i) $(T \# R) \# S = T \# (R \# S)$ (Associativity)

(ii) $R \# S = S \# R$ (Commutativity)

(iii) $\Delta_{\alpha,\beta}(T \# S) = (\Delta_{\alpha,\beta} T) \# S = T \# (\Delta_{\alpha,\beta} S)$.

(iv) If $\delta_{\alpha,\beta}$ denotes the functional on $M_{\alpha,\beta}$ defined by

$$\langle \delta_{\alpha,\beta}, \phi \rangle = 2^{\alpha-\beta} \overline{\lfloor (3\alpha+\beta)} \lim_{x \to 0^+} x^{2\beta-1} \phi(x), \qquad \phi \in M_{\alpha,\beta},$$

then $\delta_{\alpha,\beta} \in M'_{\alpha,\beta,\#}$ and $S \# \delta_{\alpha,\beta} = S$.

Proof: (i), (ii), (iii) follow immediately from Theorem 5.1.

To prove (iv), it is enough to note that

$$y^{2\beta-1}h'_{\alpha,\beta}\left(\delta_{\alpha,\beta}\right)=1.$$

This completes the proof.

REFERENCES:

1. Betancor J. J., Characterization of Hankel transformable generalized functions, Internat. J. Math. Math. Sci 14(1991), 269-274.
2. Betancor J. J. and Marrero I., Multipliers of Hankel transformable generalized functions, Comment. Math. Univ. Carolin. 33(1992), 389-401.
3. Betancor J. J. and Marrero I., The Hankel convolution and the Zemanian spaces β_μ and $\beta_{\mu'}$, Math. Nachr. 160(1993), 277-298.
4. Betancor J. J. and Marrero I., Structure and Convergence in certain spaces of distributions and the generalized Hankel convolution, Math. Japon. 38(1993), 1141-1155.
5. Betancor J. J. and Marrero I., Some properties of Hankel convolution operators, Canad. Math. Bull. 36(1993), 398-406.
6. Betancor J. J. and Marrero I., On the topology of Hankel multipliers and of Hankel convolution operators, Rend. Circ. Mat. Palermo (2) 44(1995), 469-478.
7. Betancor J. J. and Marrero I., Algebraic Characterization of convolution and multiplication operators on Hankel transformable functions and distributions spaces, Rocky Mountain J. Math. 25(1995), 1189-1204.
8. Cholewinski F. M., A Hankel convolution complex inversion theory, Mem. Amer. Math.Soc. 58(1965).
9. Eijndhoven S. J. L. Van and Kerkhof M. J., The Hankel transformation and spaces of the type W, Reports on Appl. and Num. Analysis, 10 Dept. of Maths. and Comp. Sci., Eijndhoven University of Technology, 1988.
10. Haimo D. T., Integral equations associated with Hankel convolutions, Trans. Amer. Math. Soc. 116(1965), 330-375.
11. Hasumi M., Note on the n-dimensional tempered ultra-distributions, Tohoku Math. J. 13(1961), 94-104.
12. Hirschman Jr.I.I., Variation diminishing Hankel transforms, J. Anal. Math. 8(1960/61), 307-336.
13. Koh E. L. and Li C. K., The Hankel transformation on M'_μ and it's representation, Proc. Amer. Math. Soc. 122(1994), 1085-1094.
14. Koh E. L. and Zemanian A. H., The Complex Hankel and I- transformations of generalized functions, SIAM J. Appl. Math. 16(1968), 945-957.
15. Marrero I. and Betancor J. J., Hankel convolution of generalized functions, Rend. Mat. 15(1995), 351-380.
16. Pietsch A., Nuclear Locally Convex Spaces, Springer, Berlin, 1972.
17. Treves F., Topological Vector Spaces, Distributions and Kernels, Academic Press, New York, 1967.
18. Waphare B. B., Distributional Hankel type transformation and convolution, Global J. Pure and Applied Mathematics, Vol. 5, No. 2(2009), 163-179.
19. Watson G. N., A Treaties on the theory of Bessel functions, Cambridge University Press, Cambridge, 1956.
20. Yoshinaga K., On Spaces of distributions of exponential growth, Bull. Kyushu Inst. Tech. Math. Natur. Sci. 6(1960), 1-16.
21. Zemanian A. H., A distributional Hankel transform, SIAM. J. Appl. Math. 14(1966), 561-576.
22. Zemanian A. H., The Hankel transformation of certain distributions of rapid growth, ibid. 14(1966), 678-690.
23. Zemanian A. H. Generalized Integral Transformations, Interscience Publ., New York, 1968.
24. Zielezny Z., On the spaces of convolution operators in κ'_1, Studia Math. 31(1968), 111-124.