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WEIGHTED COMPOSITION OF k - QUASI - PARANORMAL OPERATORS

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ABSTRACT

An operator $T \in B(H)$ is said to be k - quasi - paranormal operator if $\|T^{k+1}x\|^2 \le \|T^{k+2}x\| \, \|T^k x\|$ for every $x \in H$, k is a natural number. In this paper, k - quasi - paranormal composition operators on L^2 space and Hardy space is characterized.

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1. INTRODUCTION AND PRELIMINARIES

Let H be an infinite dimensional complex Hilbert space and $B(H)$ denote the algebra of all bounded linear operators acting on H. Every operator T can be decomposed into $T = U|T|$ with a partial isometry U, where $|T| = \sqrt{T^*T}$. In this paper, $T = U|T|$ denotes the polar decomposition satisfying the kernel condition $N(U) = N(|T|)$. An operator T is said to be positive (denoted $T \ge 0$) if $(Tx, x) \ge 0$ for all $x \in H$. The operator T is said to be a p - hyponormal operator if and only if $(T^*T)^p \ge (TT^*)^p$ for a positive number p.

In [23], the class of log - hyponormal operators is defined as follows: T is called log - hyponormal if it is invertible and satisfies $\log\,(T^*T)^p \ge \log\,(TT^*)^p$. Class of p - hyponormal operators and class of log hyponormal operators were defined as extension class of hyponormal operators, i.e., $T^*T \ge TT^*$. It is well known that every p - hyponormal operator is a q - hyponormal operator for $p \ge q > 0$, by the Löwner - Heinz theorem "$A \ge B \ge 0$ ensures $A^\alpha \ge B^\alpha$ for any $\alpha \in [0,1]$", and every invertible p - hyponormal operator is a log - hyponormal operator since $\log(\cdot)$ is an operator monotone function. An operator T is called paranormal if $\|Tx\|^2 \le \|T^2x\| \, \|x\|$ for all $x \in H$. It is also well known that there exists a hyponormal operator T such that T^2 is not hyponormal (see [14]).

Furuta, Ito and Yamazaki [9] introduced class $A(k)$ and absolute - k - paranormal operators for $k > 0$ as generalizations of class A and paranormal operators, respectively. An operator T belongs to class $A(k)$ if

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$\left(T^*|T|^{2k}T\right)^{\frac{1}{k+1}} \geq |T|^2$ and T is said to be absolute - k - paranormal operator if $\left\||T|^k Tx\right\| \geq \|Tx\|^{k+1}$ for every unit vector x .An operator T is called quasi class A if $T^*|T|^2 T \geq T^*|T^2|T$. Fuji, Izumino and Nakamoto [8] introduced p - paranormal operators for $p > 0$ as a generalization of paranormal operators.

Fujii, Jung, S. H. Lee, M. Y. Lee and Nakamoto [11] introduced class $A(p,r)$ as a further generalization of class $A(k)$. An operator $T \in$ class $A(p,r)$ for $p > 0$ and $r > 0$ if $\left(|T^*|^r\ |T|^{2p}\ |T^*|^r\right)^{\frac{r}{p+r}} \geq |T^*|^{2r}$ and class $AI(p,r)$ is class of all invertible operators which belong to class $A(p,r)$. Yamazaki and Yanagida [25] introduced absolute - (p,r) - paranormal operator. It is a further generalization of the classes of both absolute - k - paranormal operators and p - paranormal operators as a parallel concept of class $A(p,r)$. An operator T is said to be paranormal operator if $\|T^2x\| \geq \|Tx\|^2$ for every unit vector x . Paranormal operators have been studied by many authors [3], [10] and [16].

In [3], Ando showed that T is paranormal if and only if

$$T^{*2}T^2 - 2\lambda T^*T + \lambda^2 \geq 0 \quad \text{for all } \lambda > 0 .$$

In order to extend the class of paranormal operators and class of quasi - class A operators, Mecheri [18] introduced a new class of operators called k - quasi - paranormal operators. An operator T is called k - quasi - paranormal if $\|T^{k+1}x\|^2 \leq \|T^{k+2}x\|\ \|T^kx\|$ for all $x \in H$, where k is a natural number. A 1 - quasi - paranormal operator is quasi paranormal. The following implication gives us relations among the classes of operators.

Hyponormal $\Rightarrow$ p - hyponormal $\Rightarrow$ class A $\Rightarrow$ paranormal $\Rightarrow$ quasi - paranormal $\Rightarrow$ k - quasi – paranormal.

Hyponormal $\Rightarrow$ class A $\Rightarrow$ quasi - class A $\Rightarrow$ quasi – paranormal $\Rightarrow$ k - quasi - paranormal

Let (X, Σ, λ) be a sigma - finite measure space and let $T : X \to X$ be a non singular measurable transformation. A bounded linear operator $C\,f = f \circ T$ on $L^2(X, \Sigma, \lambda)$ is said to be a composition operator induced by T , when the measure λT^{-1} is absolutely continuous with respect to the measure λ and the Radon - Nikodym derivative $d\lambda T^{-1}/d\lambda = f_0$ is essentially bounded. The Radon - Nikodym derivative of the measure $\lambda(T^k)^{-1}$ with respect to λ is denoted by $f_o^{(k)}$, where T^k is obtained by composing T - k times.

2. k - QUASI - PARANORMAL COMPOSITION OPERATORS

In this section, we characterize k - quasi - paranormal composition operator.

Every essentially bounded complex valued measurable function f_o induces the bounded operator M_{f_0} on $L^2(\lambda)$, which is defined by $M_{f_o} f = f_0 f$ for every $f \in L^2(\lambda)$. Further $C^*C = M_{f_o}$ and $C^{*2}C^2 = M_{f_0^2}$ [21].

The following Lemma due to Harrington and Whitely [15] is well known.

Lemma 2.1 [15]: If P denote the projection of L^2 on $\overline{R(C)}$, then $C^*Cf = f_o f$ and $CC^* = (f_o \circ T)Pf$ for all $f \in L^2$ where P denote the projection of L^2 onto $\overline{R(C)}$ and $\overline{R(C)}$ = $\left\{ f \in L^2 :\ f \ \textit{is}\ T^{-1}\Sigma\ \textit{measurable} \right\}$.

The following theorem characterize k - quasi - paranormal composition operators on L^2 space.

Proposition 2.2 [18]: An operator $T \in B(H)$ is k - quasi - paranormal if and only if

$T^{*^{k+2}}T^{^{k+2}} - 2\lambda T^{*^{k+1}}T^{k+1} + \lambda^2 T^{*^{k}}T^{k} \geq 0$ for every $\lambda > 0$.

Theorem 2.3: Let $C \in B(L^2(\lambda))$. Then C is of k - quasi - paranormal operator if and only if $f_0^{(k+2)} - 2\lambda f_0^{(k+1)} + \lambda^2 f_0^{(k)} \geq 0$, a.e., where P is the projection of L^2 on $\overline{R(C)}$.

Proof: Let $C \in B(L^2(\lambda))$ is of k - quasi - paranormal operator if and only if

$$C^{*^{k+2}}C^{^{k+2}} - 2\lambda C^{*^{k+1}}C^{k+1} + \lambda^2 C^{*^{k}}C^{k} \geq 0.$$

Thus,

$$\left\langle \left(C^{*^{k+2}}C^{^{k+2}} - 2\lambda C^{*^{k+1}}C^{k+1} + \lambda^2 C^{*^{k}}C^{k} \right)\chi_E,\, \chi_E \right\rangle \geq 0$$

for every characteristic function χ_E of E in $\sum$ such that $\lambda(E) < \infty$.

Since $C^{*2}C^2 = M_{f_0^{(2)}}$ [21] and $C^*C = M_{f_0}$ [5], we have,

$$\left\langle \left(M_{f_0^{(k+2)}} - 2\lambda M_{f_0^{(k+1)}} + \lambda^2 M_{f_0^{(k)}} \right)\chi_E,\, \chi_E \right\rangle \geq 0$$

i.e., $\int_E \left(f_0^{(k+2)} - 2\lambda f_0^{(k+1)} + \lambda^2 f_0^{(k)} \right) d\lambda \geq 0$

for every E in $\sum$.

Hence C is k - quasi - paranormal operator if and only if $f_0^{(k+2)} - 2\lambda f_0^{(k+1)} + \lambda^2 f_0^{(k)} \geq 0$ a.e.

Corollary 2.4: Let $C \in B(L^2(\lambda))$ with dense range. Then $C \in$ k - quasi - paranormal operator if and only if $f_0^{(k+2)} - 2\lambda f_0^{(k+1)} + \lambda^2 f_0^{(k)} \geq 0$ a.e.

Example 2.5: Let $X = N$ the set of all natural numbers and λ be the counting measure on it. Define $T : N \to N$ by $T(1) = 1,\ T(n+m+1) = n,\ m = 0,1,2,3,...$ and $n \in N$. Since $f_0^{(k+2)} - 2\lambda f_0^{(k+1)} + \lambda^2 f_0^{(k)} \geq 0$, C is of k - quasi - paranormal composition operator.

Theorem 2.6: Let $C \in B(L^2(\lambda))$, Then $C^* \in$ k - quasi - paranormal operator if and only if $\left[\left(f_0 \circ T \right)^{(k+2)} P_1 \right] - 2\lambda \left[\left(f_0 \circ T \right)^{(k+1)} P_1 \right] + \lambda^2 \left[\left(f_0 \circ T \right)^{(k)} P_1 \right] \geq 0$ a.e, where P_1 and P_2 is the projection of L^2 onto $\overline{R(C)}$ and $\overline{R(C^2)}$ respectively.

Proof: Let C^* is of k - quasi - paranormal operator if and only if

$$C^{^{k+2}}C^{*^{k+2}} - 2\lambda C^{^{k+1}}C^{*k+1} + \lambda^2 C^{^{k}}C^{*k} \geq 0.$$

i.e., $\left\langle \left(C^{^{k+2}}C^{*^{k+2}} - 2\lambda C^{^{k+1}}C^{*k+1} + \lambda^2 C^{^{k}}C^{*k} \right) f,\, f \right\rangle \geq 0$ for every $f \in L^2$.

We have $\left\langle CC^* f,\, f \right\rangle = \left\langle (f_0 \circ T)P_1 f,\, f \right\rangle$ and $\left\langle CC^* f,\, f \right\rangle = \left\langle (f_0 \circ T)P_2 f,\, f \right\rangle$ where P_1 and P_2 are the projections of L^2 onto $\overline{R(C)}$ and $\overline{R(C^2)}$ respectively. Thus C^* is of k - quasi - paranormal operator if and only if

$\left\langle\left(\left[\left(f_0\circ T\right)^{(k+2)}P_1\right]\right)f,f\right\rangle-\left\langle\left(2\lambda\left[\left(f_0\circ T\right)^{(k+1)}P_1\right]\right)f,f\right\rangle+\left\langle\left(\lambda^2\left[\left(f_0\circ T\right)^{(k)}P_1\right]\right)f,f\right\rangle\geq 0$ for every $f\in L^2$.

i.e., $\left[\left(f_0\circ T\right)^{(k+2)}P_1\right]-2\lambda\left[\left(f_0\circ T\right)^{(k+1)}P_1\right]+\lambda^2\left[\left(f_0\circ T\right)^{(k)}P_1\right]\geq 0$ a.e.

Corollary 2.7: Let $C^*\in B(L^2(\lambda))$ with dense range. Then $C^*\in$ k - quasi - paranormal operator if and only if $\left[\left(f_0\circ T\right)^{(k+2)}\right]-2\lambda\left[\left(f_0\circ T\right)^{(k+1)}\right]+\lambda^2\left[\left(f_0\circ T\right)^{(k)}\right]\geq 0$ a.e,

3. WEIGHTED k - QUASI - PARANORMAL COMPOSITION OPERATORS

A weighted composition operator(w.c.o) induced by T is a linear transformation acting on the set of complex valued $\sum$ measurable functions f, defined as $Wf=w(f\circ T)$, w is a complex valued $\sum$ measurable function. when $w=1$, we say that W is a composition operator. Let w_k denote $w(w\circ T)(w\circ T^2)....(w\circ T^{k-1})$ so that $W^k f=w_k(f\circ T)^k$ [19]. To examine the weighted composition operators effectively Alan Lambert [17] associated conditional expectation operator E with T as $E\left(\bullet/T^{-1}\sum\right)=E\left(\bullet\right)$. $E(f)$ is defined for each non - negative measurable function $f\in L^p(1\leq p)$ and is uniquely determined by the conditions

(i) $E(f)$ is $T^{-1}\sum$ measurable.
(ii) If B is any $T^{-1}\sum$ measurable set for which $\int_B f\,d\lambda$ converges, we have $\int_B f\,d\lambda=\int_B E(f)\,d\lambda$.
The projection operator E on L^p is identity if and only if $T^{-1}\sum=\sum$. For more information [[4], [7], [12]].

Proposition 3.1 [4]: For $w\geq 0$,

(i) $W^*Wf=f_0\left[E(w^2)\right]\circ T^{-1}f$.
(ii) $WW^*f=w\left(f_0\circ T\right)E(wf)$.

Now we characterize weighted k - quasi - paranormal composition operators as follows.

Theorem 3.2: W is k - quasi - paranormal if and only if

$$\left[f_0[E(w^2)]\circ T^{-1}\right]^{k+2}-2\lambda\left[f_0[E(w^2)]\circ T^{-1}\right]^{k+1}+\lambda^2\left[f_0[E(w^2)]\circ T^{-1}\right]^{k}\geq 0 \text{ a.e.}$$

Proof: Since W is of k - quasi - paranormal, $W^{*^{k+2}}W^{k+2}-2\lambda W^{*^{k+1}}W^{k+1}+\lambda^2W^{*^k}W^k\geq 0$ and hence, $\left\langle\left(W^{*^{k+2}}W^{k+2}-2\lambda W^{*^{k+1}}W^{k+1}+\lambda^2W^{*^k}W^k\right)f,f\right\rangle\geq 0$ for all $f\in L^2$

Since $W^kf=w_k(f\circ T)^k$ and $W^{*k}f=f_0^{(k)}E(w_kf)\circ T^{-k}$, $W^{*k}W^k=f_0^{(k)}E(w_2^k)\circ T^{-k}f$ and we have $W^*Wf=f_0\left[E(w^2)\right]\circ T^{-1}f$ for $w\geq 0$ [4], and hence

$$\int_E\left\{\left[f_0[E(w^2)]\circ T^{-1}\right]^{k+2}-2\lambda\left[f_0[E(w^2)]\circ T^{-1}\right]^{k+1}+\lambda^2\left[f_0[E(w^2)]\circ T^{-1}\right]^{k}\right\}d\lambda\geq 0 \text{ for every } E\in\sum.$$

And so $\left[f_0[E(w^2)]\circ T^{-1}\right]^{k+2}-2\lambda\left[f_0[E(w^2)]\circ T^{-1}\right]^{k+1}+\lambda^2\left[f_0[E(w^2)]\circ T^{-1}\right]^{k}\geq 0$ a.e.

Corollary 3.3: Let $T^{-1}\sum = \sum$. Then W is of k - quasi - paranormal if and only if

$$\left[f_0[w^2]\circ T^{-1}\right]^{k+2} - 2\lambda\left[f_0[w^2]\circ T^{-1}\right]^{k+1} + \lambda^2\left[f_0[w^2]\circ T^{-1}\right]^{k} \geq 0 \text{ a.e.}$$

The Aluthge transform of T is the operator $\tilde{T}$ given by $\tilde{T} = |T|^{\frac{1}{2}} U |T|^{\frac{1}{2}}$ was introduced in [1] by Aluthge. More generally we may form the family of operators $\{T_s : 0 < s \leq 1\}$ where $T_s = |T|^s U |T|^{1-s}$ [2]. For a composition operator C, the polar decomposition is given by $C = U|C|$ where $|C| f = \sqrt{f_0} f$ and $Uf = \dfrac{1}{\sqrt{f_0 \circ T}} f \circ T$. In [5] Lambert has given more general Aluthge transformation for composition operators as $C_s = |C|^s U |C|^{1-s}$ and $C_s f = \left(\dfrac{f_0}{f_0\circ T}\right)^{\frac{s}{2}} f\circ T$. That is C_s is weighted composition operator with weight $\pi = \left(\dfrac{f_0}{f_0\circ T}\right)^{\frac{s}{2}}$ where $0 < s < 1$. Since C_s is a weighted composition operator it is easy to show that $|C_s| f = \sqrt{f_0\left[E(\pi)^2\circ T^{-1}\right] f}$ and $|C_s^*| f = v\ E[vf]$ where $v = \dfrac{\pi\sqrt{f_0\circ T}}{\left[E(\pi\sqrt{f_0\circ T})^2\right]^{\frac{1}{4}}}$. Also we have,

$C_s^k f = \pi_k (f\circ T^k)$,
$C_s^{*k} f = f_0^{(k)} E(\pi_k f)\circ T^{-k}$,
$C_s^{*k} C_s^k f = f_0^{(k)} E(\pi_k^2)\circ T^{-k} f$.

Corollary 3.4: If $T^{-1}\sum = \sum$, $C_s \in B(L^2(\lambda))$. Then C_s is of k - quasi - paranormal if and only if $\left[f_0[\pi^2]\circ T^{-1}\right]^{k+2} - 2\lambda\left[f_0[\pi^2]\circ T^{-1}\right]^{k+1} + \lambda^2\left[f_0[\pi^2]\circ T^{-1}\right]^{k} \geq 0$ a.e.

Proof: Since C_s is weighted composition operator with weight $\pi = \left(\dfrac{f_0}{f_0\circ T}\right)^{\frac{s}{2}}$, it follows that C_s is of k - quasi - paranormal if and only if

$$\left[f_0[\pi^2]\circ T^{-1}\right]^{k+2} - 2\lambda\left[f_0[\pi^2]\circ T^{-1}\right]^{k+1} + \lambda^2\left[f_0[\pi^2]\circ T^{-1}\right]^{k} \geq 0$$

The second Aluthge Transformation of T described by B. P. Duggal [6] is given by $\tilde{T} = |\hat{T}|^{\frac{1}{2}} V |\hat{T}|^{\frac{1}{2}}$, where $\hat{T} = V|\hat{T}|$ is the polar decomposition of $\hat{T}$.

Senthilkumar and Prasad [22] studied that the operator $\tilde{C} = |C_s|^{\frac{1}{2}} V |C_s|^{\frac{1}{2}}$, where $C_s = V|C_s|$ is the polar decomposition of the generalized Aluthge transformation $C_s : 0 < s < 1$ is a weighted composition operator with weight $w' = J^{\frac{1}{4}}\pi\left(\dfrac{\chi_{\sup J}}{J^{\frac{1}{4}}}\circ T\right)$ where $J = f_0 E(\pi^2)\circ T^{-1}$.

Corollary 3.5: If $T^{-1}\sum = \sum$, $\tilde{C} \in B(L^2(\lambda))$. Then $\tilde{C}$ is of k - quasi - paranormal if and only if $\left[f_0[w'^2]\circ T^{-1}\right]^{k+2} - 2\lambda\left[f_0[w'^2]\circ T^{-1}\right]^{k+1} + \lambda^2\left[f_0[w'^2]\circ T^{-1}\right]^{k} \geq 0$ a.e.

Proof: Since $\tilde{C}$ is weighted composition operator with weight $w' = J^{\frac{1}{4}}\pi\left(\frac{\chi_{\sup J}}{J^{\frac{1}{4}}} \circ T\right)$, it follows that $\tilde{C}$ is of k - quasi - paranormal if and only if

$$\left[f_0[w'^2]\circ T^{-1}\right]^{k+2} - 2\lambda\left[f_0[w'^2]\circ T^{-1}\right]^{k+1} + \lambda^2\left[f_0[w'^2]\circ T^{-1}\right]^{k} \geq 0 \text{ a.e.}$$

4. k - QUASI - PARANORMAL COMPOSITION OPERATORS ON WEIGHTED HARDY SPACES

The set $H^2(\gamma)$ of formal complex power series $f(z) = \sum_{n=0}^{\infty} a_n Z^n$ such that $\|f\|_{\gamma}^2 = \sum_{n=0}^{\infty} |a_n|^2 \gamma_n^2 < \infty$ is the general Hardy space of functions analytic in the unit disc with the inner product

$$\langle f, g\rangle_{\gamma} = \sum_{n=0}^{\infty} a_n \overline{b_n} \gamma_n^2$$

for f as above and $g(z) = \sum_{n=0}^{\infty} b_n Z^n$ and $\gamma = \{\gamma_n\}_{n=0}^{\infty}$ be a sequence of positive numbers with $\gamma_0 = 1$ and $\frac{\gamma_{n+1}}{\gamma_n} \to 1$ as $n \to \infty$.

If ϕ is an analytic function mapping the unit disc D into itself, we define the composition operator C_ϕ on the spaces $H^2(\gamma)$ by

$$C_\phi f = f_0 \phi$$

Though the operator C_ϕ are defined everywhere on the classical Hardy space H^2 (the case when $\gamma_n = 1$ for all n), they are not necessarily defined on all of $H^2(\gamma)$. The composition operator C_ϕ is defined on $H^2(\gamma)$ only when the function ϕ is analytic on some open set containing the closed unit disc having supremum norm strictly smaller than one [26].

The properties of composition operator on the general Hardy spaces $H^2(\gamma)$ are studied in [13], [20] and [24]. In this section, we investigate the properties of k - quasi - paranormal composition operators on general Hardy spaces $H^2(\gamma)$.

For a sequence γ as above and a point w in D, let

$$k_w\gamma(z) = \sum_{n=0}^{\infty} \frac{1}{\gamma_2^n} (\overline{w_z})^n$$

Then the function $k_w\gamma$ is a point evaluation for $H^2(\gamma)$ i.e., for f in $H^2(\gamma)$,

$$(f, k_w\gamma)_\gamma = f(w)$$

Then $k_0\gamma = 1$ and $C_\phi^* k_w \gamma = k_{\phi(w)}\gamma$.

Theorem 4.1: If C_ϕ is k - quasi - paranormal on $H^2(\gamma)$ then $\lambda = 1$.

Proof: Let C_ϕ be k - quasi - paranormal on $H^2(\gamma)$. By the definition of k - quasi - paranormal,

$$C_\phi^{*k+2} C_\phi^{k+2} - 2\lambda C_\phi^{*k+1} C_\phi^{k+1} + \lambda^2 C_\phi^{*k} C_\phi^{k} \geq 0$$

$$\left\langle \left(C_\phi^{*k+2} C_\phi^{k+2} - 2\lambda C_\phi^{*k+1} C_\phi^{k+1} + \lambda^2 C_\phi^{*k} C_\phi^{k} \right) f, f \right\rangle \geq 0 \quad \forall \quad f \in H^2(\gamma)$$

$$\left\langle \left(C_\phi^{*k+2} C_\phi^{k+2} \right) f, f \right\rangle - 2\lambda \left\langle \left(C_\phi^{*k+1} C_\phi^{k+1} \right) f, f \right\rangle + \lambda^2 \left\langle \left(C_\phi^{*k} C_\phi^{k} \right) f, f \right\rangle \geq 0$$

$$\left\langle \left(C_\phi^{k+2} f, C_\phi^{k+2} f \right) \right\rangle - 2\lambda \left\langle \left(C_\phi^{k+1} f, C_\phi^{k+1} f \right) \right\rangle + \lambda^2 \left\langle \left(C_\phi^{k} f, C_\phi^{k} f \right) \right\rangle \geq 0$$

$$\left\| C_\phi^{k+2} f \right\|^2 - 2\lambda \left\| C_\phi^{k+1} f \right\|^2 + \lambda^2 \left\| C_\phi^{k} f \right\|^2 \geq 0$$

$$\left\| C_\phi^{k+1} (C_\phi f) \right\|^2 - 2\lambda \left\| C_\phi^{k} (C_\phi f) \right\|^2 + \lambda^2 \left\| C_\phi^{k-1} (C_\phi f) \right\|^2 \geq 0$$

Let $f = k_0 \gamma$, we have

$$\left\| C_\phi^{k+1} (C_\phi k_0 \gamma) \right\|_\gamma^2 - 2\lambda \left\| C_\phi^{k} (C_\phi k_0 \gamma) \right\|_\gamma^2 + \lambda^2 \left\| C_\phi^{k-1} (C_\phi k_0 \gamma) \right\|_\gamma^2 \geq 0$$

$$\left\| C_\phi^{k+1} k_0 \gamma \right\|_\gamma^2 - 2\lambda \left\| C_\phi^{k} k_0 \gamma \right\|_\gamma^2 + \lambda^2 \left\| C_\phi^{k-1} k_0 \gamma \right\|_\gamma^2 \geq 0$$

Repeating the steps for k times we get

$$\left\| k_0 \gamma \right\|_\gamma^2 - 2\lambda \left\| k_0 \gamma \right\|_\gamma^2 + \lambda^2 \left\| k_0 \gamma \right\|_\gamma^2 \geq 0$$

$$1 - 2\lambda + \lambda^2 \geq 0 \quad \text{since } k_0 \gamma = 1$$

By elementary properties of real quadratic form we get $\lambda = 1$.

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