International Journal of Mathematical Archive-3(2), 2012, Page: 722-728
IJMA Available online through www.ijma.info ISSN 2229 – 5046

THE EDGE –TO–VERTEX MONOPHONIC NUMBER OF A GRAPH

J. John
Department of Mathematics, Government college of Engineering, Tirunelveli, 627007, India
E-mail: johnramesh1971@yahoo.co.in

K. Uma Samundesvari*
Department of Mathematics, Noorul Islam centre for Higher Education, Kumaracoil, 629175, India
E-mail: kus_krishna@yahoo.co.in

(Received on: 10-02-12; Accepted on: 28-02-12)

ABSTRACT

For a connected graph $G = (V,E)$, a monophonic set $S \subseteq E$ is called an edge – to – vertex monophonic set if every vertex of G lies on a monophonic path between two vertices in V(S). The edge -to -vertex monophonic number $m_{ev}(G)$ of G is the minimum cardinality of its edge – to – vertex monophonic sets. The edge – to – vertex monophonic number of certain classes of graphs is determined and some of its general properties are studied. Connected graphs of size $q \geq 3$ with edge – to – vertex monophonic number q and q -1 are characterized. It is shown that for positive integers r_m, d_m and $l \geq 2$ with $r_m < d_m \leq 2 r_m$, there exists a connected graph G with $rad_m G = r_m$, $diam_m G = d_m$ and $m_{ev}(G) = l$ and also shown that for every integers a, b and c with $2 \leq a \leq b \leq c$, and $c \geq 2b - a + 1$, there exists a connected graph G such that $m_{ev}(G) = a$, $g_{ev}(G) = b$ and $ß'(G) = c$, where $g_{ev}(G)$ is edge – to – vertex geodetic number and $ß'(G)$ is edge covering number of G.

Keywords: *Monophonic path, Monophonic number, Edge – to – vertex monophonic number, Geodesic, Edge – to – vertex geodetic number.*

AMS subject classification: *05C38.*

---

1. INTRODUCTION

By a *graph* $G = (V, E)$, we mean a finite undirected connected graph without loops or multiple edges. The *order* and *size* of G are denoted by p and q respectively. We consider connected graphs with at least three vertices. For basic definitions and terminologies we refer to [1].An *edge covering* of G is a subset $K \subseteq E(G)$ such that each vertex of G is end of some edge in K. The number of edges in a minimum edge covering of G, denoted by $ß'(G)$ is the edge covering number of G. For two vertices u and v in a connected graph G, the *distance* $d(u, v)$ is the length of a shortest $u - v$ path in G. An $u - v$ path of length $d(u, v)$ is called an $u - v$ *geodesic*. For a vertex v of G, the *eccentricity* $e(v)$ is the distance between v and a vertex farthest from v. The minimum eccentricity among the vertices is the *radius*, *rad G* and the maximum eccentricity is the *diameter*, *diam G* of G. For subsets A and B of $V(G)$, the *distance* $d(A, B)$ is defined as $d(A, B) = min\{d(x, y) : x \in A, y \in B\}$. An $u - v$ path of length $d(A, B)$ is called an $A - B$ *geodesic* joining the sets A, B, where $u \in A$ and $v \in B$. A vertex x is said to *lie* on an $A - B$ geodesic if x is a vertex of an $A - B$ geodesic. For $A = (u, v)$ and $B = (z, w)$ with uv and zw edges, we write an $A - B$ geodesic as $uv - zw$ geodesic and $d(A, B)$ as $d(uv, zw)$. A set $S \subseteq E$ is called an *edge-to-vertex geodetic set* if every vertex of G is either incident with an edge of S or lies on a geodesic joining a pair of edges of S. The *edge-to-vertex geodetic number* $g_{ev}(G)$ of G is the minimum cardinality of its edge-to-vertex geodetic sets and any edge-to-vertex geodetic set of cardinality $g_{ev}(G)$ is an *edge-to-vertex geodetic basis* of G. The edge-to-vertex geodetic number of a graph is introduced and studied in [6,7]. The *maximum degree* of G, denoted by $\Delta(G)$, is given by $\Delta(G) = max\{deg_G(v) : v \in V(G)\}$. $N(v) = \{u \in V(G): uv \in E(G)\}$ is called the *neighborhood of the vertex v* in G. A vertex v is an *extreme vertex* of a graph G if the subgraph induced by its neighbors is complete. An edge e of a graph G is called an *extreme edge* of G, if one of its ends is an extreme vertex of G. A *chord* of a path $u_0, u_1, u_2, \ldots, u_h$ is an edge u_iu_j, with $j \geq i + 2$. An $u - v$ path is called a *monophonic path* if it is a chord less path. For two vertices u and v in a connected graph G, the *monophonic distance* $d_m(u, v)$ is the length of the longest $u - v$ monophonic path in G. An $u - v$ monophonic path of length $d_m(u, v)$ is called an $u - v$ *monophonic*. For a vertex v of G, the *monophonic eccentricity* $e_m(v)$ is the monophonic distance between v and a vertex farthest from v. The monophonic eccentricity among the vertices is the *monophonic radius*, $rad_m(G)$ and the maximum monophonic eccentricity is the *monophonic diameter*, $diam_m(G)$. For subsets *A and B* of *V (G)*, the *monophonic distance* $d_m(A, B)$ *is*

Corresponding author: K. Uma Samundesvari*, *E-mail: kus_krishna@yahoo.co.in

defined as $d_m(A, B) = max\{d_m(x, y) : x \in A, y \in B\}$. An $u - v$ monophonic path of length $d_m(A, B)$ is called an $A - B$ *monophonic* joining the sets A, B where $u \in A$ and $v \in B$. A vertex x is said to *lie* on an $A - B$ *monophonic* if x is a vertex of an $A - B$ *monophonic*. For $A = (u, v)$ and $B = (z, w)$ with uv and zw edges, we write an $A - B$ *monophonic* as $uv - zw$ *monophonic* and $d_m(A, B)$ as $d_m(uv, zw)$. The monophonic distance is studied in [8]. A monophonic set of G is a set $M \subseteq V$ such that every vertex of G lies on a monophonic path joining some pair of vertices in M. The *monophonic number* $m(G)$ of G is the minimum order of its monophonic sets and any monophonic set of order $m(G)$ is a minimum monophonic set or simply a m-set of G. The monophonic number of a graph is studied in [2, 3, 4, 5]. For a cut – vertex v in a connected graph G and a component H of G-v, the sub graph H and the vertex v together with all edges joining v and $V(H)$ is called a *branch* of G at v. An *end-block* of G is a block containing exactly one cut-vertex of G. Thus every end-block is a branch of G. Girth is the length of the smallest cycle in G. A double star is a tree with diameter three. A *caterpillar* is a tree of order 3 or more, for which the removal of all end- vertices leaves a path.
The following theorems are used in sequel.

Theorem: 1.1. [5] Every end-edge of a connected graph G belongs to every edge-to-vertex geodetic set of G.

Theorem: 1.2. [5] For a connected graph G, $g_{ev}(G) = q$ if and only if G is a star.

Theorem: 1.3. [5] For a connected graph G with $q \geq 3$, $g_{ev}(G) = q - 1$ if and only if G is either C_3 or a double star.

2. EDGE-TO-VERTEX MONOPHONIC NUMBER OF A GRAPH G

Definition: 2.1 Let $e, f \in E(G)$. The *e- f monophonic path* is an $u - v$ monophonic path , where u is one end of e and v is one end of f .The vertex x is said to lie on a $e - f$ monophonic path if x is a vertex of $e - f$ monophonic path.

Definition: 2.2 Let $G = (V, E)$ be a connected graph with at least three vertices. A set $S \subseteq E$ is called an *edge-to-vertex monophonic set* if every vertex of G lies on a monophonic path between two vertices in $V(S)$. The *edge-to-vertex monophonic number* $m_{ev}(G)$ of G is the minimum cardinality of its edge-to-vertex monophonic sets and any edge-to-vertex monophonic set of cardinality $m_{ev}(G)$ is an m_{ev} – set of G.

Example: 2.3 For the graph G given in Figure 2.1 with $e = v_1 v_6$ and $f = v_3 v_4$, the $e - f$ monophonic paths are $P_1 : v_1 , v_2 , v_3$, $P_2: v_6 , v_1 , v_2 , v_3$, $P_3 : v_6 , v_7 , v_8 , v_3$, $P_4 : v_6 , v_5 ,v_4$, $P_5 : v_6 , v_7 , v_8 , v_3 , v_4$, $P_6 : v_6 , v_5 ,v_4 ,v_3$, and $P_7: v_6, v_1 ,v_4$. Since the vertices v_2 ,v_5 ,v_7 and v_8 lies on the $v_1 v_6$ - $v_3 v_4$ monophonic path, $S = \{ e , f\}$ is a m_{ev} – set of G and so $m_{ev}(G) = 2$

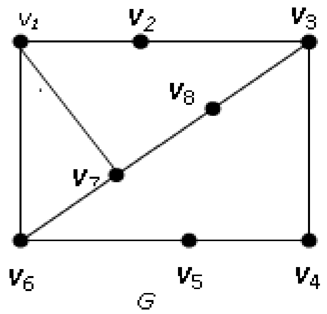


Figure: 2.1

Theorem: 2.4 If v is an extreme vertex of a connected graph G, then every edge-to-vertex monophonic set contains at least one extreme edge that is incident with v.

Proof: Let v be an extreme vertex of G. Let $e_1, e_2,.. ,e_k$ be the edges incident with v. Let S be any edge-to-vertex monophonic set of G. We claim that $e_i \in S$ for some $i(1 \leq i \leq k)$. Otherwise, $e_i \notin S$ for any $i(1 \leq i \leq k)$. Since S is an edge-to-vertex monophonic set , the vertex v lies on a monophonic path joining two elements, say, $x, y \in V(S)$. Since v is an internal vertex of a monophonic path x-y, v is not an extreme vertex of G, which is a contradiction. Hence $e_i \in S$ for some $i(1 \leq i \leq k)$.

Corollary: 2.5 Every end-edge of a connected graph G belongs to every edge-to-vertex monophonic set of G.

Proof: This follows from Theorem 2.4.

Theorem: 2.6 Let G be a connected graph with cut-vertices and S an edge-to-vertex monophonic set of G. Then every branch of G at v contains an element of S.

Proof: Assume that there is a branch B of G at a cut-vertex v such that B contains no element of S. Then by Corollary 2.5, B does not contain any end-edge of G. Hence it follows that no vertex of B is an end vertex of G. Let u be any vertex of B such that $u \neq v$ (such a vertex exists since $|V(B)| \geq 2$). Then $u \notin V(S)$ and so u lies on a $e-f$ monophonic path $P: u_1, u_2, \ldots, u, \ldots, u_t$, where u_1 is an end of e and u_t is an end of f with $e, f \in S$. Since v is a cut-vertex of G, the $u_1 - u$ and $u - u_t$ sub paths of P both contain v and so P is not a path, which is a contradiction. Hence every branch of G contains an element of S.

Corollary: 2.7 Let G be a connected graph with cut-edges and S an edge-to-vertex monophonic set of G. Then for any cut-edge e of G, which is not an end-edge, each of the two component of $G - e$ contains an element of S.

Proof: Let $e = uv$. Let G_1 and G_2 be the two component of $G - e$ such that $u \in V(G_1)$ and $v \in V(G_2)$. Since u and v are cut-vertices of G, it follows that G_1 contains at least one branch at u and G_2 contains at least one branch at v. Hence it follows from Theorem 2.6 that each of G_1 and G_2 contains an element of S.

Theorem 2.8 Let G be a connected graph and S be a m_{ev}-set of G. Then no cut edge of G which is not an end-edge of G belongs to S.

Proof: Let S be a m_{ev}- set of G. Suppose that $e = uv$ be a cut edge of G which is not an end-edge of G such that $e \in S$. Let G_1 and G_2 be the two component of $G - e$. Let $S' = S - \{uv\}$. We claim that S' is an edge-to-vertex monophonic set of G. By Corollary 2.7, G_1 contains an edge xy and G_2 contains an edge $x'y'$, where $xy, x'y' \in S$. Let z be any vertex of G. Assume without loss of generality that z belongs to G_1. Since uv is a cut edge of G, every path joining a vertex of G_1 with a vertex of G_2 contains the edge uv. Suppose that z is incident with uv or the edge xy of S or that lies on a monophonic path joining xy and uv. If z is incident with uv, then $z = u$. Let $P: y, y_1, y_2, \ldots, z = u$ be a $xy - u$ monophonic path. Let $Q: v, v_1, v_2, \ldots, y'$ be a $v - x'y'$ monophonic path. Then, it is clear that $P \cup \{uv\} \cup Q$ is a $xy - x'y'$ monophonic path .Thus z lies on the $xy - x'y'$ monophonic path. If z is incident with xy, then there is nothing to prove. If z lies on a $xy - uv$ monophonic path, say, $y, y_1, y_2 \ldots z \ldots u$, then let $v, v_1, v_2 \ldots y'$ be $v - x'y'$ monophonic path. Then clearly $y, y_1, y_2 \ldots z \ldots u, v, v_1, v_2 \ldots y'$ is a $xy - x'y'$ monophonic path. Thus z lies on a monophonic path joining a pair of edges of S'. Thus we have proved that a vertex that is incident with uv or an edge of S or that lies on a monophonic path joining xy and uv of S also is incident with an edge of S' or lies on a monophonic path joining a pair of edges of S'. Hence it follows that S' is an edge-to-vertex monophonic set such that $|S'| = |S| - 1$, which is a contradiction to S a m_{ev}- set of G. Hence the theorem follows.

In the following, we determine edge – to- vertex monophonic number of some standard graphs.

Corollary: 2.9 For any non-trivial tree T with k end-vertices, $m_{ev}(T) = k$ and the set of all end-edges of T is the unique edge-to-vertex monophonic set of T.

Proof: This follows from Corollary 2.5 and Theorem 2.8.

Theorem: 2.10 For the cycle C_p $(p \geq 4)$, $m_{ev}(C_p) = 2$

Proof: Let e, f be two independent edges of G. Then $S = \{e, f\}$ is an edge to vertex monophonic set of G so that $m_{ev}(C_p) = 2$.

Theorem: 2.11 For the complete graph $K_p (p \geq 4)$ with p even, $m_{ev}(K_p) = p/2$.

Proof: Let S be any set of $p/2$ independent edges of K_p. Since each vertex of K_p is incident with an edge of S, it follows that $m_{ev}(G) \leq p/2$. If $m_{ev}(G) < p/2$, then there exists an edge-to-vertex monophonic set S' of K_p such that $|S'| < p/2$. Therefore, there exists at least one vertex v of K_p such that v is not incident with any edge of S'. Hence v is neither incident with any edge of S' nor lies on a monophonic path of S' and so S' is not an edge-to-vertex monophonic set of G, which is a contradiction. Thus S is an edge-to-vertex monophonic basis of K_p. Hence $m_{ev}(K_p) = p/2$

Theorem: 2.12 For the complete graph $G = K_p(p \geq 5)$ with p odd, $m_{ev}(G) = \dfrac{p+1}{2}$.

Proof: Let M consist of any set of $\dfrac{p-3}{2}$ independent edges of K_p and M' consist of 2 adjacent edges of K_p, each of which is independent with the edges of M. Let $S = M \cup M'$. Since each vertex of K_p is incident with an element of S, it follows that S is an edge-to-vertex monophonic set of G so that $m_{ev}(G) \leq \dfrac{p-3}{2} + 2 = \dfrac{p+1}{2}$.

If $m_{ev}(G) < \frac{p+1}{2}$, then there exists an edge-to-vertex monophonic set S' of K_p such that $|S'| < \frac{p+1}{2}$. Therefore, there exists at least one vertex v of K_p such that v is not incident with any edge of S'. Hence the vertex v is neither incident with any edge of S' nor lies on a monophonic path of S' and so S' is not an edge-to-vertex monophonic set of G, which is a contradiction. Hence $m_{ev}(G) = \frac{p+1}{2}$.

Theorem: 2.13 For the complete bipartite graph $G = K_{m,n}(2 \le m \le n)$, $m_{ev}(G) = 2$.

Proof: Let $X = \{x_1, x_2\ldots x_m\}$ and $Y = \{y_1, y_2\ldots y_n\}$ be a bipartition of G. Let $S = \{e,f\}$, where $e \ne f$ and $e = x_i y_j$ and $f = x_k y_l$ $(1 \le j, l \le n, m, 1 \le j, k \le n)$ $i \ne k$ and $j \ne l$. Let x be a vertex of G. If $x \in Y$, then x lies on the monophonic path $x_i - x_k$. If $x \in X$, then x lies on the monophonic path $y_j - y_l$. Hence S is an edge-to- vertex monophonic set of G so that $m_{ev}(G) = 2$.

Theorem: 2.14 Let G be a connected unicyclic graph with girth greater than three. Then $mev(G)$ is either k or k+1, where k is the number of end- vertices of G.

Proof: Let C be the unicycle in G such that $|C| \ge 4$. Let W be the set of end edges of G. If $V(C)$ contain at least three cut- vertices of G, then W is an edge-to-vertex monophonic set of G and by Corollary 2.5 that $m_{ev}(G) = k$. If $V(C)$ contains exactly two cut -vertices of G, say, u, v, and if u and v are not adjacent, then W is an edge-to-vertex monophonic set of G and by Corollary 2.5 that $m_{ev}(G) = k$. If u and v are adjacent, then W is not an edge-to-vertex monophonic set of G and so by Corollary 2.5, $m_{ev}(G) \ge k + 1$. Let $x \in V(C)$ such that $xv \in E(C)$ and $x \ne u$. Then $W \cup \{xv\}$ is an edge-to-vertex monophonic set of G so that $m_{ev}(G) = k + 1$. If $V(C)$ contains one cut vertex of G, say v, then W is not an edge-to-vertex monophonic set of G and so by Corollary 2.5, $mev(G) \ge k + 1$. Since $|C| \ge 4$, there exits $x \in V(C)$ such that $d(x, v) \ge 2$. Let $y \in V(C)$ such that $xy \in E(C)$ and $y \ne v$. Then $S = W \cup \{xy\}$ is an edge-to-vertex monophonic set of G so that $mev(G) = k + 1$.

3. SOME RESULTS ON EDGE-TO-VERTEX MONOPHONIC NUMBER OF A GRAPH

Theorem: 3.1 Let G be a connected graph. Then $2 \le m_{ev}(G) \le g_{ev}(G) \le ß'(G) \le q$.

Proof: An edge-to-vertex monophonic needs at least two edges and so $m_{ev}(G) \ge 2$. Since every edge-to-vertex geodesic set of G is an edge-to-vertex monophonic set of G and so $m_{ev} \le g_{ev}(G)$. Also every edge cover of G is an edge -to-vertex geodesic set of G. We have $g_{ev}(G) \le ß'(G)$. Since $E(G)$ is the edge cover of G, we have $ß'(G) \le q$. Thus $2 \le m_{ev}(G) \le g_{ev}(G) \le ß'(G) \le q$.

Remark: 3.2 The set of two end-edges of a path P of length at least two is its unique edge-to-vertex monophonic set so that $m_{ev}(P) = 2$. For the star $K_{1,q}$ $(q \ge 2)$,it is clear that the set of all edges is the unique edge-to-vertex monophonic number set so that $m_{ev}(G) = q$. Thus the star $K_{1,q}$ $(q \ge 2)$ has the largest possible edge-to-vertex monophonic number q and the paths of length at least 2 have the smallest edge-to-vertex monophonic.

Theorem: 3.3. Let G be a connected graph. Then $m(G) \le 2\, m_{ev}(G)$.

Proof: Let S be an edge – to – vertex monophonic set of G .Since every vertex of G lies on a monophonic path between two vertices in $V(S)$, it follows that $V(S)$ is a monophonic set of G and so $m(G) \le |V(S)| \le 2\,|E(S)| = 2\, m_{ev}(G)$. Thus $m(G) \le 2\, m_{ev}(G)$.

Theorem: 3.4 Let G be a connected graph. If $d(e, f) =$ 0 or 1 for every $e, f \in E(G)$. Then $m_{ev}(G) = g_{ev}(G) = ß'(G)$.

Proof: Let S be an edge-to-vertex monophonic set of G. We show that S is an edge covering of G. Suppose that S is not an edge covering of G. Then there exists at least one vertex $v \in V(G)$ such that $v \notin V(S)$. Then v lies on a e - f monophonic path, where $e, f \in S$. Since v is an internal vertex of the e - f monophonic path, it follows that

$d(e, f) \ge 2$,which is a contradiction. Therefore S is an edge cover of G so that $m_{ev}(G) = ß'(G)$. Now it follows from Theorem 3.1 that $g_{ev}(G) = ß'(G)$. Thus $m_{ev}(G) = g_{ev}(G) = ß'(G)$.

Theorem: 3.5 The converse of the theorem 3.4 is not true. For the graph given in Figure 3.1, $m_{ev}(G) = g_{ev}(G) = ß'(G) = 6$. However $d(e, f) = 2$.

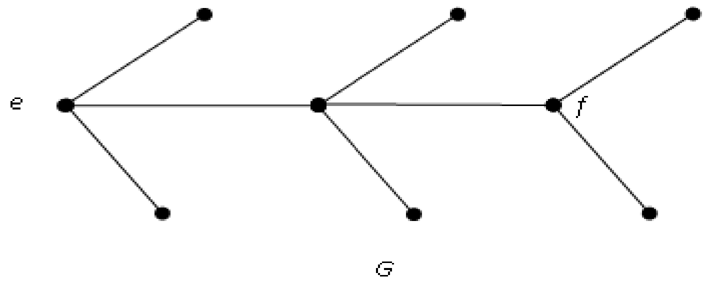


Figure: 3.1

Theorem: 3.6 Let G be a connected graph without extreme vertices. Then $m_{ev}(G) \leq q - \Delta(G) + 1$.

Proof: Let x be a vertex of G such that $deg_G(x) = \Delta(G)$ and $N(x) = \{y_1, y_2,\ldots, y_\Delta\}$.Since x is not an extreme vertex of G , xy_i $(1 \leq i \leq \Delta)$ is not an extreme edge of G. Now $S = E(G) – \{xy_2 , xy_3 , \ldots , xy_\Delta\}$ is an edge – to – vertex monophonic set of G so that $m_{ev}(G) \leq q - \Delta(G) + 1$.

Theorem: 3.7 Let G be a connected non- complete graph and U is a minimum cutest of G such that each component of $G – U$ has at least one edge. Then $m_{ev}(G) \leq q – m\, k(G)$, where m is the number of component of $G – U$.

Proof: Suppose G is non- complete, it is clear that $1 \leq k(G) \leq p$ -2. Let $U = \{u_1, u_2\ldots u_k\}$ be a minimum cutest of G.
Let $G_1, G_2, \ldots, G_m$ $(m \geq 2)$ be the components of $G – U$. Then every vertex u_i $(1 \leq i \leq k)$ is adjacent to at least one vertex of G_j , for every j $(1 \leq j \leq m)$. Let x_{ij} be a vertex of G_j $(1 \leq i \leq k)$ $(1 \leq j \leq m)$ which is adjacent to u_i $(1 \leq i \leq k)$. Then $S = E(G) – \{u_1x_{11}, u_1x_{12},\ldots,u_{1m}, u_2x_{21}, u_2x_{22}, \ldots, u_2x_{2m}, u_k x_{k1}, u_k x_{k2,\ldots,} u_k x_{km}\}$ is an edge – to – vertex monophonic set of G so that $m_{ev}(G) \leq q – m\, k(G)$.

The following Theorems 3.8 and 3.9 characterize graphs for which $m_{ev}(G) = q$ and $m_{ev}(G) = q – 1$ respectively.

Theorem: 3.8 For a connected graph G, $m_{ev}(G) = q$ if and only if $g_{ev}(G) = q$.

Proof: Let $m_{ev}(G) = q$. Then by Theorem 3.1, $g_{ev}(G) = q$. Conversely, let $g_{ev}(G) = q$. Then by Theorem 1.2, G is a star.

Now it follows from Corollary 2.9 that $m_{ev}(G) = q$.

Theorem: 3.9 For a connected graph G with $q \geq 3$, $m_{ev}(G) = q$-1 if and only if $g_{ev}(G) = q$ -1.

Proof: Let $m_{ev}(G) = q -1$. Then by Theorem 3.1, $g_{ev}(G) = q$ or $q – 1$. If $g_{ev}(G) = q$, then by Theorem 1.2,G is a star. But by Corollary 2.9, $m_{ev}(G) = q$, which is a contradiction. Therefore $g_{ev}(G) = q$ -1.Conversely, let $g_{ev}(G) = q$ -1.Then by Theorem 1.3, G is either C_3 or a double star. If G is C_3 , then $m_{ev}(G) = 2 = q$-1. If G is a double star , then by Corollary 2.9 that $m_{ev}(G) = q$-1. Hence the theorem.

4. EDGE – TO – VERTEX MONOPHONIC NUMBER AND MONOPHONIC DIAMETER OF A GRAPH

Theorem: 4.1 For a connected graph G with $q \geq 2$, $m_{ev}(G) \leq q – d_m + 2$, where d_m is the monophonic diameter of G.

Proof: Let u and v be vertices of G for which $d_m(u, v) = d_m$, where d_m is the monophonic diameter of G and let $P : u = v_0, v_1, v_2, ..., v_{dm} = v$ be a $u – v$ path of length d_m. Let $e_i = v_{i-1} v_i$ $(1 \leq i \leq d_m)$. Let $S = E (G) – \{ v_1v_2, v_2 v_3, \ldots, v_{dm-2}\, v_{dm-1} \}$.
Let x be any vertex of G. If $x = v_i$ $(1 \leq i \leq d_m$-$1)$, then x lies on the $e_1 – e_{dm}$ monophonic path P_1: $v_1, v_2\ldots v_{dm-1}$.
If $x \neq v_i$ $(1 \leq i \leq d_m – 1)$, then x is incident with an edge of S. Therefore, S is an edge-to-vertex monophonic set of G. Consequently, $m_{ev}(G) \leq | S | = q – d_m +2$.

Remark: 4.2 The bound in Theorem 4.1 is sharp. For the star $G = K_{1,q}(q \geq 2)$, $d_m = 2$ and $m_{ev}(G) = q$, by Corollary 2.9, so that $m_{ev}(G) = q – d_m + 2$.

Theorem: 4.3 Let G be a connected graph without extreme vertices. Then $m_{ev}(G) \leq q – d_m$, where d_m is the monophonic diameter of G.

Proof: Let u and v be vertices of G for which $d_m(u, v) = d_m$, where d_m is the monophonic diameter of G and let $P : u = v_0, v_1, v_2, ..., v_{dm} = v$ be a $u – v$ monophonic path of length d_m. Let $e_i = v_{i-1} v_i$ $(1 \leq i \leq d_m)$.

Let $S = E(G) - \{e_2, e_3, \ldots, e_{dm-1}\} - \{e, f\}$ where $e = v_0 y$ and $f = v_{dm} z$, where $y \neq v_1$ *and* $z \neq v_{dm-1}$. Let x be any vertex of G. If $x = v_i$ $(1 \leq i \leq d_m - 1)$, then x lies on the $e_1 - e_{dm}$ monophonic path P_1: $v_1, v_2, \ldots, v_{dm-1}$. If $x \neq v_i$ $(1 \leq i \leq d_m - 1)$, then $x \in V(S)$. Therefore, S is an edge-to-vertex monophonic set of G. Consequently, $m_{ev}(G) \leq |S| = q - d_m$.

Theorem: 4.4 For any y non-trivial tree T with $q \geq 2$, $m_{ev}(T) = q - d_m + 2$ if and only if T is a caterpillar.

Proof: Let P: $v_0, v_1, v_2, \ldots, v_{dm-1}, v_{dm} = v$ be a diametral path of length d_m. Let $e_i = v_{i-1} v_i$ $(1 \leq i \leq d_m)$ be the edges of the diametral path P. Let k be the number of end-edges of T and l be the number of internal edges of T other than e_i $(2 \leq i \leq d_m\text{-}1)$.Then $d_m - 2 + l + k = q$. By Corollary 2.9, $m_{ev}(T) = k$ and so $m_{ev}(T) = q - d_m + 2 - l$.

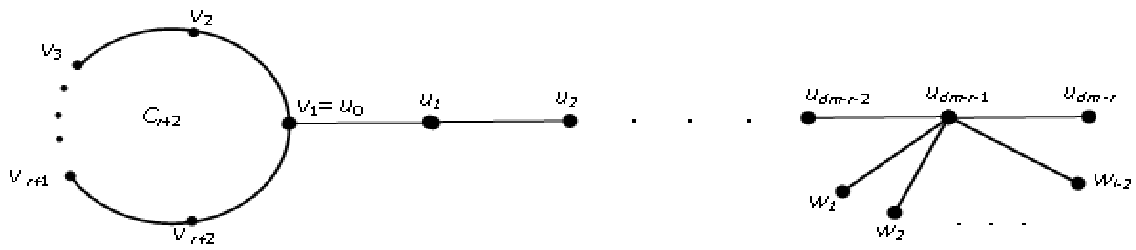
Hence $m_{ev}(T) = q - d_m + 2$ if and only if $l = 0$, if and only if all internal vertices of T lie on the diametral path P, if and only if T is a caterpillar.

For every connected graph G, $rad\ G \leq diam\ G \leq 2\ rad\ G$. Ostrand[9] showed that every two positive integers a and b with $a \leq b \leq 2a$ are realizable as the radius and diameter , respectively, of some connected graphs. Now, Ostrand's theorem can be extended so that the edge - to - vertex monophonic number can also be prescribed.

Theorem: 4.5 For positive integers r_m, d_m and $l \geq 2$ with $r_m < d_m \leq 2\, r_m$, there exists a connected graph G with $rad_m\ G = r_m$, $diam_m\ G = d_m$ and $m_{ev}(G) = l$.

Proof: When $r_m = 1$, we let $G = K_{1,l}$. Then the result follows from Corollary 2.9.

Let $r_m \geq 2$. Let $C_{r+2} : v_1, v_2, \ldots, v_{r+2}$ be a cycle of length $r + 2$ and let P_{dm-r+1}: $u_0, u_1, u_2, \ldots, u_{dm}\text{-}r$ be a path of length $d_m - r_m + 1$. Let H be a graph obtained from C_{r+2} and P_{dm-r+1} by identifying v_1 in C_{r+2} and u_0 in P_{dm-r+1}. Now add l-2 new vertices $w_1, w_2, \ldots, w_{l-2}$ to H and join each w_i $(1 \leq i \leq l\text{-}2)$ to the vertex u_{dm-r-1} and obtain the graph G of Figure 4.1.Then $rad_m\ G = r_m$, $diam_m\ G = d_m$.



G

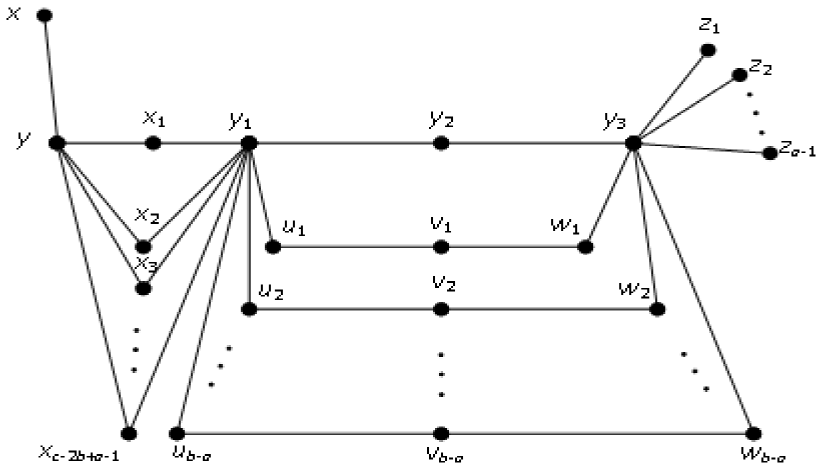
Figure: 4.1

Let $S = \{u_{dm-r-1} w_1, u_{dm-r-1} w_2, \ldots, u_{dm-r-1} w_{l-2}, u_{dm-r-1} u_{dm-r}\}$ be the set of all end-edges of G. By Corollary 2.5, S is contained in every edge - to - vertex monophonic set of G. It is clear that S is not an edge - to - vertex monophonic set of G. However, the set $S \cup \{e\}$, where $e \in \{v_2 v_3, v_3 v_4, \ldots, v_r v_{r+1}, v_{r+1} v_{r+2}\}$ is an edge - to - vertex monophonic set of G so that $m_{ev}(G) = l - 1 + 1 = l$.

5. REALIZATION THEOREM

Theorem: 5.1 For every integers a , b and c with $2 \leq a \leq b \leq c$, and $c \geq 2b - a + 1$, there exists a connected graph G such that $m_{ev}(G) = a$, $g_{ev}(G) = b$ and $\beta'(G) = c$.

Proof: Let P_3: y_1, y_2, y_3 be a path of order 3. For each integer i with $1 \leq i \leq b - a$, let Q_i: u_i , v_i , w_i $(1 \leq i \leq b - a)$ be a path of order 3. Let H be a graph obtained from P_3 and Q_i $(1 \leq i \leq b - a)$ by joining each u_i $(1 \leq i \leq b - a)$ with y_1 and each w_i $(1 \leq i \leq b - a)$ with y_3 and also adding $a - 1$ new vertices z_i $(1 \leq i \leq a - 1)$ and joining each z_i $(1 \leq i \leq a - 1)$ with y_3 . Let G be a graph obtained from H by adding new vertices x ,y and x_i $(1 \leq i \leq c - 2b + a - 1)$, and joining each x_i $(1 \leq i \leq c - 2b + a - 1)$ with y and y_1 and x with y. The graph G is given in Figure 5.1. First show that $m_{ev}(G) = a$. Let $S = \{y_3 z_1, y_3 z_2, \ldots y_3 z_{a-1}, x y\}$ be the set of all end edges of G. By Corollary 2.9, S is a subset of every edge – to – vertex monophonic set of G and so $m_{ev}(G) \geq a$. Now it is clear that S is an edge – to – vertex monophonic set of G so that $m_{ev}(G) = a$. Next show that $g_{ev}(G) = b$. By Theorem 1.1, S is a subset of every edge – to – vertex geodetic set of G. It is clear that S is not an edge – to – vertex geodetic set of G. Let H_i : $\{u_i vi, vi, w_i\}$, $(1 \leq i \leq b - a)$. It is easily observed that every edge – to – vertex geodetic set of G contains at least one edge from each H_i ,$(1 \leq i \leq b - a)$ and so $g_{ev}(G) \geq a + b - a = b$. Now $S' = S \cup \{v_1 w_1, v_2 w_2, v_3 w_3, \ldots v_{b-a} w_{b-a}\}$ is an edge – to – vertex geodetic set of G, so that $g_{ev}(G) = b$. Next show that $ß'(G) = c$. Let Q_i: $\{\{ u_i\, vi,\ vi, w_i\}, \{y_1 u_i,\ v_i, w_i\}, \{y_3 w_i,\ u_i\, vi\}\}$ $(1 \le i \le b - a)$ and $F_i : \{ y_1\, x_i , y\, x_i \}$, $(1 \le i \le c - 2b + a - 1)$ and $M = \{y_1\, y_2 , y_2\, y_3\}$. We have to show that a set W of edges is a minimum edge covering of G if and only if W has the following five properties.(1) S is a subset of W, (2) W contains exactly one element of each Q_i $(1 \le i \le b - a)$, (3) W contains exactly one element of M , (4) W contains exactly one element each F_i , $(1 \le i \le c - 2b + a - 1)$, (5) $V(M) \cap V(y_1 x_i) \ne x_i$ and $V(M) \cap V(y x_i) \ne x_i$. Obviously, the properties (1) and (3) are true. Suppose that W does not contain an element of Q_i, $(1 \le i \le b - a)$. Then the vertices u_i, v_i and w_i does not lie on the edge covering of G .Therefore the property (2) holds good. Suppose that W does not contain an element of F_i $(1 \le i \le c - 2b + a - 1)$. Then the vertices x_i $(1 \le i \le c - 2b + a - 1)$ does not lie on the edge covering of G, which is a contradiction .Therefore W has the property (4). Suppose that property (5) is not satisfied. Then $\beta'(G)$ is not a minimum edge covering of G. Therefore from the above said five conditions , it is clear that $\beta'(G) \ge a + 2b - 2a + c - 2b + a - 1 + 1 = c$ .Now $S'' = S \cup \{ u_1 v_1 , u_2 v_2 , u_3 v_3 , \ldots , u_{b-a} v_{b-a} , v_1 w_1 , v_2 w_2 , , \ldots , v_{b-a} w_{b-a} , y_1 y_2 , y_1 x_1 , y_1 x_2 , , \ldots , y_1 x_{c-2b+a-1} \}$ is an edge cover of G so that $ß'(G) = c$.



G

Figure: 5.1

REFERENCES

[1] F. Buckley and F. Harary, *Distance in Graphs*, Addition- Wesley, Redwood City, CA, 1990.

[2] Carmen Hernando, Tao Jiang, Merce Mora, Ignacio. M. Pelayo and Carlos Seara, On the Steiner, geodetic and hull number of graphs, *Discrete Mathematics* 293 (2005) 139 - 154.

[3] Esamel M. paluga, Sergio R. Canoy, Jr., Monophonic numbers of the join and Composition of connected graphs, *Discrete Mathematics* 307 (2007) 1146 - 1154.

[4] J. John and S. Panchali, The Upper Monophonic Number of a Graph,*International Journal of Mathematical Combinatorics* 4(2010), 46-52.

[5] Mitre C. Dourado, Fabio protti and Jayme. L. Szwarcfiter, Algorithmic Aspects of Monophonic Convexity, *Electronic Notes in Discrete Mathematics* 30(2008) 177-182.

[6] A. P. Santhakumaran and J. John, The Edge – to- Vertex Geodetic Number of a Graph –Serdica Mathematical Journal (Accepted).

[7] A. P. Santhakumaran and J. John, On the Edge – to- Vertex Geodetic Number of a Graph, *Miskolc Mathematical Notes* (Accepted).

[8] A. P. Santhakumaran and P.Titus, Monophonic Distance in Graphs, *Discrete Algorithms and Applications* 3(2011) 159-169.

[9] P. A. Ostrand, Graphs with specified radius and diameter, *Discrete Mathematics* 4(1973), pp.71-75.

****