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gα-separation axioms

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ABSTRACT

In this paper we discuss new separation axioms using gα-open sets.

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1. Introduction:

Norman Levine introduced generalized closed sets in 1970. After him various Authors studied different versions of generalized sets and related topological properties. Recently V.K. Sharma and the author of the present paper defined separation axioms for g-open; gs-open; sg-open; rg-open sets and studied their basic properties.

Definition 1.1: $A \subseteq X$ is called generalized closed[resp: regular generalized; generalized regular]{briefly: g-closed; rg-closed; gα-closed}if cl{A}⊆U whenever A⊆U and U is open[resp: regular open, open] and generalized[resp: regular generalized; generalized regular] open if its complement is generalized[resp: regular generalized; generalized regular] closed.

Note 1: The class of regular open sets, open sets, g-open sets, gα-open sets are denoted by RO(X), τ(X), αO(X), and gαO(X) respectively. Clearly $RO(X) \subset \tau(X) \subset \alpha O(X) \subset g\alpha O(X)$.

Note 2: $A \in g\alpha O(X, x)$ means A is generalized α-open neighborhood of X containing x.

Definition 1.2: $A \subset X$ is called clopen[resp: gα-clopen] if it is both open[resp: gα-open] and closed[resp: gα-closed]

Definition 1.3: A function $f: X \to Y$ is said to be
(i) g-continuous [resp: gα-continuous] if inverse image of closed set is g-closed [resp: gα-closed] and g-irresolute [resp: gα-irresolute] if inverse image of g-closed [resp: gα-closed] set is g-closed [resp: gα-closed]
(iii) gα-open if the image of open set gα-open
(iv) gα-homeomorphism [resp: gαc-homeomorphism] if f is bijective, gα-continuous [resp: gα-irresolute] and f^{-1} is gα-continuous [resp: gα-irresolute]

Definition 1.4: X is said to be
(i) compact [resp: nearly compact, semi-compact, g-compact, gα-compact] if every open[resp: regular-open, semi-open, g-open, gα-open] cover has a finite sub cover.
(ii) T_0 [resp: rT_0, sT_0, g_0] space if for each $x \neq y \in X$ ∃ $U \in \tau(X)$[resp: RO(X); SO(X); GO(X)] containing either x or y.
(iii) T_1 [resp: rT_1, g_1,]{T_2 [resp: rT_2, g_2,]} space if for each $x \neq y \in X$ ∃ {disjoint} U, $V \in \tau(X)$ [resp: RO(X); GO(X)], G and H containing x and y respectively.
(iv) $T_{1/2}$ [resp: $rT_{1/2}$, $sT_{1/2}$] if every generalized [resp: regular generalized, semi-generalized] closed set is closed [resp: regular-closed, semi-closed]

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2. gα-Continuity and product spaces:

Theorem 2.1: Let Y and $\{X_\alpha:\alpha\in I\}$ be Topological Spaces. Let f: Y→ ΠX_α be a function. If f is gα-continuous, then $\pi_\alpha\bullet f$: Y→ X_α is gα-continuous.

Converse of the above theorem is not true in general as shown by the following Example:

Example 2.1: Let X = {p, q, r, s}; τ_X = {ϕ, {p}, {q}, {s}, {p, q}, {p, s},{q, s}, {p, q, r}, {p, q, s}, X}, $Y_1 = Y_2$ = {a, b}; τ_{Y1} = {ϕ, {a}, Y_1}; τ_{Y2} = {ϕ, {a}, Y_2}; Y = $Y_1\times Y_2$ = {(a, a), (a, b), (b, a), (b, b)} and τ_Y = {ϕ, {(a, a)}, {(a, a), (a, b)}, {(a, a), (b, a)}, {(a, a), (a, b), (b, a)}, $Y_1\times Y_2$}.
Define f: X→ Y by f(p) = (a, a), f(q) = (b, b), f(r) = (a, b), f(s) = (b, a). It is easy to see that $\pi_1\bullet f$ and $\pi_2\bullet f$ are gα-continuous. However {(b, b)} is closed in Y but f^{-1}({(b, b)}) = {q} is not gα-closed in X. Therefore f is not gα-continuous.

Theorem 2.2: If Y is $rT_{1/2}$ and $\{X_\alpha:\alpha\in I\}$ be Topological Spaces. Let f: Y→ ΠX_α be a function, then f is gα-continuous iff $\pi_\alpha\bullet f$: Y→ X_α is gα-continuous.

Corollary 2.3: (i) Let f_α: $X_\alpha\to Y_\alpha$ be a function and let f: $\Pi X_\alpha\to \Pi Y_\alpha$ be defined by $f(x_\alpha)_{\alpha\in I} = (f_\alpha(x_\alpha))_{\alpha\in I}$. If f is gα-continuous then each f_α is gα-continuous.
(ii) For each α, let X_α be $rT_{1/2}$ and let f_α: $X_\alpha\to Y_\alpha$ be a function and let f: $\Pi X_\alpha\to \Pi Y_\alpha$ be defined by $f(x_\alpha)_{\alpha\in I} = (f_\alpha(x_\alpha))_{\alpha\in I}$, then f is gα-continuous iff each f_α is gα-continuous.

3. $g\alpha_i$ spaces i = 0, 1, 2:

Definition 3.1: X is said to be
(i) a $g\alpha_0$ space if for each pair of distinct points x, y of X, there exists a gα-open set G containing either of the point x or y.
(ii)a $g\alpha_1$ [resp: $g\alpha_2$] space if for each pair of distinct points x, y of X there exists [resp: disjoint] gα-open sets G and H containing x and y respectively.

Note 3:
(i) $rT_i\to T_i\to \alpha_i\to g\alpha_i$, i = 0, 1, 2. but the converse is not true in general.
(ii)X is $g\alpha_2$ → X is $g\alpha_1$ → X is $g\alpha_0$.

Example 3.1: Let X = {a, b, c} and τ = {ϕ, X}, then X is $g\alpha_i$ but not rT_0 and T_0, i = 0, 1, 2.for i = 0, 1, 2.

Example 3.2: Let X = {a, b, c, d} and τ = {ϕ, {b}, {a, b}, {b, c}, {a, b, c}, X} then X is not $g\alpha_i$ for i = 0, 1, 2.

Remark 3.1: If X is $rT_{1/2}$ then rT_i and $g\alpha_i$ are one and the same for i = 0,1,2.

Theorem 3.1: The following are true
(i) Every [resp: regular open] open subspace of $g\alpha_i$ space is $g\alpha_i$ for i = 0, 1, 2.
(ii)The product of $g\alpha_i$ spaces is again $g\alpha_i$ for i = 0, 1, 2.
(iii) gα-continuous image of T_i[resp: rT_i] spaces is $g\alpha_i$ for i = 0, 1, 2.
(iv) X is $g\alpha_0$ iff ∀ x∈ X, ∃ U∈ gαO(X) containing x such that the subspace U is $g\alpha_0$.
(v) X is $g\alpha_0$ iff distinct points of X have disjoint gα-closures.
(vi) If X is $g\alpha_1$ then distinct points of X have disjoint gα-closures.

Theorem 3.2: The following are equivalent:
(i) X is $g\alpha_1$.
(ii) Each one point set is gα-closed.
(iii)Each subset of X is the intersection of all gα-open sets containing it.
(iv) For any x∈ X, the intersection of all gα-open sets containing the point is the set {x}.

Theorem 3.3: Suppose x is a gα-limit point of a subset of A of a $g\alpha_1$ space X. Then every neighborhood of x contains infinitely many distinct points of A.

Theorem 3.4: The following are true
(i) X is $g\alpha_2$ iff the intersection of all gα-closed, gα-neighborhoods of each point of the space is reduced to that point.

(ii) If to each point x∈X, there exist a gα-closed, gα-open subset of X containing x which is also a $g\alpha_2$ subspace of X, then X is $g\alpha_2$.
(iii) If X is $g\alpha_2$ then the diagonal Δ in X×X is gα-closed.
(iv) In $g\alpha_2$-space, gα-limits of sequences, if exists, are unique.
(v) In a $g\alpha_2$ space, a point and disjoint gα-compact subspace can be separated by disjoint gα-open sets.
(vi) Every gα-compact subspace of a $g\alpha_2$ space is gα-closed.

Corollary 3.1: The following are true
(i) In a T_1 [resp: rT_1; g_1] space, each singleton set is gα-closed.
(ii) If X is T_1 [resp: rT_1; g_1] then distinct points of X have disjoint gα-closures.
(iii)If X is T_2 [resp: rT_2; g_2] then the diagonal Δ in X×X is gα-closed.
(iv) Show that in a T_2 [resp: rT_2; g_2] space, a point and disjoint compact[resp: nearly-compact; g-compact] subspace can be separated by disjoint gα-open sets
(v) Every compact [resp: nearly-compact; g-compact] subspace of a T_2 [resp: rT_2; g_2] space is gα-closed.

Theorem 3.5: The following are true
(i) If *f:* X→ Y is injective, gα-irresolute and Y is $g\alpha_i$ then X is $g\alpha_i$, i = 0, 1, 2.
(ii) If *f:* X→ Y is injective, gα-continuous and Y is T_i then X is $g\alpha_i$, i = 0, 1, 2.
(iii) If *f:* X→ Y is injective, r-irresolute[r-continuous] and Y is rT_i then X is $g\alpha_i$, i = 0, 1, 2.
(iv) The property of being a space is $g\alpha_0$ is a gα-Topological property.
(v) Let *f:* X → Y is a gαc-homeomorphism, then X is $g\alpha_i$ if Y is $g\alpha_i$, i = 0, 1, 2.
(vi) Let X be T_1 and *f*: X → Y be gα-closed surjection. Then X is $g\alpha_1$.
(vii) Every gα-irresolute map from a gα-compact space into a $g\alpha_2$ space is gα-closed.
(viii) Any gα-irresolute bijection from a gα-compact space onto a $g\alpha_2$ space is a gαc-homeomorphism.
(ix) Any gα-continuous bijection from a gα-compact space onto a $g\alpha_2$ space is a gα-homeomorphism.

Theorem 3.6: The following are equivalent:
(i) X is $g\alpha_2$.
(ii) For each pair x ≠ y∈ X ∃ a gα-open, gα-closed set V such that x∈V and y∉V, and
(iii)For each pair x ≠ y∈ X ∃ *f:* X→ [0, 1] such that *f*(x) = 0 and *f*(y) = 1 and *f* is gα-continuous.

Theorem 3.7: If *f:* X→ Y is gα-irresolute and Y is $g\alpha_2$ then
(i) the set A = {(x_1, x_2): *f*(x_1) = *f*(x_2)} is gα-closed in X× X.
(ii)G(*f*), gαaph of *f*, is gα-closed in X× Y.

Theorem 3.8: If *f*: X→ Y is gα-open and A = {(x_1, x_2): *f*(x_1) = *f*(x_2)} is closed in X×X. Then Y is $g\alpha_2$.

Theorem 3.9: Let Y and {X_α:α∈ I} be Topological Spaces. If *f:* Y→ Π X_α be a gα-continuous function and Y is $rT_{1/2}$, then Π X_α and each X_α are $g\alpha_i$, i = 0,1,2.

Theorem 3.10: Let X be an arbitrary space, R an equivalence relation in X and *p*: X → X/R the identification map. If R⊂ X× X is gα-closed in X× X and *p* is gα-open map, then X/R is $g\alpha_2$.

Proof: Let *p*(X), *p*(y) be distinct members of X/R. Since x and y are not related, R⊂ X× X is gα-closed in X× X. There are gα-open sets U and V such that x∈ U, y∈ V and U× V⊂ R^c. Thus {*p*(U), *p*(V)} are disjoint and also gα-open in X/R since *p* is gα-open.

Theorem 3.11: The following four properties are equivalent:
(i) X is $g\alpha_2$
(ii) Let x∈ X. For each y ≠ x, ∃ U∈ gαO(X) such that x∈ U and y∉ gαcl(U)
(iii)For each x∈ X, ∩{ gαcl(U)/U∈ gαO(X) and x∈ U} = {x}.
(iv) The diagonal Δ = {(x, x)/x∈X} is gα-closed in X× X.

Proof: (i) ⇒ (ii) Let x∈X and y ≠ x. Then there are disjoint gα-open sets U and V such that x∈U and y∈V. Clearly V^c is gα-closed, gαcl(U) ⊂ V^c, y∉V^c and therefore y∉ gαcl(U).

(ii) ⇒ (iii) If y ≠ x, then ∃ U∈ gαO(X) s.t. x∈U and y∉ gαcl(U). So y∉∩{ gαcl(U)/U∈ gαO(X) and x∈U}.

(iii) ⇒ (iv) We prove Δ^c is gα-open. Let (x, y) ∉Δ. Then y ≠ x and ∩{ gαcl(U)/U∈ gαO(X) and x∈U} = {x} there is some U∈ gαO(X) with x∈U and y∉ gαcl(U). Since U∩(gαcl(U))^c = ϕ, U×(gαcl(U))^c is a gα-open set such that (x, y)∈U×(gαcl(U))^c⊂Δ^c.

(iv) ⇒ (i) $y \neq x$, then $(x, y) \notin \Delta$ and thus there exist gα-open sets U and V such that $(x, y) \in U \times V$ and $(U \times V) \cap \Delta = \phi$. Clearly, for the gα-open sets U and V we have; $x \in U$, $y \in V$ and $U \cap V = \phi$.

4. gα-R_i spaces; i = 0,1:.

Definition 4.1: Let $x \in X$. Then
(i) gα-kernel of x is defined and denoted by $Ker_{\{g\alpha\}}\{x\} = \cap\{U : U \in g\alpha O(X) \text{ and } x \in U\}$
(ii) $Ker_{\{g\alpha\}}F = \cap\{U : U \in g\alpha O(X) \text{ and } F \subset U\}$

Lemma 4.1: Let $A \subset X$, then $Ker_{\{g\alpha\}}\{A\} = \{x \in X : g\alpha cl\{x\} \cap A \neq \phi.\}$

Lemma 4.2: Let $x \in X$. Then $y \in Ker_{\{g\alpha\}}\{x\}$ iff $x \in g\alpha cl\{y\}$.

Proof: Suppose that $y \notin Ker_{\{g\alpha\}}\{x\}$. Then $\exists$ $V \in g\alpha O(X)$ containing x such that $y \notin V$. Therefore we have $x \notin g\alpha cl\{y\}$. The proof of converse part can be done similarly.

Lemma 4.3: For any points $x \neq y \in X$, the following are equivalent:
(i) $Ker_{\{g\alpha\}\{g\alpha\}}\{x\} \neq Ker_{\{g\alpha\}}\{y\}$;
(ii) $g\alpha cl\{x\} \neq g\alpha cl\{y\}$.

Proof: (i) ⇒ (ii): Let $Ker_{\{g\alpha\}}\{x\} \neq Ker_{\{g\alpha\}}\{y\}$, then $\exists$ $z \in X$ such that $z \in Ker_{\{g\alpha\}}\{x\}$ and $z \notin Ker_{\{g\alpha\}}\{y\}$. From $z \in Ker_{\{g\alpha\}}\{x\}$ it follows that $\{x\} \cap g\alpha cl\{z\} \neq \phi \Rightarrow x \in g\alpha cl\{z\}$. By $z \notin Ker_{\{g\alpha\}}\{y\}$, we have $\{y\} \cap g\alpha cl\{z\} = \phi$. Since $x \in g\alpha cl\{z\}$, $g\alpha cl\{x\} \subset g\alpha cl\{z\}$ and $\{y\} \cap g\alpha cl\{x\} = \phi$. Therefore $g\alpha cl\{x\} \neq g\alpha cl\{y\}$. Now $Ker_{\{g\alpha\}}\{x\} \neq Ker_{\{g\alpha\}}\{y\} \Rightarrow g\alpha cl\{x\} \neq g\alpha cl\{y\}$.

(ii) ⇒ (i): If $g\alpha cl\{x\} \neq g\alpha cl\{y\}$. Then $\exists$ $z \in X$ such that $z \in g\alpha cl\{x\}$ and $z \notin g\alpha cl\{y\}$. Then $\exists$ a gα-open set containing z and therefore containing x but not y, namely, $y \notin Ker_{\{g\alpha\}}\{x\}$. Hence $Ker_{\{g\alpha\}}\{x\} \neq Ker_{\{g\alpha\}}\{y\}$.

Definition 4.2: X is said to be
(i) gα-R_0 iff $g\alpha cl\{x\} \subseteq G$ whenever $x \in G \in g\alpha O(X)$.
(ii) weakly gα-R_0 iff $\cap\, g\alpha cl\{x\} = \phi$.
(iii) gα-R_1 iff for $x, y \in X$ such that $g\alpha cl\{x\} \neq g\alpha cl\{y\}$ $\exists$ disjoint U; $V \in g\alpha O(X)$ such that $g\alpha cl\{x\} \subseteq U$ and $g\alpha cl\{y\} \subseteq V$.

Example 4.1: Let X = {a, b, c, d} and $\tau = \{\phi, \{b\}, \{a, b\}, \{b, c\}, \{a, b, c\}, X\}$, then X is not weakly gα R_0 and not gαR_i, i = 0, 1.

Remark 4.1:
(i) r-$R_i \Rightarrow R_i \Rightarrow \alpha R_i \Rightarrow g\alpha R_i$, i = 0, 1.
(ii) Every weakly-R_0 space is weakly gα R_0.

Lemma 4.1: Every gαR_0 space is weakly gαR_0.

Converse of the above Theorem is not true in general by the following Examples.

Example 4.2: Let X = {a, b, c, d} and $\tau = \{\phi, \{a\}, \{b\}, \{d\}, \{a, b\}, \{a, d\}, \{b, d\}, \{a, b, c\}, \{a, b, d\}, X\}$. Clearly, X is weakly gαR_0, since $\cap\, g\alpha cl\{x\} = \phi$. But it is not gα$R_0$, for $\{a\} \subset X$ is gα-open and $g\alpha cl\{a\} = \{a, b\} \not\subset \{a\}$.

Theorem 4.1: Every gα-regular space X is gα_2 and gα-R_0.

Proof: Let X be gα-regular and let $x \neq y \in X$. By Lemma 4.1, {x} is either gα-open or gα-closed. If {x} is gα-open, {x} is gα-open and hence gα-clopen. Thus {x} and X - {x} are separating gα-open sets. Similarly for {x} is gα-closed, {x} and X - {x} are separating gα-closed sets. Thus X is gα_2 and gα-R_0.

Theorem 4.2: X is gα-R_0 iff given $x \neq y \in X$; $g\alpha cl\{x\} \neq g\alpha cl\{y\}$.

Proof: Let X be gα-R_0 and let let $x, \neq y \in X$. Suppose U is a gα-open set containing x but not y, then $y \in g\alpha cl\{y\} \subset X - U$ and so $x \notin g\alpha cl\{y\}$. Hence $g\alpha cl\{x\} \neq g\alpha cl\{y\}$.

Conversely, let x, ≠ y∈ X such that gαcl{x} ≠ gαcl{y}⇒ gαcl{x}⊂ X- gαcl{y} = U(say) a gα-open set in X. This is true for every gαcl{x}. Thus ∩ gαcl{x}⊆ U where x∈ gαcl{x}⊆ U∈ gαO(X), which in turn implies ∩ gαcl{x}⊆ U where x∈ U∈ gαO(X). Hence X is $g\alpha R_0$.

Theorem 4.3: X is weakly $g\alpha R_0$ iff $Ker_{\{g\alpha\}}\{x\} \neq X$ for any x∈X.

Proof: Let x_0∈ X such that $ker_{\{g\alpha\}}\{x_0\} = X$. This means that x_0 is not contained in any proper gα-open subset of X.

Thus x_0 belongs to the gα-closure of every singleton set. Hence x_0∈∩ gαcl{x}, a contradiction.

Conversely assume $Ker_{\{g\alpha\}}\{x\} \neq X$ for any x∈ X. If there is an x_0∈ X such that x_0∈∩{ gαcl{x}}, then every gα-open set containing x_0 must contain every point of X. Therefore, the unique gα-open set containing x_0 is X. Hence $Ker_{\{g\alpha\}}\{x_0\} = X$, which is a contradiction. Thus X is weakly gα-R_0.

Theorem 4.4: The following are equivalent:
(i) X is gα-R_0 space.
(ii) For each x∈ X, gαcl{x}⊂ $Ker_{\{g\alpha\}}\{x\}$
(iii)For any gα-closed set F and a point x∉ F, ∃ U∈ gαO(X) such that x∉U and F⊂U.
(iv) Each gα-closed set F can be expressed as F = ∩{G: G is gα-open and F⊂G}.
(v) Each gα-open set G can be expressed as G = ∪{A: A is gα-closed and A⊂G}.
(vi) For each gα-closed set F, x∉ F implies gα-cl{x}∩ F = ϕ.

Proof: **(i) ⇒ (ii)** For any x∈ X, we have $Ker_{\{g\alpha\}}\{x\}$ = ∩{U: U∈ gαO(X) and x∈U}. Since X is gα-R_0, each gα-open set containing x contains gαcl{x}. Hence gαcl{x}⊂ $Ker_{\{g\alpha\}}\{x\}$.

(ii) ⇒ (iii) Let x∉F∈ gαC(X). Then for any y∈F; gαcl{y}⊂F and so x∉ gαcl{y}⇒ y∉ gαcl{x} that is ∃ U_y∈ gαO(X) such that y∈U_y and x∉U_y ∀ y∈F. Let U = ∪{U_y: U_y is gα-open, y∈U_y and x∉U_y}. Then U is gα-open such that x∉U and F⊂U.

(iii) ⇒ (iv) Let F be any gα-closed set and N = ∩{G: G is gα-open and F⊂ G}. Then F⊂N → (1).

Let x∉ F, then by (iii) ∃ G∈ gαO(X) such that x∉G and F⊂G.

Hence x∉N which implies x∈N ⇒ x∈F. Hence N⊂F → (2).

Therefore from (1) and (2), each gα-closed set F = ∩{G: G is gα-open and F⊂G}

(iv) ⇒ (v) obvious.

(v) ⇒ (vi) Let x∉F∈ gαC(X). Then X-F = G is a gα-open set containing x. Then by (v), G can be expressed as the union of gα-closed sets A contained in G, and so there is an M∈ gαC(X) such that x∈M⊂G; and hence gαcl{x}⊂G which implies gαcl{x}∩ F = ϕ.

(vi) ⇒ (i) Let x∈G∈ gαO(X). Then x∉(X-G), which is a gα-closed set. Therefore by (vi) gαcl{x}∩(X-G) = ϕ, which implies that gαcl{x}⊆ G. Thus X is $g\alpha R_0$ space.

Theorem 4.5: Let *f*: X → Y be a gα-closed one-one function. If X is weakly gα-R_0, then so is Y.

Theorem 4.6: If X is weakly gα-R_0, then for every space Y, X× Y is weakly gα-R_0.

Proof: ∩ gαcl{(x,y)} ⊆∩{ gαcl{x}× gαcl{y}} = ∩ [gαcl{x}]× [gαcl{y}] ⊆ ϕ× Y = ϕ. Hence X× Y is $g\alpha R_0$.

Corollary 4.1:
(i) If X and Y are weakly $g\alpha R_0$, then X× Y is weakly $g\alpha R_0$.
(ii) If X and Y are (weakly-)R_0, then X× Y is weakly $g\alpha R_0$.
(iii)If Xand Y are $g\alpha R_0$, then X× Y is weakly $g\alpha R_0$.
(iv) If X is $g\alpha R_0$ and Y are weakly R_0, then X× Y is weakly $g\alpha R_0$.

Theorem 4.7: X is $g\alpha R_0$ iff for any x, y∈ X, gαcl{x} ≠ gαcl{y}⇒ gαcl{x}∩ gαcl{y} = ϕ.

Proof: Let X is $g\alpha R_0$ and x, y∈X such that gαcl{x} ≠ gαcl{y} .Then ∃ z∈ gαcl{x} such that z∉ gαcl{y} (or z∈gαcl{y}) such that z∉gαcl{x}. There exists V∈gαO(X) such that y∉V and z∈V; hence x∈V. Therefore, x∉gαcl{y}.

Thus x∈[gαcl{y}]^c∈gαO(X), which implies gαcl{x}⊂[gαcl{y}]^c and gαcl{x}∩gαcl{y} = ϕ. The proof for otherwise is similar.

Sufficiency: Let x∈V∈gαO(X). We show that gαcl{x}⊂V. Let y∉V, i.e., y∈V^c. Then x ≠ y and x∉ gαcl{y}. Hence gαcl{x} ≠ gαcl{y}. But gαcl{x}∩ gαcl{y} = ϕ. Hence y∉ gαcl{x}. Therefore gαcl{x}⊂V.

Theorem 4.8: X is $g\alpha R_0$ iff for any points x, y∈X, $Ker_{\{g\alpha\}}\{x\} \neq Ker_{\{g\alpha\}}\{y\} \Rightarrow Ker_{\{g\alpha\}}\{x\} \cap Ker_{\{g\alpha\}}\{y\} = \phi$.

Proof: Suppose X is $g\alpha R_0$. Thus by Lemma 4.3 for any x, y∈X if $Ker_{\{g\alpha\}}\{x\} \neq Ker_{\{g\alpha\}}\{y\}$ then gαcl{x} ≠ gαcl{y}. Assume that $z \in Ker_{\{g\alpha\}}\{x\} \cap Ker_{\{g\alpha\}}\{y\}$. By $z \in Ker_{\{g\alpha\}}\{x\}$ and Lemma 4.2, it follows that x∈ gαcl{z}. Since x∈ gαcl{z}, gαcl{x} = gαcl{z}. Similarly, we have gαcl{y} = gαcl{z} = gαcl{x}. This is a contradiction. Therefore, we have $Ker_{\{g\alpha\}}\{x\} \cap Ker_{\{g\alpha\}}\{y\} = \phi$.

Conversely, let x, y∈X, s.t. gαcl{x} ≠ gαcl{y}, then by Lemma 4.3, $Ker_{\{g\alpha\}}\{x\} \neq Ker_{\{g\alpha\}}\{y\}$. Hence by hypothesis $Ker_{\{g\alpha\}}\{x\} \cap Ker_{\{g\alpha\}}\{y\} = \phi$ which implies gαcl{x}∩ gαcl{y} = ϕ Because z∈ gαcl{x} implies that $x \in Ker_{\{g\alpha\}}\{z\}$ and therefore $Ker_{\{g\alpha\}}\{x\} \cap Ker_{\{g\alpha\}}\{z\} \neq \phi$ Therefore by Theorem 4.7 X is a $g\alpha R_0$ space.

Theorem 4.9: The following are equivalent:
(i) X is a gα-R_0 space.
(ii) For any A ≠ ϕ and G∈ gαO(X) such that A∩G ≠ ϕ ∃ F∈ gαC(X)such that A∩F ≠ ϕ and F⊂G.

Proof: (i) ⇒ (ii): Let A ≠ ϕ and G∈ gαO(X) such that A∩G ≠ ϕ. There exists x∈A∩G. Since x∈G∈ gαO(X), gαcl{x}⊂G. Set F = gαcl{x}, then F∈ gαC(X), F⊂G and A∩F ≠ ϕ

(ii) ⇒ (i): Let G∈ gαO(X) and x∈ G. By (2), gαcl{x}⊂ G. Hence X is gα-R_0.

Theorem 4.10: The following are equivalent:
(i) X is a gα-R_0 space;
(ii) x∈ gαcl{y} iff y∈ gαcl{x}, for any points x and y in X.

Proof: (i) ⇒ (ii): Assume X is $g\alpha R_0$. Let x∈ gαcl{y} and D be any gα-open set such that y∈D. Now by hypothesis, x∈D. Therefore, every gα-open set which contain y contains x. Hence y∈ gαcl{x}.

(ii) ⇒ (i): Let U be a gα-open set and x∈U. If y∉ U, then x∉ gαcl{y} and hence y∉ gαcl{x}. This implies that gαcl{x}⊂U. Hence X is $g\alpha R_0$.

Theorem 4.11: The following are equivalent:
(i) X is a $g\alpha R_0$ space;
(ii) If F is gα-closed, then F = $Ker_{\{g\alpha\}}$ (F);
(iii) If F is gα-closed and x∈ F, then $Ker_{\{g\alpha\}}\{x\} \subseteq F$;
(iv) If x∈ X, then $Ker_{\{g\alpha\}}\{x\} \subset$ gαcl{x}.

Proof: (i) ⇒ (ii): Let x∉F∈ gαC(X) ⇒ (X-F)∈ gαO(X) and contains x. For X is $g\alpha R_0$, gαcl({x})⊂(X-F). Thus gαcl({x})∩ F = ϕ and x∉$Ker_{\{g\alpha\}}$ (F). Hence $Ker_{\{g\alpha\}}$ (F) = F.

(ii) ⇒ (iii): A⊂B ⇒ $Ker_{\{g\alpha\}}$ (A)⊂$Ker_{\{g\alpha\}}$ (B). Therefore, by (2) $Ker_{\{g\alpha\}}\{x\} \subset Ker_{\{g\alpha\}}$ (F) = F.

(iii) ⇒ (iv): Since x∈ gαcl{x} and gαcl{x} is gα-closed, by (3) $Ker_{\{g\alpha\}}\{x\} \subset$ gαcl{x}.

(iv) ⇒ (i): Let x∈ gαcl{y}. Then by Lemma 4.2 $y \in Ker_{\{g\alpha\}}\{x\}$. Since x∈ gαcl{x} and gαcl{x} is gα-closed, by (iv) we obtain $y \in Ker_{\{g\alpha\}}\{x\} \subseteq$ gαcl{x}. Therefore x∈ gαcl{y} implies y∈ gαcl{x}. The converse is obvious and X is $g\alpha R_0$.

Corollary 4.2: The following are equivalent:
(i) X is $g\alpha R_0$.
(ii) gαcl{x} = $Ker_{\{g\alpha\}}\{x\}$∀ x∈ X.

Proof: Follows from Theorems 4.4 and 4.11.

Recall that a filterbase F is called gα-convergent to a point x in X, if for any gα-open set U of X containing x, there exists B∈ F such that B⊂ U.

Lemma 4.4: Let x and y be any two points in X such that every net in X gα-converging to y gα-converges to x. Then x∈ gαcl{y}.

Theorem 4.12: The following are equivalent:
(i) X is a $g\alpha R_0$ space;
(ii) If x, y∈X, then y∈ gαcl{x} iff every net in X gα-converging to y gα-converges to x.

Proof:
(i) ⇒ (ii): Let x, y∈X such that y∈ gαcl{x}. Suppose that $\{x_\alpha\}_{\alpha\in I}$ is a net in X such that $\{x_\alpha\}_{\alpha\in I}$ gα-converges to y. Since y∈ gαcl{x}, by Thm. 4.7 we have gαcl{x} = gαcl{y}. Therefore x∈ gαcl{y}. This means that $\{x_\alpha\}_{\alpha\in I}$ gα-converges to x.

Conversely, let x, y∈ X such that every net in X gα-converging to y gα-converges to x. Then x∈ gα-cl{y}[by 4.4]. By Thm. 4.7, we have gαcl{x} = gαcl{y}. Therefore y∈ gαcl{x}.

(ii) ⇒ (i): Let x, y∈X such that gαcl{x}∩ gαcl{y} ≠ ϕ. Let z∈ gαcl{x}∩ gαcl{y}. So ∃ a net $\{x_\alpha\}_{\alpha\in I}$ in gαcl{x} such

that $\{x_\alpha\}_{\alpha\in I}$ gα-converges to z. Since z∈ gαcl{y}, then $\{x_\alpha\}_{\alpha\in I}$ gα-converges to y. It follows that y∈ gαcl{x}. Similarly we obtain x∈ gαcl{y}. Therefore gαcl{x} = gαcl{y}. Hence X is $g\alpha R_0$.

Theorem 4.13:
(i) Every subspace of $g\alpha R_1$ space is again $g\alpha R_1$.
(ii) Product of any two $g\alpha R_1$ spaces is again $g\alpha R_1$.
(iii) X is $g\alpha R_1$ iff given x ≠ y∈ X, gαcl{x} ≠ gαcl{y}.
(iv) Every $g\alpha_2$ space is $g\alpha R_1$.

The converse of 4.13(iv) is not true. However, we have the following result.

Theorem 4.14: Every $g\alpha_1$ and $g\alpha R_1$ space is $g\alpha_2$.

Proof: Let x ≠ y∈X. Since X is $g\alpha_1$; {x} and {y} are gα-closed sets such that gαcl{x} ≠ gαcl{y}. Since X is $g\alpha R_1$, there exists disjoint gα-open sets U and V such that x∈U; y∈V. Hence X is $g\alpha_2$.

Corollary 4.3: X is $g\alpha_2$ iff it is $g\alpha R_1$ and $g\alpha_1$.

Theorem 4.15: The following are equivalent
(i) X is gα-R_1.
(ii) ∩ gαcl{x} = {x}.
(iii)For any x∈ X, intersection of all gα-neighborhoods of x is {x}.

Proof:
(i) ⇒ (ii) Let y ≠ x∈X such that y∈ gαcl{x}. Since X is $g\alpha R_1$, ∃ U∈ gαO(X) such that y∈U, x∉U or x∈U, y∉U. In either case y∉ gαcl{x}. Hence ∩ gαcl{x} = {x}.

(ii) ⇒ (iii) If y ≠ x∈X, then x∉∩ gαcl{y}, so there is a gα-open set containing x but not y. Therefore y does not belong to the intersection of all gα-neighborhoods of x. Hence intersection of all gα-neighborhoods of x is {x}.

(iii) ⇒ (i) Let x ≠ y∈X. by hypothesis, y does not belong to the intersection of all gα-neighborhoods of x and x does not belong to the intersection of all gα-neighborhoods of y, which implies gαcl{x} ≠ gαcl{y}. Hence X is gα-R_1.

Theorem 4.16: The following are equivalent:
(i) X is gα-R_1.
(ii) For each pair x, y∈X with gαcl{x} ≠ gαcl{y}, ∃ a gα-open, gα-closed set V s.t. x∈V and y∉V, and
(iii)For each pair x, y∈X with gαcl{x} ≠ gαcl{y}, ∃ *f*: X→[0, 1] s.t. *f*(x) = 0 and *f*(y) = 1 and *f* is gα-continuous.

Theorem 4.17:
(i) If X is gα-R_1, then X is gα-R_0.
(ii) X is gα-R_1 iff for x, y∈ X, $Ker_{\{g\alpha\}}\{x\} \neq Ker_{\{g\alpha\}}\{y\}$, ∃ disjoint U; V∈ gαO(X) such that gαcl{x}⊂U and gαcl{y}⊂V.

5. gα-C_i and gα-D_i spaces, i = 0,1,2:

Definition 5.1: X is said to be a
(i) gα-C_0 space if for each pair of distinct points x, y of X there exists a gα-open set G whose closure contains either of the point x or y.
(ii) gα-C_1 [resp: gα-C_2] space if for each pair of distinct points x, y of X there exists disjoint gα-open sets G and H such that closure of G containing x but not y and closure of H containing y but not x.

Note 4: gα-C_2 ⇒ gα-C_1 ⇒ gα-C_0. Converse need not be true in general:

Example 5.1:
(i) Let X = {a, b, c} and τ = {ϕ, X}, then X is gαC_i for i = 0, 1, 2.
(ii) Let X = {a, b, c, d} and τ = {ϕ, {b}, {a, b}, {b, c}, {a, b, c}, X} then X is not gαC_i for i = 0, 1, 2.

Theorem 5.1:
(i) Every subspace of gα-C_i space is gα-C_i.
(ii) Every $g\alpha_i$ spaces is gα-C_i.
(iii) Product of gα-C_i spaces are gα-C_i.
(iv) Let X be any gα-C_i space and A⊂X then A is gα-C_i iff (A, $\tau_{/A}$) is gα-C_i.
(v) If X is gα-C_1 then each singleton set is gα-closed.
(vi) In an gα-C_1 space disjoint points of X has disjoint gα- closures.

Definition 5.2: A⊂X is called a gα-Difference(Shortly gαD-set) set if there are two U, V∈ gαO(X) such that U ≠ X and A = U-V.

Clearly every gα-open set U different from X is a gαD-set if A = U and V = ϕ.

Definition 5.3: X is said to be a
(i) gα-D_0 if for any pair of distinct points x and y of X there exist a gαD-set in X containing x but not y or a gαD-set in X containing y but not x.
(ii) gα-D_1 [resp: gα-D_2]if for any pair of distinct points x and y in X there exists disjoint gαD-sets G and H in X containing x and y respectively.

Remark 5.2: (i) If X is rT_i, then it is $g\alpha_i$, i = 0, 1, 2 and converse is false.
(ii) If X is $g\alpha_i$, then it is $g\alpha_{\{i-1\}}$, i = 1, 2.
(iii) If X is $g\alpha_i$, then it is gα-D_i , i = 0, 1, 2.
(iv) If X is gα-D_i, then it is gα-$D_{\{i-1\}}$, i = 1, 2.

Theorem 5.2: The following statements are true:
(i) X is gα-D_0 iff it is $g\alpha_0$.
(ii) X is gα-D_1 iff it is gα-D_2.

Corollary 5.1: If X is gα-D_1, then it is $g\alpha_0$.

Proof: Remark 5.1(iv) and Theorem 5.1(vi)

Definition 5.4: A point x∈ X which has X as the unique gα-neighborhood is called gα.c.c point.

Theorem 5.3: For an $g\alpha_0$ space X the following are equivalent:
(i) X is gα-D_1;
(ii) X has no gα.c.c point.

Proof: (i) ⇒ (ii) Since X is gα-D_1, then each point x of X is contained in a gαD-set O = U - V and thus in U. By Definition U ≠ X. This implies that x is not a gα.c.c point.

(ii) ⇒ (i) If X is $g\alpha_0$, then for each x ≠ y∈ X, at least one of them, x (say) has a gα-neighborhood U containing x and not y. Thus U which is different from X is a gαD-set. If X has no gα.c.c point, then y is not a gα.c.c point. This means that there exists a gα-neighborhood V of y such that V ≠ X. Thus y∈ V-U but not x and V-U is a gαD-set. Hence X is gα-D_1.

Definition 5.5: X is gα-symmetric if for x and y in X, x ∈ gαcl{y} implies y ∈ gαcl{x}.

Theorem 5.4: X is gα-symmetric iff {x} is gα-closed for each x∈ X.

Proof: Assume that x∈ gαcl{y} but y∉ gαcl{x}. This means that $[g\alpha cl\{x\}]^c$ contains y. This implies that gαcl{y}⊂ $[g\alpha cl\{x\}]^c$. Now $[g\alpha cl\{x\}]^c$ contains x which is a contradiction.
Conversely, suppose that {x}⊂E∈gαO(X) but gαcl{x}⊄E. This means that gαcl{x} and E^c are not disjoint. Let y belongs to their intersection. Now we have x∈ gαcl{y}⊂E^c and x∉E. But this is a contradiction.

Corollary 5.2: If X is a $g\alpha_1$, then it is gα-symmetric.

Proof: Follows from Theorem 2.2(ii) and Theorem 5.4.

Corollary 5.3: The following are equivalent:
(i) X is gα-symmetric and $g\alpha_0$
(ii) X is $g\alpha_1$.

Proof: By Corollary 5.2 and Remark 5.1 it suffices to prove only (i) ⇒ (ii). Let x ≠ y and by $g\alpha_0$, we may assume that $x\in G_1\subset\{y\}^c$ for some $G_1\in g\alpha O(X)$.Then x∉gαcl{y} and hence y∉gαcl{x}. There exists a $G_2\in g\alpha O(X)$ such that $y\in G_2 \subset\{x\}^c$ and X is a $g\alpha_1$ space.

Theorem 5.5: For a gα-symmetric space X the following are equivalent:
(i) X is $g\alpha_0$; (ii) X is gα-D_1; (iii) X is $g\alpha_1$.

Proof: (i) ⇒ (iii) Corollary 5.3 and **(iii) ⇒ (ii) ⇒ (i)** Remark 5.1.

Theorem 5.6: If *f*: X→ Y is a gα-irresolute surjective function and E is a gαD-set in Y, then the inverse image of E is a gαD-set in X.

Proof: Let E be a gαD-set in Y. Then there are gα-open sets U_1 and U_2 in Y such that E = U_1 - U_2 and U_1 ≠ Y. By the gα-irresoluteness of *f*, $f^{-1}(U_1)$ and $f^{-1}(U_2)$ are gα-open in X. Since U_1 ≠ Y, we have $f^{-1}(U_1)$ ≠ X.

Hence $f^{-1}(E) = f^{-1}(U_1) - f^{-1}(U_2)$ is a gα-D-set.

Theorem 5.7: (i) If Y is gα-D_1 and *f*: X → Y is gα-irresolute and bijective, then X is gα-D_1.
(ii) X is gα-D_1 iff for each pair of x ≠ y in X there exist a gα-irresolute surjective function *f*: X→ Y, where Y is a gα-D_1 space such that *f*(x) and *f*(y) are distinct.

Corollary 5.4: Let $\{X_\alpha/\alpha\in I\}$ be any family of spaces. If X_α is gα-D_1 for each α∈I, then ΠX_α is gα-D_1.

Proof: Let $(x_\alpha) \neq (y_\alpha)$ in ΠX_α. Then there exists an index β∈I s. t. $x_\beta \neq y_\beta$. The natural projection $P_\beta: \Pi X_\alpha \to X_\beta$ is almost continuous and almost open and $P_\beta((x_\alpha)) = P_\beta((y_\alpha))$. Since X_β is gα-D_1, ΠX_α is gα-D_1.

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