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# On Legendre Curve in an LP-Sasakian manifold with coefficient $\alpha$

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***ABSTRACT***

*In the present paper the torsion of the Legendre curve in an LP-Sasakian manifold with a coefficient $\alpha$ is obtained. Necessary and Sufficient conditions for Legendre curve having parallel mean curvature vector, having proper mean curvature vector and biharmonic condition is derived. It is also obtained that the necessary and sufficient condition for a Legendre curve in an LP-Sasakian manifold with coefficient $\alpha$ is of type $AW(k), k = 1,2,3$.*

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## INTRODUCTION

Dimitric (1992) studied biharmonic curves in the Euclidean space and proved that the only biharmonic curves in an Euclidean space are straight line, i.e. totally geodesic. Chen (1991) defined a Riemannian submanifold with vanishing Laplacian of mean curvature vector $\Delta H$ as a biharmonic submanifold.

The class of submanifolds satisfying the following condition in Riemannian space form has been investigated by many geometers Barros (1995), and Chen (1996),

$$\Delta H = \lambda H$$

$$\Delta^{\perp} H = \lambda H$$

where $\lambda$ is the real valued fuction, $H$ is the mean curvature vector field and $\Delta^{\perp}$ denotes the Laplacian of the curve in the normal bundle. The Laplacian in the normal bundle of $H$, $\Delta^{\perp} H$, is an ingredient of the normal part of $\Delta H$ to $M^n$ and $\Delta^{\perp} H = 0$ is less restrictive then $\Delta H = 0$. Arslan and Özgür (1999) investigated the curve of type $AW(k)$and the concepts of submanifolds of type $AW(k)$ are defined by Arslan and West (1995).

Blair and Baikoussis (1994) introduced the notion of Legendre curves in a contact metric manifold. Recently, De and his co-authors (2002) studied the LP-Sasakian manifolds with a coefficient $\alpha$ and obtained several interesting result in this manifold.

The object of the present paper is to study Legendre curves in an LP-Sasakian manifold with coefficient $\alpha$. After preliminaries, in section 2 it is proved that a Legendre curve in an LP-Sasakian manifold with coefficient $\alpha$ is a Frenet curve of order 3 and its torsion is always $\alpha$. Section 3, gives the necessary and sufficient condition for a Legendre curves in an LP-Sasakian manifold with coefficient $\alpha$ having parallel and proper mean curvature vector. In section 4, it is find that the necessary and sufficient condition for a Legendre curve in an LP-Sasakian manifold with coefficient $\alpha$ is of type $AW(1), AW(2)$ and also $AW(3)$.

## 1. PRELIMINARIES

Let $M^n$ be an $n$-dimensional differentiable manifold endowed with a (1,1) tensor field $\varphi$, a contravariant vector field $\xi$, a covariant vector field $\eta$ and a Lorentzian metric $g$ of type (0,2) such that for each $p \epsilon M^n$, the tensor $g_p: T_p M \times T_p M \to R$ is a non-degenerate inner product of signature $(-, +, +, \dots, +)$, where $T_p M$ denote the tangent vector space of $M^n$ at $p$ and $R$ is the real number space, which satisfies

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$\eta(\xi) = -1$ , $\phi^2 X = X + \eta(X)\xi$, (1.1)

$g(X,\xi) = \eta(X)$, $g(\phi X, \phi Y) = g(X,Y) + \eta(X)\eta(Y)$ (1.2)

for all vector field $X$, $Y$. Then such a structure $(\phi,\xi,\eta,g)$ is called Lorentzian almost paracontact structure and the manifold $M^n$ with the structure $(\phi,\xi,\eta,g)$ is called Lorentzian almost paracontact manifold [13]. In the Lorentzian almost paracontact manifold $M$, the following relations holds [13]:

$\phi\xi = 0$, $\eta(\phi X) = 0$, (1.3)

$\Omega(X,Y) = \Omega(Y,X)$, where $\Omega(X,Y) = g(X,\phi Y)$. (1.4)

In the Lorentzian almost paracontact manifold $M^n$, if the relations

$\nabla_Z\Omega(X,Y) = \alpha[\{g(X,Z) + \eta(X)\eta(Z)\}\eta(Y) + \{g(Y,Z) + \eta(Y)\eta(Z)\}\eta(X)]$, $(\alpha \neq 0)$ (1.5)

and $\Omega(X,Y) = \frac{1}{\alpha}(\nabla_X\eta)Y$, (1.6)

hold, where $\nabla$ denote the operator of covariant differentiation with respect to the Lorentzian metric $g$, then $M^n$ is called an LP-Sasakian manifold with a coefficient $\alpha$ [11]. An LP-Sasakian manifold with a coefficient 1 is called LP-Sasakian manifold [13].

In a Lorentzian manifold $M^n$, if we assume that $\xi$ is a unit torse-forming vector field, then we have the following:

$(\nabla_X\eta)Y = \alpha\{g(X,Y) + \eta(X)\eta(Y)\}$, (1.7)

where $\alpha$ is non-zero scalar function.

Let $\gamma(s)$ be a curve in a Riemannian manifold $M^n$ parameterized by the arc length. The curve $\gamma$ is called a Frenet curve of order $r$ if there exist orthonormal vector fields $E_1, \ldots, E_r$ along $\gamma$ such that

$\dot\gamma = E_1$, $\nabla_{\dot\gamma}E_1 = k_1E_2$, $\nabla_{\dot\gamma}E_2 = -k_1E_1 + k_2E_3$, …..,$\nabla_{\dot\gamma}E_r = -k_{r-1}E_{r-1}$,

where $k_1, \ldots k_{r-1}$ are positive smooth function of s, and $\nabla$ is Levi-Civita connection.

A 1-dimentional integral submanifold of a contact manifold is called a Legendre curve. Belkhelfa and its co-author (2002) proved that a 3-dimentional contact metric manifold is Sasakian if and only if the torsion of its Legendre curves is equal to 1. In [3], it was also shown that for a 3-dimensional manifold $M^n$ endowed with the contact metric structure $(\phi,\xi,\eta,g,\epsilon)$, $M^n$ is Sasakian if and only if the torsion of its Legendre curves is equal to $\epsilon$.

## 2. LEGENDRE CURVE IN AN LP-SASAKIAN MANIFOLD WITH COEFFICIENT *α:*

Let $\gamma(s)$ be a curve in an LP-Sasakian manifold with a coefficient $\alpha$. Then its associated Frenet frame is $\{\dot\gamma,\ \phi\dot\gamma,\ \xi\}$, so that we have the following equations:

$\nabla_{\dot\gamma}\dot\gamma = k\phi\dot\gamma$ (2.1)

$\nabla_{\dot\gamma}\phi\dot\gamma = -k\dot\gamma + \alpha\xi$ (2.2)

$\nabla_{\dot\gamma}\xi = \alpha\phi\dot\gamma$ (2.3)

Hence, we have the following:

**Theorem (2.1):** In an LP-Sasakian manifold with a coefficient $\alpha$, a Legendre curve is Frenet curve of order 3 and its torsion is always an $\alpha$.

By virtue of (2.1), (2.2) and (2.3) we can state the following Lemma.

**Lemma (2.1):** Let $\gamma(s)$ be a curve in an LP-Sasakian manifold with a coefficient $\alpha$, then

$\nabla_{\dot\gamma}\nabla_{\dot\gamma}\dot\gamma = k^2\dot\gamma + \dot k\phi\dot\gamma + \alpha k\xi$ (2.4)

$\nabla_{\dot\gamma}\nabla_{\dot\gamma}\nabla_{\dot\gamma}\dot\gamma = 3k\dot k\dot\gamma + (k^3 + \ddot k + \alpha^2 k)\phi\dot\gamma + 2\alpha\dot k\xi$ (2.5)

 

**Definition:** Chen (1984) proved the following identity:

$\Delta H = -\nabla_{\dot\gamma}\nabla_{\dot\gamma}\nabla_{\dot\gamma}\dot\gamma,$

where $H$ is the mean curvature vector. Moreover, the Laplacian of the mean curvature in the normal bundle is defined by

$\Delta^{\perp} H = -\Delta^{\perp}_{\dot\gamma}\Delta^{\perp}_{\dot\gamma}\ \Delta^{\perp}_{\dot\gamma}\ \dot\gamma,$

where $\Delta^{\perp}$ denotes the normal connection in the normal bundle.

A curve $\gamma(s)$ in a Riemannian manifold $M^n$ is called a curve with proper mean curvature vector field [9] if $\Delta H = \lambda H$, where $\lambda$ is a function. In particular, if $\Delta H = 0$, then it becomes a biharmonic curve [10].

A curve $\gamma(s)$ is known to be a curve with proper mean curvature vector field in the normal bundle [4] if $\Delta^{\perp} H = \lambda H$, where $\Delta^{\perp} H$ is the Laplacian of the mean curvature in the normal bundle and $\lambda$ is a function. In particular, if $\Delta^{\perp} H = 0$ then it reduces to a curve with harmonic mean curvature vector field in the normal bundle [4].

**Theorem (2.2):** Let $\gamma(s)$ be a curve in an LP-Sasakian manifold with a coefficient $\alpha$, then $\gamma(s)$ has parallel mean curvature vector field if and only if $k = 0$.

**Proof:** The proof is obvious from (2.4).

**Theorem (2.3):** Let $\gamma(s)$ be a curve in an LP-Sasakian manifold with a coefficient $\alpha$, then $\gamma(s)$ is a curve with proper mean curvature vector field if and only if either $k = 0$ or $\lambda$ is constant equal to$-(\alpha^2 + k^2)$.

**Proof:** In view of (2.5), the condition $\Delta H = \lambda H$ gives

$$-3k\dot{k}\dot\gamma - \left(k^3 + \ddot{k} + \alpha^2 k\right)\phi\dot\gamma - 2\alpha\dot{k}\xi = \lambda k\phi\dot\gamma, \tag{2.6}$$

which implies that

(1) $k\dot{k} = 0$
(2) $k^3 + \ddot{k} + \alpha^2 k + \lambda k = 0$
(3) $\alpha\dot{k} = 0$

From (3), we have $k = c$, where $c$ is a constant. Then in view of (2), we find that either $c = 0\ or \quad \lambda = -(c^2 + k^2)$.

The converse is straightforward.

As a corollary, we have the following result:

**Corollary (2.1):** A Legendre curve in an LP-Sasakian manifold with coefficient $\alpha$ is biharmonic if and only if its curvature is zero.

**Theorem (2.4):** Let $\gamma(s)$ be a curve in an LP-Sasakian manifold with a coefficient $\alpha$, then $\gamma(s)$ is a curve with proper mean curvature vector field in the normal bundle if and only if either $k = 0$ or $k$ is non zero constant and $\lambda = -\alpha^2$.

**Proof:** From (2.4), we have

$$\left(\Delta_{\dot\gamma} H\right)^{\perp} = \dot{k}\phi\dot\gamma + \alpha k\xi \tag{2.7}$$

From the above equation, we obtained the following equation.

$$\Delta_{\dot\gamma}\left(\left(\Delta_{\dot\gamma} H\right)^{\perp}\right) = k\dot{k}\dot\gamma + \left(\ddot{k} + \alpha^2 k\right)\phi\dot\gamma + 2\alpha\dot{k}\xi,$$

which gives

$$\Delta^{\perp} H = -\left(\ddot{k} + \alpha^2 k\right)\phi\dot\gamma - 2\alpha\dot{k}\xi. \tag{2.8}$$

 

The condition $\Delta^{\perp}H = \lambda H$ gives

$$-(\ddot{k} + \alpha^2 k)\phi\dot{\gamma} - 2\alpha\dot{k}\xi = \lambda k\phi\dot{\gamma}$$

which implies that
(1) $\ddot{k} + \alpha^2 k + \lambda k = 0,$
(2) $\dot{k} = 0.$

From (2), it follows that $k$ is some constant $c$. Then from (1), it follows that $c(\alpha^2 + \lambda) = 0$, which implies that either $c = 0$ *or* $c \neq 0$ and $\lambda = -\alpha^2$.

The converse follows easily.

In particular, we can state the following:

**Corollary (2.2):** A Legendre curve in an LP-Sasakian manifold with coefficient $\alpha$ is with harmonic mean curvature vector field in the normal form if and only if $k = 0$.

## 3. LEGENDRE CURVE IN AN LP-SASAKIAN MANIFOLD WITH COEFFICIENT $A$ OF TYPE $AW(k)$, $k = 1, 2, 3$:

A Frenet curve $\gamma(s)$ is said to be [1]
(i) of type $AW(1)$ if $N_3 = 0$ (3.1)

(ii) of type $AW(2)$ if $\|N_2\|^2 N_3 = \langle N_3, N_2\rangle N_2$ (3.2)

(iii) of type $AW(3)$ if $\|N_1\|^2 N_3 = \langle N_3, N_1\rangle N_1$ (3.3)

where $N_1 = (\ddot{\gamma})^{\perp}$, $N_2 = (\dddot{\gamma})^{\perp}$, $N_3 = (\gamma^{iv})^{\perp}$.

Let $\gamma(s)$ be a Legendre curve in an LP-Sasakian manifold with a coefficient $\alpha$, then from (2.1), (2.4) and (2.5) we get

$$N_1 = k\phi\dot{\gamma} \tag{3.4}$$

$$N_2 = \dot{k}\phi\dot{\gamma} + \alpha k\xi \tag{3.5}$$

$$N_3 = (k^3 + \ddot{k} + \alpha^2 k)\phi\dot{\gamma} + 2\alpha\dot{k}\xi \tag{3.6}$$

**Theorem (3.1):** A Legendre curve in an LP-Sasakian manifold with a coefficient $\alpha$ is of type $AW(1)$ if and only if $k = 0$.

**Proof:** If a Legendre curve $\gamma(s)$ in an LP-Sasakian manifold with a coefficient $\alpha$ is of type $AW(1)$, then from (3.1) and (3.6) we have
(1)$k^3 + \ddot{k} + \alpha^2 k = 0,$
(2)$\dot{k} = 0.$

The statement (2) implies that $k$ is constant which in view of (1) becomes zero.

The converse is easily verified.

**Theorem (3.2):** A Legendre curve in an LP-Sasakian manifold with a coefficient $\alpha$ is of type $AW(2)$ if and only if either $k = 0$ or $k$ satisfies the differential equation

$$2\alpha\dot{k}^2 - \alpha k(k^3 + \ddot{k} + \alpha^2 k) = 0.$$

**Proof:** Putting the value from (3.5) and (3.6) in (3.2), we get

$$\{2\alpha^2 k\dot{k} + (k^3 + \ddot{k} + \alpha^2 k)\dot{k}\}\alpha k = (\dot{k}^2 + \alpha^2 k^2)2\alpha\dot{k} \tag{3.7}$$

$$\{2\alpha^2 k\dot{k} + (k^3 + \ddot{k} + \alpha^2 k)\dot{k}\}\dot{k} = (\dot{k}^2 + \alpha^2 k^2)(k^3 + \ddot{k} + \alpha^2 k). \tag{3.8}$$

If $k = 0$, then in view of (3.7) and (3.8), the Legendre curve becomes of type $AW(2)$. If $k \neq 0$ and the Legendre curve is of type $AW(2)$, then from (3.6) and (3.7), we obtain

$$\left(\dot{k}^2 + \alpha^2 k^2\right)\left\{2\alpha\dot{k}^2 - \alpha k\left(k^3 + \ddot{k} + \alpha^2 k\right)\right\} = 0 \tag{3.9}$$

Since $k \neq 0$ so $\left(\dot{k}^2 + \alpha^2 k^2\right)$ cannot vanish.

Therefore, we have

$$2\alpha\dot{k}^2 - \alpha k\left(k^3 + \ddot{k} + \alpha^2 k\right) = 0,$$

which proves the theorem.

**Theorem (3.3):** A Legendre curve in an LP-Sasakian manifold with a coefficient $\alpha$ is of type $AW(3)$ if and only if $k$ is constant.

**Proof:** In view of (3.4), (3.6) and (3.3) the condition for a Legendre curve in an LP-Sasakian manifold with a coefficient $\alpha$ is of type $AW(3)$ is equivalent to the following relation

$$k^2\left\{\left(k^3 + \ddot{k} + \alpha^2 k\right)\phi\dot{\gamma} + 2\alpha\dot{k}\xi\right\} = \left\{\left(k^3 + \ddot{k} + \alpha^2 k\right)k\right\}k\phi\dot{\gamma}$$

which gives $\dot{k} = 0$, i.e. $k$ is constant.

Hence the theorem.

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