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ON THE FORCING HULL AND FORCING GEODETIC NUMBERS OF GRAPHS

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ABSTRACT

In this paper, we prove that, for any non-negative integers a, b, c and d with $a < c < d$, $b < d$, $c > a + 1$ and $d > b + c - a$, there exists a connected graph G such that $f_h(G) = a$, $f_g(G) = b$, $h(G) = c$ and $g(G) = d$, where $f_h(G)$, $f_g(G)$, $h(G)$ and $g(G)$ are the forcing hull number, the forcing geodetic number, the hull number and the geodetic number of a graph respectively. This result solves a problem of Li-Da Tong [Li-Da Tong, The forcing hull and forcing geodetic numbers of graphs, Discrete Applied Mathematics, 157 (2009), 1159-1163].

Keywords: *hull number, geodetic number, forcing hull number, forcing geodetic number.*

AMS Subject Classification: *05C12.*

1. INTRODUCTION

By a graph $G = (V,E)$, we mean a finite undirected connected graph without loops or multiple edges. The order and size of G are denoted by p and q respectively. For basic graph theoretic terminology, we refer to Harary [1, 8]. A convexity on a finite set V is a family C of subsets of V, convex sets which are closed under intersection and which contains both V and the empty set. The pair (V, E) is called a convexity space. A finite graph convexity space is a pair (V, E), formed by a finite connected graph $G = (V, E)$ and a convexity C on V such that (V, E) is a convexity space satisfying that every member of C induces a connected subgraph of G. Thus, classical convexity can be extended to graphs in a natural way. We know that a set X of R^n is convex if every segment joining two points of X is entirely contained in it. Similarly a vertex set W of a finite connected graph is said to be convex set of G if it contains all the vertices lying in a certain kind of path connecting vertices of W[2,7]. The distance $d(u,v)$ between two vertices u and v in a connected graph G is the length of a shortest u–v path in G. An u–v path of length $d(u,v)$ is called an u–v geodesic. A vertex x is said to lie on a u-v geodesic P if x is a vertex of P including the vertices u and v. For two vertices u and v, let $I[u,v]$ denotes the set of all vertices which lie on $u - v$ geodesic. For a set S of vertices, let $I[S] = \cup_{u,v \in S} I[u, v]$. The set S is *convex* if $I[S] = S$. Clearly if $S = \{v\}$ or $S = V$, then S is convex. The *convexity* number, denoted by $C(G)$, is the cardinality of a maximum proper convex subset of V. The smallest convex set containing S is denoted by $I_h(S)$ and called the *convex hull* of S. Since the intersection of two convex sets is convex, the convex hull is well defined. Note that $S \subseteq I[S] \subseteq I_h(S) \subseteq V$. A subset $S \subseteq V$ is called a *geodetic set* if $I[S] = V$ and a *hull set* if $I_h(S) = V$. The *geodetic number* $g(G)$ of G is the minimum order of its geodetic sets and any geodetic set of order $g(G)$ is a *minimum geodetic set* or simply a *g- set* of G. Similarly, the *hull number* $h(G)$ of G is the minimum order of its hull sets and any hull set of order $h(G)$ is a *minimum hull set* or simply a *h- set* of G. The geodetic number of a graph is studied in [1, 5, 9] and the hull number of a graph is studied in [1,6]. It was shown in [9] that determining the geodetic number of a graph is NP-hard problem. A vertex v of G is said to be a *geodetic vertex* of G if v belongs to every minimum geodetic set of G. A subset $T \subseteq W$ is called a *forcing subset* for W if W is the unique minimum geodetic set containing T. A forcing subset for W of minimum cardinality is a *minimum forcing subset* of W. The *forcing geodetic number* of W, denoted by $f(W)$, is the cardinality of a minimum forcing subset of W. The *forcing geodetic number* of G, denoted by $f(G)$, is $f(G) = \min\{f(W)\}$, where the minimum is taken over all minimum geodetic sets W in G. The forcing geodetic number of a graph was introduced in [3]. A vertex v of G is said to be a *hull vertex* of G if v belongs to every minimum hull set of G.A subset $T \subseteq S$ is called a *forcing subset* for S if S is the unique minimum hull set containing T. A forcing subset for S of minimum cardinality is a *minimum forcing subset of S*. The *forcing hull number* of S, denoted by $f_h(S)$, is the cardinality of a minimum forcing subset of S. The *forcing hull number* of G, denoted by $f_h(G)$,is $f_h(G)=\min\{f_h(S)\}$, where the minimum

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is taken over all minimum hull sets S in G. The forcing hull number of a graph was introduced in [4] and further studied in [10]. A vertex v of G is said to be an extreme *vertex* of G if the subgraph induced by its neighbors is complete. In [6] Chartrand and Zhang raised the question, for which pair of integers a, b there exists a connected graph G with $f_h(G) = a$ and $f_g(G) = b$. In [10] Li-Da Tong proved that for every pairs a, b of nonnegative integers, there exists a connected a graph G with $f_h(G) = a$ and $f_g(G) = b$ and raised the question, for every integers a, b ,c and d with $a \leq c \leq d$, $b \leq d$, $c \geq 2$, does there exists a connected graph G with $f_h(G) = a$, $f_g(G) = b$, $h(G) = c$ and $g(G) = d$. In this paper it is answered that, for every non negative integers a, b, c and d with $a < c < d$, $b < d$, $c > a + 1$ and $d > c + b - a$, there exists a connected graph G such that $f_h(G) = a$, $f_g(G) = b$, $h(G) = c$ and $g(G) = d$.

Theorem 1.1: [5, 6] If v is an extreme vertex of a graph G , then v belongs to every hull set and geodetic set of G.

Theorem 1.2: [1] For a connected graph G, $h(G) = p$ if and only if $G = K_p$.

Theorem 1.3: [4] Let G be a connected graph. Then
(a) $f_h(G) = 0$ if and only if G has a unique *h-set*
(b) $f_h(G) \leq h(G) - |W|$,where W is the set of all hull vertices of G.

Theorem 1.4: [1] For a connected graph G, $g(G) = p$ if and only if $G = K_p$.

Theorem 1.5: [3] Let G be a connected graph. Then
(a) $f_g(G) = 0$ if and only if G has a unique *g-set*.
(b) $f_g(G) \leq g(G) - |W|$. and W is the set of all geodetic vertices of G.

2. SPECIAL GRAPHS

In this section, we present some graphs from which various graphs arising in theorems are generated using identification.

Let $U_i : \alpha_i, \beta_i, g_i, h_i, \alpha_i$ $(1 \leq i \leq a)$ be a copy of cycle C_4. Let V_i be the graph obtained from $\boldsymbol{U_i}$ by adding a new vertex n_i and the edges $\beta_i n_i$, $n_i h_i$ $(1 \leq i \leq a)$. The graph Z_a is obtained from V_i's by identifying α_i of V_i and g_{i-1} of V_{i-1} $(2 \leq i \leq a)$.

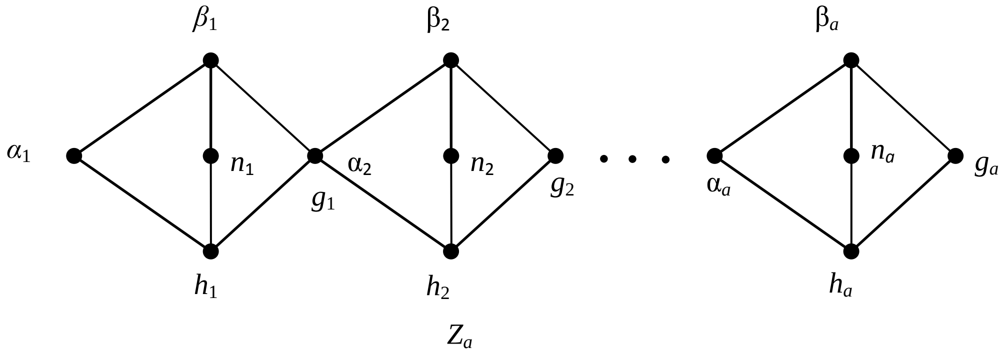


Figure: 2.1

The graph G_a in Figure 2.2 is obtained from F_i 's by identifying the vertices t_i of F_i and r_{i-1} of F_{i-1} $(2 \leq i \leq a)$, where $F_i : s_i, t_i, u_i, v_i, r_i, s_i$ $(1 \leq i \leq a)$ be a copy of cycle C_5.

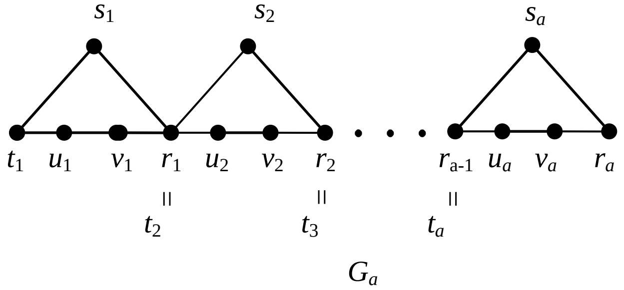
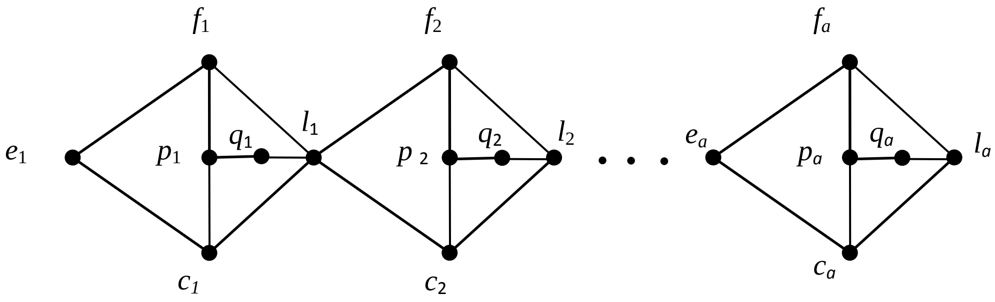


Figure: 2.2

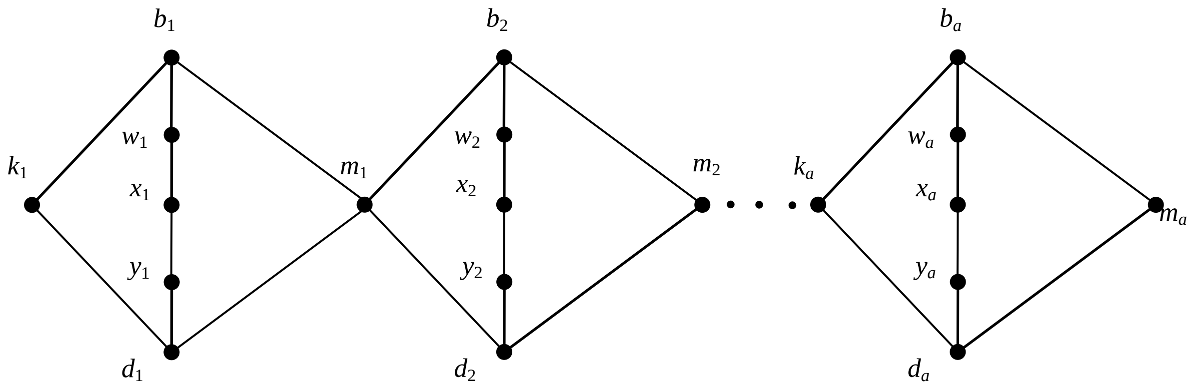
Let $J_i : e_i, f_i, l_i, c_i, e_i$ $(1 \le i \le a)$ be a copy of cycle C_4.Let R_i be the graph obtained from J_i by adding two new vertices p_i, q_i and the edges p_ic_i, p_if_i , p_iq_i ,q_il_i $(1 \le i \le a)$. The graph L_a in Figure 2.3 is obtained from R_i's by identifying $e_i$of R_i and l_{i-1} of R_{i-1} $(2 \le i \le a)$.



L_a

Figure: 2.3

Let $P_i : k_i, b_i, m_i, d_i$, k_i $(1 \le i \le a)$ be a copy of cycle C_4. Let Q_i be the graph obtained from P_i by adding three new vertices w_i, x_i, y_i and the edges w_ib_i, w_ix_i, x_iy_i, y_id_i, $(1 \le i \le a)$. The graph T_a is obtained from Q_i' s by identifying k_i of Q_i and m_{i-1} of Q_{i-1} $(2 \le i \le a)$.



T_a

Figure: 2.4

3. SOME REALIZATION RESULTS

Theorem 3.1: For every pair a, b of integers with $2 \le a \le b$, there exists a connected graph G such that $f_h(G) = f_g(G) = 0$, $h(G) = a$ and $g(G) = b$.

Proof: If $a = b$, let $G = K_a$. Then by Theorems1.3 (a) and 1.2, $f_h(G) = 0$ and $h(G) =$ a. Also by Theorems1.4 and 1.5(a) that $g(G) = b$ and $f_g(G) = 0$. For $a < b$, let G be the graph obtained from Z_{b-a} by adding new vertices x, $z_1,z_2,...,z_{a-1}$ and joining the edges α_1x, $g_{b-a}z_1$, $g_{b-a}z_2$,.,.,., $g_{b-a}z_{a-1}$. Let $Z = \{x, z_1, z_2, .. .,z_{a-1}\}$ be the set of end vertices of G. It is clear that Z is a hull set of G and so by Theorem 1.1, Z is the unique h-set of G so that $h(G) = a$, and hence by Theorem 1.3(a), $f_h(G)$=0. Since the vertices n_i $(1\le i \le a)$ do not lie on any geodesic joining a pair of vertices in Z, we see that Z is not a geodetic set of G. Now it is easily seen that $W = Z \cup \{n_1,n_2,\ldots,n_{b-a}\}$ is the unique g-set of G so that $g(G) =|W| =$ b and hence by Theorems 1.5(a), $f_g(G)$=0.

Theorem: 3.2 For every integers a, b and c with $0 \le a < b \le c$, and $b > a$+1, there exists a connected graph G such that $f_g(G) = 0$, $f_h(G) = a$, $h(G) = b$ and $g(G) = c$.

Proof:
Case 1: If $a = 0$, then the graph G constructed in Theorem 3.1 satisfies the requirements of this theorem.

Case 2: $a \ge 1$.

Sub case2a: $b = c$. Let G be the graph obtained from T_a by adding new vertices x, z_1, z_2,...,z_{b-a-1} and joining the edges xk_1, m_az_1,m_az_2,.,.,.,m_az_{b-a-1}. Let $Z = \{x, z_1, z_2, .. .,z_{b-a-1}\}$ be the set of end vertices of G. Let W be any geodetic set of G. Then by Theorem 1.1, $Z \subseteq W$. It is clear that Z is not a geodetic set of G. For $1\leq i \leq a$. let $H_i = \{w_i, x_i, y_i\}$. We observe that every g-set of G must contain only the vertex x_i from each H_i $(1\leq i \leq a)$so that $g(G)\geq b-a+a = b$. Now, $W=Z \cup\{x_1,x_2,\ldots,x_a\}$ is a geodetic set of G so that $g(G)\leq b-a+a=b$. Thus $g(G)= b$. Also it is easily seen that W is the unique g-set of G and so by Theorem 1.5(a), $f_g(G)=0$.Now it is clear that Z is not a hull set of G. We observe that every h-set of G must contain at least one vertex from each H_i $(1\leq i \leq a)$ so that $h(G)\geq b-a+a = b$. Now, $S=Z \cup \{w_1, w_2,\ldots,w_a\}$ is a hull set of G so that $h(G)\leq b-a+a=b$. Thus $h(G)= b$. Next, we show that $f_h(G)=a$. Since every h-set contains Z, it follows from Theorem 1.3(b) that $f_h(G) \leq h(G)-|Z|=b-(b-a)=a$. Now, since $h(G) = c$ and every *h-set* of G contains Z, it is easily seen that every *h-set* S is of the form $Z \cup \{d_1, d_2,\ldots,d_a\}$, where $d_i \in H_i(1\leq i \leq a)$. Let T be any proper subset of S with $|T|< a$. Then it is clear that there exists some j such that $T \cap H_j = \Phi$, which shows that $f_h(G) = a$.

Sub case 2b: $b < c$. Let G be the graph obtained from T_a and Z_{c-b} by identifying vertex m_a of T_a and α_1 of Z_{c-b} and adding the new vertices x, z_1, z_2... z_{b-a-1} and joining the edges xk_1,$g_{c-b}z_1$,$g_{c-b}z_2$,...$g_{c-b}z_{b-a-1}$. Let $Z = \{x,z_1,z_2, ..., z_{b-a-1}\}$ be the set of end vertices of G. Let W be any geodetic set of G. Then by Theorem 1.1, $Z\subseteq W$. It is clear Z is not a geodetic set of G. For $1\leq i \leq a$. let $H_i = \{w_i, x_i, y_i\}$. We observe that every g-set of G must contain only the vertex x_i from each $H_i(1\leq i \leq a)$ and only the vertex x_i from each $H_i(1\leq i \leq a)$ and only the vertex n_i $(1\leq i\leq c-b)$ so that $g(G)\geq b-a+a+c-b=c$.

Now $W=Z\cup\{x_1,x_2,\ldots,x_a\}\cup \{n_1,n_2,\ldots,n_{c-b}\}$ is a geodetic set of G so that $g(G)\leq b-a+a+c-b=c$. Thus $g(G)= c$. Also it is easily seen that W is the unique g-set of G and so by Theorem1.5 (a) $f_g(G)=0$. It is clear that Z is not a hull set of G. We observe that every h-set of G must contain at least one vertex from each $H_i(1\leq i \leq a)$ so that $h(G) \geq b-a+a =b$. Now, $S=Z \cup\{w_1,w_2,\ldots,w_a\}$ is a hull set of G so that $h(G)\leq b-a+a=b$. Thus $h(G)= b$. Next, we show that $f_h(G) = a$. Since every h-set contains Z, it follows from Theorem 1.3(b) that $f_h(G) \leq h(G)-|Z|=b-(b-a) = a$. Now, since $h(G) = b$ and every h-set of G contains Z, it is easily seen that every h-set S is of the form $Z \cup \{d_1,d_2,\ldots,d_a\}$,where $d_i \in H_i(1\leq i \leq a)$. Let T be any proper subset of S with $|T|< a$. Then it is clear that there exists some j such that $T \cap H_j = \Phi$, which shows that $f_h(G) = a$.

Theorem 3.3: For every integers a, b and c with $0 \leq a < b < c$, $b \geq 2$ and $c > a + b$, there exists a connected graph G such that $f_h(G) = 0$, $f_g(G) = a$, $h(G) = b$ and $g(G) = c$.

Case 1: $a = 0$. Then the graph G constructed in Theorem 3.1 satisfies the requirements of this theorem.

Case 2: $a \geq 1$. Let G be the graph obtained from L_a and Z_{c-b-a} by identifying the vertex l_a of L_a and α_1 of Z_{c-b-a} and adding new vertices x, z_1, z_2,...,z_{b-1} and joining the edges e_1x, $g_{c-b-a}z_1$, $g_{c-b-a}z_2$,..,$g_{c-b-a}z_{b-1}$. Let $Z = \{x, z_1, z_2, ..., z_{b-1}\}$be the set of end vertices of G. It is clear that Z is a hull set of G and so by Theorem 1.1, Z is the unique *h-set* of G so that $h(G) = b$, and hence by Theorem 1.3(a), $f_h(G)=0$. Next we show that $g(G) = c$. Let W be any geodetic set of G

Then by Theorem 1.1, $Z \subseteq W$. It is clear Z is not a geodetic set of G. For $1\leq i \leq a$, let $Q_i = \{p_i, q_i\}$.

We observe that every *g-set* of G must contain at least one vertex from each Q_i $(1\leq i \leq a)$ and each n_i $(1\leq i \leq c-b-a)$ so that $g(G)\geq b+a+c-b-a = c$. Now, $W=Z \cup\{p_1,p_2,\ldots,p_a\}\cup\{n_1,n_2,\ldots,n_{c-b-a}\}$ is a geodetic set of G so that $g(G)\leq b+a+c-b-a=c$. Thus $g(G) =c$. Next, we show that $f_g(G)=a$. Since every *g-set* contains $W_1=Z\cup \{n_1, n_2,\ldots,n_{c-b-a}\}$, it follows from Theorem 1.5(b) that $f_g(G) \leq g(G)-|W_1|=c-(c-a) = a$. Now, since $g(G)=c$ and every g-set of G contains W_1, it is easily seen that every g-set W is of the form $Z\cup\{n_1,n_2,\ldots,n_{c-b-a}\} \cup \{d_1,d_2,\ldots,d_a\}$, where $d_i \in Q_i(1\leq i \leq a)$. Let T be any proper subset of W with $|T|<a$. Then it is clear that there exists some j such that $T \cap Q_j = \Phi$, which shows that $f_g(G) = a$.

Theorem 3.4: For every integers a, b and c with $0\leq a < b \leq c$ and $b > a + 1$, there exists a connected graph G such that $f_h(G) = f_g(G) = a$, $h(G) = b$ and $g(G) = c$.

Proof:

Case 1: $a = 0$ Then the graph G constructed in Theorem 3.1 satisfies the requirements of this theorem.

Case2: $a \geq 1$.

Sub case 2a: $b = c$. Let G be the graph obtained from G_a by adding new vertices x, z_1, z_2, ..., z_{b-a-1} and joining the edges xt_1, r_az_1,r_az_2, ..., r_az_{b-a-1}. Let $Z = \{x,z_1,z_2, ..., z_{b-a-1}\}$ be the set of end vertices of G. By Theorem1.1, every g-set of G

contains Z. Let $F_i = \{u_i, v_i\}$ $(1 \le i \le a)$. First, we show that $h(G) = b$. Since the vertices u_i, v_i do not lie on the geodesic joining any pair of vertices of Z, it is clear that Z is not a hull set of G. We observe that every h-set of G must contain at least one vertex from each F_i $(1 \le i \le a)$. Thus, $h(G) \ge b - a + a = b$. On the other hand, since the set $S = Z \cup \{v_1, v_2, \ldots, v_a\}$ is a hull set of G, it follows that $h(G) \le |S| = b$. Hence $h(G) = b$. Next, we show that $f_h(G) = a$. By Theorem 1.1, every hull set of G contains Z and so it follows from Theorem 1.3(b) that $f_h(G) \le h(G) - |Z| = a$. Now, since $h(G) = b$ and every h-set of G contains Z, it is easily seen that every h-set S is of the form $Z \cup \{c_1, c_2, \ldots, c_a\}$, where $c_i \in F_i(1 \le i \le a)$. Let T be any proper subset of S with $|T| < a$. Then it is clear that there exists some j such that $T \cap F_j = \Phi$, which shows that $f_h(G) = a$. By similar way we can prove that $g(G) = b$ and $f_g(G) = a$.

Sub case 2b: $b < c$. Let G be the graph obtained from G_a and Z_{c-b} by identifying the vertex r_a of G_a and α_1 of Z_{c-b} and then adding new vertices x, $z_1, z_2, \ldots, z_{b-a-1}$ and joining the edges xt_1, $g_{c-b}z_1$, $g_{c-b}z_2$, …, $g_{c-b}z_{b-a-1}$. First, we show that $h(G) = b$. Since the vertices u_i, v_i do not lie on the geodesic joining any pair of vertices of Z, it is clear that Z is not a hull set of G. Let $F_i = \{u_i, v_i\}$. We observe that every h-set of G must contain at least one vertex from each F_i $(1 \le i \le a)$. Thus, $h(G) \ge b - a + a = b$. On the other hand, since the set $S = Z \cup \{v_1, v_2 \ldots v_a\}$ is a hull set of G, it follows that $h(G) \le |S| = b$.

Hence $h(G) = b$. Next, we show that $f_h(G) = a$. By Theorem 1.1, every hull set of G contains Z and so it follows from Theorem 1.3(b) that $f_h(G) \le h(G) - |Z| = a$. Now, since $h(G) = b$ and every h-set of G contains Z, it is easily seen that every h-set S is of the form $Z \cup \{c_1, c_2, \ldots, c_a\}$, where $c_i \in F_i(1 \le i \le a)$. Let T be any proper subset of S with $|T| < a$. Then it is clear that there exists some j such that $T \cap F_j = \Phi$, which shows that $f_h(G) = a$. Next, we show that $g(G) = c$. Since the vertices u_i, v_i, n_i do not lie on the geodesic joining any pair of vertices of Z, it is clear that Z is not a geodetic set of G. We observe that every *g-set* of G must contain at least one vertex from each F_i $(1 \le i \le a)$ and each n_i $(1 \le i \le c\text{-}b)$. Thus, $g(G) \ge b - a + a + c - b = c$. On the other hand, since the set $W = Z \cup \{v_1, v_2 \ldots v_a\} \cup \{n_1, n_2, \ldots, n_{c-b}\}$ is a geodetic set of G, it follows that $g(G) \le |W| = c$. Hence $g(G) = c$. Next, we show $f_g(G) = a$. Since every *g- set* of G contains $W_1 = Z \cup \{n_1, n_2, \ldots, n_{c-b}\}$ and so it follows from Theorem 1.5 (b) that $f_g(G) \le g(G) - |W_1| = a$. Now, since $g(G) = c$ and every g-set of G contains W_1, it is easily seen that every g-set W is of the form $W_1 \cup \{c_1, c_2, \ldots, c_a\}$, where $c_i \in F_i(1 \le i \le a)$. Let T be any proper subset of W with $|T| < a$. Then it is clear that there exists some j such that $T \cap F_j = \Phi$, which shows that $f_g(G) = a$.

Theorem 3.5: For every integers a, b, c and d with $0 \le a \le b < c \le d$, d and $c > b + 1$, there exists a connected graph G such that $f_g(G) = a$, $f_h(G) = b$, $h(G) = c$ and $g(G) = d$.

Proof:
Case 1: $a = b = 0$. Then the graph G constructed in Theorem 3.1 satisfies the requirements of this theorem.

Case 2: $a = 0$, $b \ge 1$. Then the graph G constructed in Theorem 3.2 satisfies the requirements of this theorem.

Case 3: $1 \le a = b$. Then the graph G constructed in Theorem 3.4 satisfies the requirements of this theorem.

Case 4: $1 \le a < b$.

Sub case 4a: $c = d$. Let G be the graph obtained from G_a and T_{b-a} by identifying the vertex r_a of G_a and k_1 of T_{b-a} and then adding new vertices x, $z_1, z_2, \ldots, z_{c-b-1}$ and joining the edges xt_1, $m_{b-a}z_1$, $m_{b-a}z_2, \ldots, m_{b-a}z_{c-b-1}$. Let $Z = \{x, z_1, z_2, \ldots, z_{c-b-1}\}$ be the set of end vertices of G. Let $F_i = \{u_i, v_i\}$ and $H_i = \{w_i, x_i, y_i\}$. It can be easily seen that any h-set of G is of the form $S = Z \cup \{c_1, c_2, \ldots, c_a\} \cup \{d_1, d_2, \ldots, d_{b-a}\}$, where $c_i \in F_i(1 \le i \le a)$ and $d_j \in H_j(1 \le j \le b\text{-}a)$. Then as in earlier theorems it can be seen that $f_h(G) = b$ and $h(G) = c$. Any g-set is of the form $W = Z \cup \{x_1, x_2, \ldots, x_{b-a}\} \cup \{c_1, c_2, \ldots, c_a\}$, where $c_i \in F_i$ $(1 \le i \le a)$. Then as in earlier theorems it can be seen that $f_g(G) = a$ and $g(G) = c$.

Sub case 4b: $c < d$ Let G_1 be the graph obtained from G_a and T_{b-a} by identifying the vertex r_a of G_a and k_1 of T_{b-a}. Now let G be the graph obtained from G_1 and Z_{d-c} by identifying the vertex m_{b-a} of G_1 and α_1 of Z_{d-c} and then adding new vertices x, $z_1, z_2, \ldots, z_{c-b-1}$ and joining the edges xt_1, $g_{d-c}z_1, g_{d-c}z_2, \ldots, g_{d-c}z_{c-b-1}$. Let $Z = \{x, z_1, z_2, \ldots, z_{c-b-1}\}$ be the set of end vertices of G. Let $F_i = \{u_i, v_i\}$ and $H_i = \{w_i, x_i, y_i\}$. It can be easily seen that any h-set of G is of the form $S = Z \cup \{c_1, c_2, \ldots, c_a\} \cup \{d_1, d_2, \ldots, d_{b-a}\}$, where $c_i \in F_i$ $(1 \le i \le a)$ and $d_j \in H_j$ $(1 \le j \le b\text{-}a)$. Then as in earlier theorems it can be seen that $f_h(G) = b$ and $h(G) = c$. Any *g-set* is of the form $W = Z \cup \{x_1, x_2, \ldots, x_{b-a}\} \cup \{n_1, n_2, \ldots, n_{d-c}\} \cup \{c_1, c_2, \ldots, c_a\}$, where $c_i \in F_i$ $(1 \le i \le a)$. Then as in earlier theorems it can be seen that $f_g(G) = a$ and $g(G) = d$.

Theorem 3.6: For every integers a, b, c and d with $0 \le a < c < d$ and $a \le b < d$, $d > c+b-a$ and $c > a+1$, there exists a connected graph G such that $f_h(G) = a$, $f_g(G) = b$, $h(G) = c$ and $g(G) = d$.

Proof:

Case 1: $a = b = 0$. Then the graph G constructed in of Theorem 3.1 satisfies the requirements of this theorem.

Case 2: $a = 0$, $b \ge 1$. Then the graph G constructed in Theorem 3.3 satisfies the requirements of this theorem.

Case 3: $1 \le a = b$. Then the graph G constructed in Theorem 3.4 satisfies the requirements of this theorem**.**

Case 4: $1 \le a < b$. Let G_1 be the graph obtained from G_a and L_{b-a} by identifying the vertex r_a of G_a and e_1 of L_{b-a}. Now let G be the graph obtained from G_1 and $Z_{d-c-b+a}$ by identifying the vertex l_{b-a} of G_1 and α_1 of $Z_{d-c-b+a}$ and then adding new vertices $x, z_1, z_2,...,z_{c-a-1}$ and joining the edges $xt_1, g_{d-c-b+a}z_1, g_{d-c-b+a}z_2,.,.,., g_{d-c-b+a}z_{c-a-1}$. Let $Z = \{x, z_1, z_2... z_{c-a-1}\}$ be the set of end vertices of G. Let $F_i = \{u_i, v_i\}$. It is clear that any h-set is of the form $S= Z \cup \{c_1, c_2\dots c_a\}$, where $c_i \in F_i(1 \le i \le a)$.Then as in earlier theorems it can be seen that $f_h(G) = a$ and $h(G) = c$. Let $Q_i = \{p_i, q_i\}$. It is clear that any *g-set* is of the form $W= Z \cup \{n_1, n_2\dots n_{d-c-b+a}\} \cup \{c_1, c_2, \dots, c_a\} \cup \{d_1, d_2, \dots, d_{b-a}\}$, where $c_i \in F_i\,(1 \le i \le a)$ and $d_j \in Q_j$ $(1 \le j \le b-a)$. Then as in earlier theorems it can be seen that $f_g(G) = b$ and $g(G) = d$.

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