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DUAL STONE ALMOST DISTRIBUTIVE LATTICES

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ABSTRACT

In this paper, the dual Stonity of a dual pseudo-complemented ADL is defined and proved that it is a sub ADL. The concept of a dual Stone Almost Distributive Lattice is introduced. Several necessary and sufficient conditions for an Almost Distributive Lattice (ADL) to become a dual Stone ADL are obtained.

Keywords: *Almost Distributive Lattice (ADL); Birkhoff center; Principal ideal; Dual pseudo-complementation; Dual Stonity; Dual Stone ADL.*

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1. INTRODUCTION

Boolean algebra is a homomorphic image of the algebra $C(X,2)$ of all continuous functions mappings of a compact Hausdorff and totally disconnected topological space X onto the discrete two element Boolean algebra 2. In place of the algebra 2, if we take a dense bounded distributive lattice D, then $C(X,D)$ and its homomorphic images represent a special class of pseudo-complemented distributive lattices, namely Stone lattices. A pseudo-complemented distributive lattice A is said to be a Stone lattice if $x^* \vee x^{**} = 0^*$ for all $x \in A$. The concept of Almost Distributive Lattice(ADL) was introduced in [5] as a common abstraction of the most of the existing ring theoretic generalization of a Boolean algebra on one hand and the class of distributive lattice on the other. Later, in [6], pseudo-complementation in an ADL was introduced and the properties of Stone ADL were studied in [7]. Unlike in lattices, the dual of an ADL in not an ADL in general. For this reason, we introduced the concept of dual pseudo- complemented ADL (Dual PCADL) in [3] and studied its properties. In this paper, we define the concept of dual Stonity is a dual PCADL and derive the inter-relation between different dual pseudo-complementation in an ADL A and observe that the dual Stonity of A under different pseudo-complementations is same. We derive necessary and sufficient conditions for an ADL become a dual Stone ADL.

2. PRELIMINARIES

In this section, we give the necessary definitions and important properties of an ADL taken from [5] for ready reference.

Definition 2.1: [5] An algebra $(A,V,\Lambda,0)$ of type (2, 2, 0) is called an Almost Distributive Lattice (ADL) if it satisfies the following axioms:

(i) $x \vee 0 = x$

(ii) $0 \wedge x = 0$

(iii) $(x \vee y) \wedge z = (x \wedge z) \vee (y \wedge z)$

(iv) $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$

(v) $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$

(vi) $(x \vee y) \wedge y = y$ for all $x, y, z \in A$.

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A non-empty subset I of an ADL A is called an ideal of A if $x \vee y \in I$ and $x \wedge a \in I$ for any $x, y \in I$ and $a \in A$. The principal ideal of A generated by x is denoted by $(x]$ and it is given by $(x] = \{x \wedge a \mid a \in A\}$. The set $PI(A)$ of all principal ideals of A forms a distributive lattice under the operations $\vee$, $\wedge$ defined by $(x] \vee (y] = (x \vee y]$ and $(x] \wedge (y] = (x \wedge y]$ in which $(0]$ is the least element. If A has a maximal element m, then $(m]$ is the greatest element of $PI(A)$. We extended many existing concept in the class of distributive lattices to the class of ADL, through the principal ideal lattice $PI(A)$. For other properties of ADLs we refer to [5]. In [6], the concept of Pseudo-Complementation in an ADL was defined as follows.

Definition 2.2: [6] Let $(A, V, \Lambda, 0)$ be an ADL. Then a unary operation $x \to x^*$ on A is called a pseudo-complementation on A if, for any $x, y \in A$, it satisfies the following conditions:

(i) $x \wedge y = 0 \Rightarrow x^* \wedge y = y$.

(ii) $x \wedge x^* = 0$.

(iii) $(x \vee y)^* = x^* \wedge y^*$.

An ADL $(A, V, \Lambda, 0)$ together with a pseudo-complement is called a pseudo- complementation on A. The concept of a Stone ADL is defined in [7] as follows.

Definition 2.3: [7] Let $(A, V, \Lambda, 0)$ be an ADL with a pseudo-complement $*$. Then A is called a Stone Almost Distributive Lattice (Stone ADL) if, for any $x \in A$, $x^* \vee x^{**} = 0^*$.

For other properties of PCADL, Stone ADL we refer to [6], [7].

3. DUAL STONITY DS (A)

The concept of pseudo-complementation in an ADL A was introduced in [6] and it was observed that, the set A^* of all pseudo-complementation of elements of A forms a Boolean algebra with respect to the same partial ordering of A. A pseudo-complemented ADL $(A, \vee, \wedge, *, 0$ is called a Stone ADL if $x^* \vee x^{**} = 0^*$ for all $x \in A$. We begin with the following.

Definition 3.1: [3] Let $(A, V, \Lambda, 0)$ be an ADL. Then a unary operation $*$ on A is called a dual pseudo-complementation on A if, for any $x, y \in A$, it satisfies the following conditions:

d₁: if $x \vee y = m$, then $(x_* \vee y) \wedge m = y \wedge m$.

d₂: $x \vee x_*$ is a maximal element of A.

d₃: $(x \wedge y)_* = x_* \vee y_*$.

An ADL A with a dual pseudo-complementation is called a dual Pseudo-Complemented Almost Distributive Lattice (or simply dual PCADL). We state the important properties of dual PCADL, taken from [3] in the following theorem.

Theorem 3.2: [3] Let A be an ADL and $*$ be a dual pseudo-complementation on A. For any $x, y \in A$, we have the following:

(i) $x_* = x_{***}$.

(ii) $(x \vee y)_{**} = x_{**} \vee y_{**}$.

(iii) $(x \wedge m)_* = x_*$.

(iv) If $x \leq y$, then $y_* \leq x_*$.

Unlike in distributive lattices, an ADL can have more than one dual pseudo-complementation. The following theorem is taken from [3] will be used in this paper.

Theorem 3.3: [3] Let A be an ADL and $*$ and $\perp$ be a dual pseudo-complementation on A. For any $x, y \in A$, we have the following:

(i) $x_{*\perp} = x_{\perp\perp}$.

(ii) $x_* = y_* \Leftrightarrow x_\perp = y_\perp$.

(iii) $x_* \wedge x_{**} = 0 \Leftrightarrow x_\perp \wedge x_{\perp\perp} = 0$.

For other properties of a dual PCADL, we refer to [3]. The following definition of dual annihilator of a subset of A is taken from [4].

Definition 3.4: [4] Let A be an ADL with a maximal element m and S is any non-empty subset of A. Write $S^+ = \{a \in A \mid s \vee a$ is maximal for all $s \in S\}$. Then S^+ is called the dual annihilator of S in A and it can be verified that S^+ is a filter of A. We write s^+ for S^+ if $S = \{s\}$.

Now we prove the following theorem.

Theorem 3.5: Let A be a dual PCADL. Then, for any $x \in A, x^+ = [x_*)$.

Proof: Since, for any $x \in A, x \vee x_*$ is maximal, we get $x_* \in x^+$ and hence $[x_*) \subseteq x^+$. On the other hand, if $y \in x^+$, then $x \vee y$ is maximal and hence $(x_* \vee y) \wedge m = y \wedge m$.

Now $y \wedge x_* = y \wedge m \wedge x_* = (x_* \vee y) \wedge m \wedge x_* = x_*$ and hence $y \vee x_* = y$. Thus $y \in [x_*)$. Hence $x^+ \subseteq [x_*)$.

Corollary 3.6: Let A be an ADL with a maximal element m and $*$, $\perp$ be dual pseudo-complementations on A. Then, for any $x \in A, [x_*) = [x_\perp)$.

If $*$ is a dual pseudo-complementation on A, then it was proved in [3], that $A_* = \{x_* \wedge m \mid x \in A\}$ is a Boolean algebra under operations V, $\overline{\Lambda}$ where for any $x_* \wedge m, y_* \wedge m \in A_*, (x_* \wedge m)\overline{\Lambda}(y_* \wedge m) = (x_{**} \vee y_{**})_* \wedge m$.

Now we prove the following.

Theorem 3.7: Let A be an ADL with a dual pseudo-complementation $*$. Then the map $f : A \to A_*$ defined by $f(x) = x_{**} \wedge m$ is an epimorphism.

Proof: Let $x, y \in A$. Then

$$\begin{aligned} f(x \wedge y) &= [x \wedge y]_{**} \wedge m \\ &= [x_* \vee y_*]_{***} \wedge m \\ &= [x_{***} \vee y_{***}]_* \wedge m \\ &= (x_{**} \wedge m)\overline{\Lambda}(y_{**} \wedge m) \\ &= f(x)\overline{\Lambda} f(y) \end{aligned}$$

and $f(x \vee y) = (x \vee y)_{**} \wedge m = (x_{**} \wedge m) \vee (y_{**} \wedge m) = f(x) \vee f(y)$.

Also $f(x_*) = x_{***} \wedge m = x_* \wedge m$. Hence f is an epimorphism.

Theorem 3.8: Let A be an ADL with two dual pseudo-complementations $*$ and $\perp$. Then the map $f : A_* \to A_\perp$ defined by $f(x_* \wedge m) = x_\perp \wedge m$, is an isomorphism of the Boolean algebras A_* and $A_\perp$.

Proof: By Theorem 3.3, the map $f : A_* \to A_\perp$ defined by $f(x_* \wedge m) = x_\perp \wedge m$ for all $x_* \wedge m \in A_*$, is a bijection. Let $x, y \in A$. Then

$$\begin{aligned} f\left((x_* \wedge m)\overline{\Lambda}(y_* \wedge m)\right) &= f\left((x_{**} \vee y_{**})_* \wedge m\right) \\ &= (x_{**} \vee y_{**})_\perp \wedge m \\ &= (x \vee y)_{**\perp} \wedge m \\ &= (x \vee y)_{\perp\perp\perp} \wedge m \\ &= (x_{\perp\perp} \vee y_{\perp\perp})_\perp \wedge m \\ &= (x_\perp \wedge m)\overline{\Lambda}(y_\perp \wedge m) \\ &= f(x_* \wedge m)\overline{\Lambda} f(y_* \wedge m) \end{aligned}$$

and $f\left((x_* \wedge m) \vee (y_* \wedge m)\right) = f\left((x \wedge y)_* \wedge m\right) = (x \wedge y)_\perp \wedge m = (x_\perp \wedge m) \vee (y_\perp \wedge m)$.

$$= f(x_* \wedge m) \vee f(y_* \wedge m)$$

Definition 3.9: Let A be a dual PCADL with a maximal element m. Then the set $\{x \in A \mid x_* \wedge x_{**} = 0\}$ is called the Dual Stonity of A and it is denoted by $DS(A)$.

In view of Theorem 3.3, we get that the set $DS(A)$ is independent of the dual pseudo-complementation on A. Now we prove the following.

Theorem 3.10: Let A be a dual PCADL. Then $DS(A)$ is a sub ADL of A, closed under dual pseudo-complementation.

Proof: If $x \in DS(A)$, then $x_{**} \wedge x_{***} = x_{**} \wedge x_* = 0$ and hence $x_* \in DS(A)$. Let $x, y \in A$. Then

$$\begin{aligned} (x \wedge y)_* \wedge (x \wedge y)_{**} &= (x_* \vee y_*) \wedge (x \wedge y)_{**} \\ &= \left\{\left(x_* \wedge (x \wedge y)_{**}\right) \vee \left(y_* \wedge (x \wedge y)_{**}\right)\right\} \\ &\leq (x_* \wedge x_{**}) \vee (y_* \wedge y_{**}) = 0 \end{aligned}$$

and

$$\begin{aligned} (x \vee y)_* \wedge (x \vee y)_{**} &= (x \vee y)_* \wedge (x_{**} \vee y_{**}) \\ &= \left[(x \vee y)_* \wedge x_{**}\right] \vee \left[(x \vee y)_* \wedge y_{**}\right] \\ &\leq (x_* \wedge x_{**}) \vee (y_* \wedge y_{**}) = 0. \end{aligned}$$

Therefore $x \wedge y, x \vee y \in DS(A)$. Also, since $0_* \wedge 0_{**} = 0_* \wedge 0 = 0$, we get $0 \in DS(A)$. Hence $DS(A)$ is a sub ADL of A.

4. DUAL STONE ADL

The concept of Stone lattice was given by T.P. Speed. Later, this concept was extended to the class of ADLs in [7]. In this section, we introduce the concept of a dual Stone ADL and study its properties. We derive necessary and sufficient conditions for an Dual PCADL to become a Dual Stone ADL. We begin with the following definition.

Definition 4.1: An ADL A with a dual pseudo-complementation * is called a dual Stone ADL if $DS(A) = A$, or equivalently, $x_* \wedge x_{**} = 0$ for all $x \in A$. Here afterwards A stands for an ADL $(A, V, \Lambda, 0)$ with a dual pseudo-complementation * and with a maximal element m.

Theorem 4.2: A is a dual Stone ADL if and only if $\left(DS\left(A\right)\right)_* = A_*$.

Proof: Suppose $\left(DS\left(A\right)\right)_* = A_*$. Let $x \in A$. Then $x_* \in A_* = \left(DS\left(A\right)\right)_*$ and hence $x_{**} \wedge x_{***} = x_* \wedge x_{**} = 0$. Thus $x \in DS\left(A\right)$. Therefore A is a dual Stone ADL. Converse is trivial. □

Theorem 4.3: The following are equivalent:
(i) A is a dual Stone ADL.
(ii) $x^+ \vee y^+ = A$ whenever $x, y \in A$ and $x \vee y$ is maximal.
(iii) $x_* \wedge y_* = 0$ whenever $x, y \in A$ and $x \vee y$ is maximal.

Proof: (i) $\Rightarrow$ (ii): suppose A is a dual Stone ADL and let $x, y \in A$ such that $x \vee y$ is maximal. Then $\left(x \vee y\right) \wedge m = m$ and hence $\left(x_* \vee y\right) \wedge m = y \wedge m$. Thus $y \wedge x_* = x_*$.

Now $\left(x_{**} \vee y\right) \wedge m = \left[\left(y \wedge x_*\right)_* \vee y\right] \wedge m = \left(x_{**} \vee y_* \vee y\right) \wedge m = m.$ So that $x_{**} \vee y$ is maximal and hence $x_{**} \in y^+$. Since $x \vee x_*$ is maximal, we get $x_* \in x^+$. Thus $0 = x_* \wedge x_{**} \in x^+ \vee y^+$.

Hence $x^+ \vee y^+ = A$. (ii) $\Rightarrow$ (iii): Let $x, y \in A$ such that $x \vee y$ is maximal. Then 0 = $s \wedge t$ for some $s \in x^+$ and $t \in y^+$. So that $s \vee x$ is maximal and hence $s \wedge x_* = x_*$. Similarly, $t \wedge y_* = y_*$. Hence $x_* \wedge y_* = s \wedge x_* \wedge t \wedge y_* = 0$. (iii) $\Rightarrow$ (i): If $x \in A$, then $x \vee x_*$ is maximal, and hence $x_* \wedge x_{**} = 0$. Thus A is a dual Stone ADL.

Theorem 4.4: Let A be a dual Stone ADL. Then, for any $x, y \in A$, $\left(x_* \vee y_*\right)_* \wedge m = x_{**} \wedge y_{**} \wedge m$.

Proof: Since $x_* \wedge m \le \left(x_* \vee y_*\right) \wedge m$, we get $\left(x_* \vee y_*\right)_* \wedge m \le x_{**} \wedge m$. Similarly, we get $\left(x_* \vee y_*\right)_* \wedge m \le y_{**} \wedge m.$ Hence $\left(x_* \vee y_*\right)_* \wedge m \le x_{**} \wedge y_{**} \wedge m.$ On the other hand, we have

$$x_{**} \wedge y_{**} \wedge \left(x_* \vee y_*\right) = \left(x_{**} \wedge y_{**} \wedge x_*\right) \vee \left(x_{**} \wedge y_{**} \wedge y_*\right) = 0.$$

$$\Rightarrow \left(x_{**} \wedge y_{**} \wedge \left(x_* \vee y_*\right)\right) \vee \left(x_* \vee y_*\right)_* = 0 \vee \left(x_* \vee y_*\right)_*$$

$$\Rightarrow [x_{**} \wedge y_{**} \vee (x_* \vee y_*)_*] \wedge [(x_* \vee y_*) \vee (x_* \vee y_*)_*] \wedge m = (x_* \vee y_*)_* \wedge m$$

$$\Rightarrow \left(x_{**} \wedge y_{**} \vee \left(x_* \vee y_*\right)_*\right) \wedge m = \left(x_* \vee y_*\right)_* \wedge m.$$

Hence $x_{**} \wedge y_{**} \wedge m \le \left(x_* \vee y_*\right)_* \wedge m.$ Thus we get the result.

Theorem 4.5: A is a dual Stone ADL if and only if $\left(x \vee y\right)_* \wedge m = x_* \wedge y_* \wedge m$ for all $x, y \in A$.

Proof: Suppose A is a dual Stone ADL and $x, y \in A$. Then

$$\begin{aligned}\left(x \vee y\right)_* \wedge m &= \left(x \vee y\right)_{***} \wedge m \\ &= \left(x_{**} \vee y_{**}\right)_* \wedge m \\ &= x_{***} \wedge y_{***} \wedge m \text{ (by theorem 4.4)} \\ &= x_* \wedge y_* \wedge m .\end{aligned}$$

Conversely, suppose, for any $x, y \in A$, $\left(x \vee y\right)_* \wedge m = x_* \wedge y_* \wedge m$. Since $x \vee x_*$ is maximal, we get $\left(x \vee x_*\right) \wedge m = m$. Then $m_* \wedge m = \left(\left(x \vee x_*\right) \wedge m\right)_* \wedge m$

$$=\left(\left(x\vee x_*\right)_*\vee m_*\right)\wedge m$$
$$=\left(x_*\wedge x_{**}\right)\wedge m\,.$$

Therefore $x_*\wedge x_{**}=0$ and hence A is a dual Stone ADL.

Theorem 4.6: Let A be a dual Stone ADL. Write $A_*=\left\{x_*\wedge m\mid x\in A\right\}$. Then $(A_*,\vee,\wedge,*,0,0_*\wedge m)$ is a Boolean algebra.

Proof: Clearly, A_* is a bounded distributive lattice. Let $x_*\wedge m\in A_*$. Now $x_*\wedge m\wedge x_{**}\wedge m=x_*\wedge x_{**}\wedge m=0$ and $\left(x_*\wedge m\right)\vee\left(x_{**}\wedge m\right)=\left(x_*\wedge x_{**}\right)\wedge m=m.$ Hence A_* is a Boolean algebra.

Now we prove the following.

Note 4.7: Since the dual of a distributive lattice is again a distributive lattice, we say that a distributive lattice $\left(A,V,\Lambda\right)$ is a dual Stone lattice if its dual $\left(A,\Lambda,V\right)$ is a Stone lattice. Now we prove the following.

Theorem 4.8: Let A be a dual PCADL with a maximal element m. Then the following are equivalent:
(i) A is a dual Stone ADL.
(ii) $\left[a,a\vee m\right]$ is a dual Stone lattice for all $a\in A$.
(iii) $\left[0,m\right]$ is a dual Stone lattice.

Proof: (i) ⟹ (ii): Suppose A is a dual Stone ADL and $a\in A$. For any $x\in A$, define $x_\perp=a\vee\left(x_*\wedge m\right)$. Then by Theorem 3.11 [3], we get that $\perp$ is a dual pseudo-complementation on $\left[a,a\vee m\right]$. Now for any $x\in\left[a,a\vee m\right]$, $x_\perp\wedge x_{\perp\perp}=\left[a\vee\left(x_*\wedge m\right)\right]\wedge\left[a\vee\left(x_{**}\wedge m\right)\right]=a\vee\left(x_*\wedge x_{**}\wedge m\right)=a\vee 0=a$ .Hence $\left[a,a\vee m\right]$ is a dual Stone lattice. (ii) ⇒(iii) is trivial. (iii) ⇒ (i): Suppose $\left[0,m\right]$ is a dual Stone lattice under the unary operation $\perp$. For $x\in A$, define $x_*=\left(x\wedge m\right)_\perp$. Then by Theorem 3.11 [3], we get A is a dual PCADL. Now, for any $x\in A$, $x_*\wedge x_{**}=\left(x\wedge m\right)_\perp\wedge\left(\left(x_\perp\wedge m\right)_\perp\wedge m\right)_\perp=x_\perp\wedge\left(x_\perp\wedge m\right)_\perp=x_\perp\wedge x_{\perp\perp}=0$. Hence A is a dual Stone ADL.

Finally, we conclude this paper with the following.

Theorem 4.9: Let A be an ADL. Then A is a dual Stone ADL if and only if $PI\left(A\right)$ is a dual Stone lattice.

Proof: Suppose A is a dual Stone ADL. For any $x\in A$. define $\left(x\right]_+$ = (x_*]. Then by Theorem 3.12 [3], we get $PI\left(A\right)$ is a dual pseudo-complemented lattice. Now $\left(x\right]_+\cap\left(x\right]_{++}$ = (x_*] ∩ (x_{**}] = ($x_*\wedge x_{**}$] = (0]. Hence $PI\left(A\right)$ is a dual Stone lattice. Conversely, suppose ($PI\left(A\right)$, ∨ , ∧ , +) is a dual Stone lattice. For $x\in A$, define $x_*=a\wedge m$ where $\left(x\right]_+$ = (a]. Then by Theorem 3.12 [3], we get that A is a dual PCADL. Let $x\in A$. Suppose $\left(x\right]_+$ = (a] and $\left(x\right]_{++}=\left(a\right]_+$ = (b]. Since $\left(x\right]_+\cap\left(x\right]_{++}$ = (0]. we get $a\wedge b$ = 0. Since ($a\wedge m$] = (a], we get that $x_*\wedge x_{**}=a\wedge m\wedge b\wedge m$ = 0. Hence A is a dual Stone ADL.

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