International Journal of Mathematical Archive-3(2), 2012, Page: 675-680
IJMA Available online through www.ijma.info ISSN 2229 – 5046

SOME NUMERICAL ALGORITHMS FOR MINIMIZATION OF UNCONSTRAINED OPTIMIZATION PROBLEMS

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(Received on: 26-01-12; Accepted on: 24-02-12)

ABSTRACT

In this paper, we propose few numerical algorithms for minimization of unconstrained non-linear functions by using Modified Newton's method. Then comparative study among the six new algorithms which are cubically convergent with the classical Newton's algorithm is established by means of examples.

Key words: *Non-linear functions; Minimization; Newton's method; Third-order convergence.*

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1. INTRODUCTION

In recent years, many problems in business situations and engineering designs have been modeled as an optimization problem for taking optimal decisions. Numerical optimization techniques have made deep in to almost all branches of engineering and mathematics. Several methods [8, 9, 12, 13] are available for solving unconstrained minimization problems. These methods can be classified in to two categories as non gradient and gradient methods. The non gradient methods require only the objective function values but not the derivatives of the function in finding minimum. The gradient methods require, in addition to the function values, the first and in some cases the second derivatives of the objective function. Since more information about the function being minimized is used through the use of derivatives, gradient methods are generally more efficient than non gradient methods. All the unconstrained minimization methods are iterative in nature and hence they start from an initial trial solution and proceed towards the minimum point in a sequential manner.

Vinay Kanwar et. al. [14] introduced new algorithms called, external touch technique and orthogonal intersection technique for solving the non linear equations. Recently, Kou Jisheng et. al. [7] introduced several new algorithms of third order which are the modification of Newton's method for solving non-linear equation.

In this paper, we introduce few new algorithms for minimization of non linear functions by using Modified Newton's method. Then comparative study is established among the new algorithms with Newton's algorithm by means of examples.

2. NEW ALGORITHM

Consider the nonlinear optimization problem: Minimize $\{ f(x),\ x \in R,\ f : R \to R \}$ where f a non-linear twice differentiable function.

Consider the function $G(x) = x - (g(x)/g'(x))$ where $g(x) = f'(x)$. Here $f(x)$ is the function to be minimized. $G'(x)$ is defined around the critical point x^* of $f(x)$ if $g'(x^*) = f''(x^*) \neq 0$ and is given by

$$G'(x) = g(x)g''(x)/g'(x).$$

If we assume that $g''(x^*) \neq 0$, we have $G'(x^*) = 0$ iff $g(x^*) = 0$.

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In this article, we consider iterative methods to optimize the non linear equations g(x) = 0 where $g : D \subset R \to R$ for an open interval D. It is clear that the order of convergence of classical Newton's method for minimization of non linear function is two[10]. Recently, some modified Newton's methods with cubic convergence have been developed in [1, 2, 11, 15], by considering different quadrature formulae for the computation of the integral arising from Newton's theorem

$$g(x) = g(x_n) + \int_{x_n}^{x} g'(t)\,dt \qquad (2.1).$$

Weerakoon and Fernando[15] rederive the classical Newton's method by the rectangular rule to compute the integral of (2.1) and derived the following Modified Newton's method with third order convergence by the trapezoidal rule.

$$x_{n+1} = x_n - \frac{2\,g(x)}{g'(x^*_{n+1}) + g'(x_n)} \quad \text{where} \quad x^*_{n+1} = x_n - \frac{g(x_n)}{g'(x_n)} \qquad (2.2)$$

since $g(x) = f'(x)$, equation (2.2) becomes

New Algorithm - I

$$x_{n+1} = x_n - \frac{2\,f'(x)}{f''(x^*_{n+1}) + f''(x_n)} \quad \text{where} \quad x^*_{n+1} = x_n - \frac{f'(x_n)}{f''(x_n)} \qquad (2.3)$$

The midpoint rule for the integral (2.1) gives that [1, 11]

$$x_{n+1} = x_n - \frac{g(x_n)}{g'((x_n + x^*_{n+1})/2)} \qquad (2.4)$$

The algorithm (2.4) has also been derived in [4] independently and the multivariate case is treated in [3, 5].

since $g(x) = f'(x)$, equation (2.4) becomes

New Algorithm - II

$$x_{n+1} = x_n - \frac{f'(x_n)}{f''((x_n + x^*_{n+1})/2)} \qquad (2.5)$$

In [6], instead of using the Newton's theorem for y = f(x). Homeier uses it for the inverse function x(y)

$$x(y) = x(y_n) + \int_{y_n}^{y} x'(t)\,dt \qquad (2.6)$$

to obtain a class of cubically convergent Newton type methods, the best efficient one of which is

$$x_{n+1} = x_n - \frac{g(x_n)}{2}\left(\frac{1}{g'(x_n)} + \frac{1}{g'(x^*_{n+1})}\right) \qquad (2.7)$$

since $g(x) = f'(x)$, equation (2.7) becomes

New Algorithm – III

$$x_{n+1} = x_n - \frac{f'(x_n)}{2}\left(\frac{1}{f''(x_n)} + \frac{1}{f''(x^*_{n+1})}\right) \qquad (2.8)$$

Equation (2.2) can be rewritten as

$$x_{n+1} = x_n - \frac{g(x)}{(g'(x^*_{n+1}) + g'(x_n))/2} \tag{2.9}$$

since $g(x) = f'(x)$, equation (2.9) becomes

New Algorithm – IV

$$x_{n+1} = x_n - \frac{f'(x)}{(f''(x^*_{n+1}) + f''(x_n))/2} \tag{2.10}$$

The iterative algorithm (2.4) can be viewed as obtained by using the midpoint value $g'((x_n + x^*_{n+1})/2)$ instead of arithmetic mean of $g'(x_n)$ and $g'(x^*_{n+1})$ in (2.9). Here we apply this idea to (2.7) and we obtain new modification of Newton' method.

We rewrite the equation (2.7) as

$$x_{n+1} = x_n - \frac{g(x_n)}{2}\left(\frac{1}{g'(x_n)} + \frac{1}{2((g'(x^*_{n+1}) + g'(x_n))/2) - g'(x_n)}\right) \tag{2.11}$$

since $g(x) = f'(x)$, equation (2.11) becomes

New Algorithm - V

$$x_{n+1} = x_n - \frac{f'(x_n)}{2}\left(\frac{1}{f''(x_n)} + \frac{1}{2((f''(x^*_{n+1}) + f''(x_n))/2) - f''(x_n)}\right) \tag{2.12}$$

In the equation (2.11), we replace the arithmetic mean $(g'(x^*_{n+1}) + g'(x_n))/2$ with the midpoint value $g'((x_n + x^*_{n+1})/2)$

$$x_{n+1} = x_n - \frac{g(x_n)}{2}\left(\frac{1}{g'(x_n)} + \frac{1}{2(g'(x_n + x^*_{n+1})/2) - g'(x_n)}\right) \tag{2.13}$$

since $g(x) = f'(x)$, equation (2.13) becomes

New Algorithm - VI

$$x_{n+1} = x_n - \frac{f'(x_n)}{2}\left(\frac{1}{f''(x_n)} + \frac{1}{2(f''(x_n + x^*_{n+1})/2) - f''(x_n)}\right) \tag{2.14}$$

3. CONVERGENCE ANALYSIS

The convergence analysis of new algorithms I to V are of cubically convergence since the computation of the integral (2.1) involves trapezoidal rule and in the integral (2.6) Homeizer[6] uses it for the inverse function x(y) to obtain a class of cubically convergent. The following theorem shows that the order of convergence of new algorithm VI is also cubically convergent.

Theorem 3.1: Assume that the function $g: D \subset R \to R$ has a simple root $\alpha \in D$, where D is an open interval. If g(x) has first, second and third derivatives in the interval D, then the method defined by (2.13) converges cubically.

Proof: The proof of this theorem follows as in [7].

Hence the convergence analysis of New Algorithm-VI follows from the above theorem.

4. NUMERICAL ILLUSTRATION

Example 4.1: Consider the function $f(x) = x^3 - 2x - 5$. The minimized value of the function is 0.816497. The following table depicts the number of iterations needed to converge to the minimized value for all the new algorithms with three initial values $x_0 = 1$, $x_0 = 2$ and $x_0 = 3$.

Table – I: shows a comparison between the New iterative Algorithms and Newton's Algorithms

Sl. No	Methods	For initial value x_0 =1.000000	For initial value x_0 =2.000000	For initial value x_0 =3.000000
1	Newton's Algorithm	3	5	5
2	New Algorithm-I	2	3	3
3	New Algorithm-II	2	3	3
4	New Algorithm-III	2	3	3
5	New Algorithm-IV	2	3	3
6	New Algorithm-V	2	2	2
7	New Algorithm-VI	2	3	3

Example 4.2: Consider the function $f(x) = xe^x - 1$. The minimized value of the function is -1. The following table depicts the number of iterations needed to converge to the minimized value for all the new algorithms with three initial values $x_0 = 1$, $x_0 = 2$ and $x_0 = 3$.

Table – II: shows a comparison between the New iterative Algorithms and Newton's Algorithms

Sl. No	Methods	For initial value x_0 =1.000000	For initial value x_0 =2.000000	For initial value x_0 =3.000000
1	Newton's Algorithm	7	8	10
2	New Algorithm-I	5	6	7
3	New Algorithm-II	5	5	6
4	New Algorithm-III	4	5	5
5	New Algorithm-IV	5	6	7
6	New Algorithm-V	4	5	5
7	New Algorithm-VI	3	4	4

Example 4.3: Consider the function $f(x) = x^5 + x^4 + 4x^2 - 15$. The minimized value of the function is 0.0000.

The following table depicts the number of iterations needed to converge to the minimized value for all the new algorithms with three initial values $x_0 = 1$, $x_0 = 2$ and $x_0 = 3$.

Table – III: shows a comparison between the New iterative Algorithms and Newton's Algorithms

Sl. No	Methods	For initial value x_0 =1.000000	For initial value x_0 =2.000000	For initial value x_0 =3.000000
1	Newton's Algorithm	5	6	8
2	New Algorithm-I	4	5	6
3	New Algorithm-II	3	4	5
4	New Algorithm-III	3	4	5
5	New Algorithm-IV	4	5	6
6	New Algorithm-V	3	4	5
7	New Algorithm-VI	3	4	4

Example 4.4: Consider the function $f(x) = x^4 - x - 10$. The minimized value of the function is 0.629961. The following table depicts the number of iterations needed to converge to the minimized value for all the new algorithms with three initial values $x_0 = 1$, $x_0 = 2$ and $x_0 = 3$.

Table – IV: shows a comparison between the New iterative Algorithms and Newton's Algorithms

Sl. No	Methods	For initial value x_0 =1.000000	For initial value x_0 =2.000000	For initial value x_0 =3.000000
1	Newton's Algorithm	4	6	7
2	New Algorithm-I	3	4	5
3	New Algorithm-II	3	4	5
4	New Algorithm-III	3	4	4
5	New Algorithm-IV	3	4	5
6	New Algorithm-V	3	4	4
7	New Algorithm-VI	3	3	4

Example 4.5: Consider the function $f(x) = e^x - 3x^2$. The minimized value of the function is 0.20448. The following table depicts the number of iterations needed to converge to the minimized value for all the new algorithms with three initial values $x_0 = -1$, $x_0 = 0$, and $x_0 = 1$.

Table – V: shows a comparison between the New iterative Algorithms and Newton's Algorithms

Sl. No	Methods	For initial value x_0 =−1.000000	For initial value x_0 =0.000000	For initial value x_0 =1.000000
1	Newton's Algorithm	3	3	4
2	New Algorithm-I	2	2	2
3	New Algorithm-II	2	2	3
4	New Algorithm-III	2	2	3
5	New Algorithm-IV	2	2	2
6	New Algorithm-V	2	2	3
7	New Algorithm-VI	2	2	2

5. CONCLUSION

In this paper, we have introduced six numerical algorithms namely, New Algorithm –I, New Algorithm – II, New Algorithm – III, New Algorithm – IV, New Algorithm – V, New Algorithm – VI for minimization of non linear functions. From the above illustrations it is clear that the rate of convergence of these new algorithms is faster than Newton's Algorithm. In real life problems, the variables can not be chosen arbitrarily rather they have to satisfy certain specified conditions called constraints. Such problems are known as constrained optimization problems. In near future, we have a plan to extend the proposed new algorithms to constrained optimization problems.

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