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SUPERSOLVABLE, SIMPLICIAL AND EULERIAN IN THE LATTICE OF WEAK CONGRUENCES

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ABSTRACT

In this paper, we study supersolvable, simplicial and Eulerian in the lattice of weak congruences. We are going to prove that if L_n is a chain of n elements then $C_w(L_n)$ is supersolvable. Next we prove that $C_w(L)$ is simplicial if and only if L is a two-element chain. For any lattice L, $C_w(L)$ is not dual simplicial. Also prove that for any lattice L, $C_w(L)$ is not Eulerian.

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1. INTRODUCTION

The study on weak congruences in algebras have started in the seventies by H. Draskovicova [8], M. Kolibiar, F. Sik, T. D. Mai[15] and others, in the name of quasicongruences. Also, reflexive and symmetric relations called tolerances play an important role in algebra and applications. This concept occurs in automata theory [1], biology questions [21], linguistics [10] and it is used for some "inaccuracy" in abstract algebra.

Since then so many authors have made contributions to the weak congruence theory which is a symmetric and transitive relations satisfying the substitution property. For example, one can refer to [2], [4], [6], etc.

Several authors have attempted to characterize the structure of an algebra $\mathcal{A}$ in terms of its lattice of weak congruences $C_w(\mathcal{A})$. For example, Gradimir Vojvodic and Branimir Seselja have proved that for an algebra, $C_w(\mathcal{A})$ is modular if and only if (i) $\mathcal{A}$ satisfied the CEP and CIP and (ii) Con $\mathcal{A}$ and Sub $\mathcal{A}$ are modular[20]. In the case of a lattice L, Andreja Tepavcevic[17], etc., has proved that $C_w(L)$ is semimodular if and only if L is a two-element chain. These result have motivated us to look for which $C_w(L)$ satisfy still weaker condition supersolvable, simplicial and Eulerian.

R. P. Stanley [20] introduced the notion of a supersolvable lattice. In this chapter, we study supersolvablity in the lattice of weak congruences. Here we prove that $C_w(L_n)$ is supersolvable. Using the pattern of the proof we conclude that $C_w(B_n)$, $C_w(M_n)$ and more generally, $C_w(L)$ for any bounded lattice L, is supersolvable.

Several authors have made contributions to the field of Eulerain lattices, one can see for example, V. K. Santhi[12] and A. Vethamancikam[21], Bayer and Billera[2]. In particular, many basic results and properties of Eulerian posets were eleborately first studied by V. K. Santhi in her thesis. Also, she dealt with the product of two Eulerian posets and construction of an Eulerian poset from Eulerian posets of smaller ranks. In her thesis we can find so many results in lower Eulerian and semi Eulerian posets. A. Vethamanickam's subsequent work on Eulerian lattices inspired us very much for further study. His work on Eulerian lattices, strongly uniform Eulerian lattices and pleasant Eulerian posets are of great inspiration to us. In this section, we examine Eulerian property in the lattice of weak congruences.

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2. PRELIMINARY NOTES

An algebra is a pair ($\mathcal{A}$, F) where $\mathcal{A}$ is a non-empty set and F is a set of finitary operations on $\mathcal{A}$. A weak congruence relation on $\mathcal{A}$ is a symmetric and transitive relation ρ on $\mathcal{A}$ which satisfies the substitution property namely, for each n-ary operation $f_i \in F$, whenever $a_j \rho b_j$, j = 1, 2,…n…$f_i(a_1, a_2,…,a_n) \rho f_i(b_1, b_2,…,b_n)$, $a_j, b_j \in \mathcal{A}$, j = 1, 2,…,n.

In other words, a weak congruence relation on $\mathcal{A}$ is a symmetric and transitive sub algebra of $\mathcal{A}^2$. The set of all weak congruence relations on $\mathcal{A}$ denoted by $C_w(\mathcal{A})$ becomes an algebraic lattice under the set inclusion [20].

The diagonal relation Δ = {(x, x) | x ∈ L} is always a co-distributive element in a weak congruence lattice, i.e., for all ρ, θ ∈ $C_w(\mathcal{A})$, the following holds:

$$\Delta \wedge (\rho \vee \theta) = (\Delta \wedge \rho) \vee (\Delta \wedge \theta)$$

The filter $[\Delta) = \Delta\uparrow = \{\Theta \in C_w(\mathcal{A}) \mid \Theta \geq \Delta\}$ is isomorphic to the lattice of congruences Con $\mathcal{A}$, and the ideal $(\Delta] = \Delta\downarrow = \{\Theta \in C_w(\mathcal{A}) \mid \Theta \leq \Delta\}$, consisting of all the diagonal relations is isomorphic with Sub $\mathcal{A}$, under the mapping ρ → {x | x ρ x}[20]. The atoms are always join-irreducible and the co-atoms are always meet-irreducible in any lattice. If L is a chain then Sub (L) is Boolean [13]. In C_w (L), Sub (L) is atomic [13]. It is easily seen that Con (L) is Boolean in the case when L is a chain [13].

We use the following notations throughout the paper:

$C_w(L)$ is the set of all weak congruences on a lattice L.

Sub (L) is the set of all sublattices of L.

Con (L) is the set of all congruences on L.

Definition: 2.1. An equivalence relation Θ (that is, a reflexive, symmetric, and transitive binary relation) on a lattice L is called a congruence relation on L iff $(a_0, b_0) \in \Theta$ and $(a_1, b_1) \in \Theta$ imply that $(a_0 \wedge a_1, b_0 \wedge b_1) \in \Theta$ and $(a_0 \vee a_1, b_0 \vee b_1) \in \Theta$ (Substitution Property).

Definition: 2.2. [20] A binary relation Θ on a lattice L is called a weak congruence relation, if it is a symmetric and transitive binary relation satisfying the substitution property, that is $a_0, b_0, a_1, b_1 \in L$, $(a_0, b_0) \in \Theta$ and $(a_1, b_1) \in \Theta$ imply that $(a_0 \wedge a_1, b_0 \wedge b_1) \in \Theta$ and $(a_0 \vee a_1, b_0 \vee b_1) \in \Theta$.

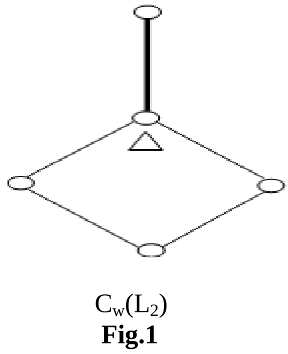
Definition: 2.3. A lattice L is said to be supersolvable, if it contains a maximal chain called an M-chain in which every element is modular. An element a of L is said to be modular, if for all x, y ∈ L, a ≤ y implies $a \vee (x \wedge y) = (a \vee x) \wedge y$.

Definition: 2.4. Let P be a poset with 0. P is said to be simplicial if for every element t ∈ P, [0, t] is Boolean. A dual simplicial poset is defined dually.

Definition: 2.5. Let P be a finite graded poset. A poset P is said to be Eulerian if the Mobius function on P satifies the following condition $\mu(x, y) = (-1)^{l(x,y)}$ for all x ≤ y in P, where $l(x, y) = \rho(y) - \rho(x)$ and ρ is the rank function of P. An equivalent definition for an Eulerian poset is "A finite graded poset P is Eulerian if and only if all intervals [x, y] of length l ≥ 1 in P contain an equal number of elements of odd and even ranks".

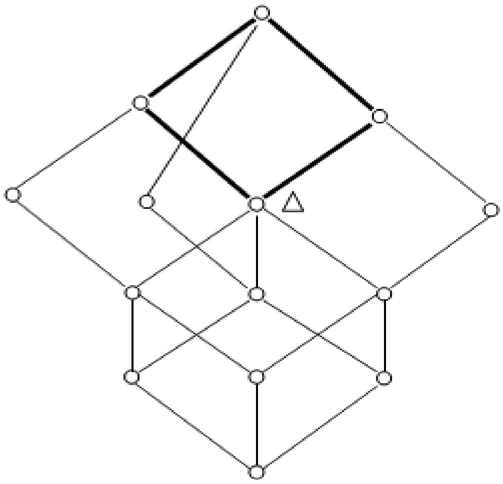
We produce below the lattice structures of weak congruences of chains up to four elements and Boolean lattice rank 2.

1. The lattice of weak congruences $C_w(L_2)$ of a two-element chain L_2 is given in Fig. 1[17].



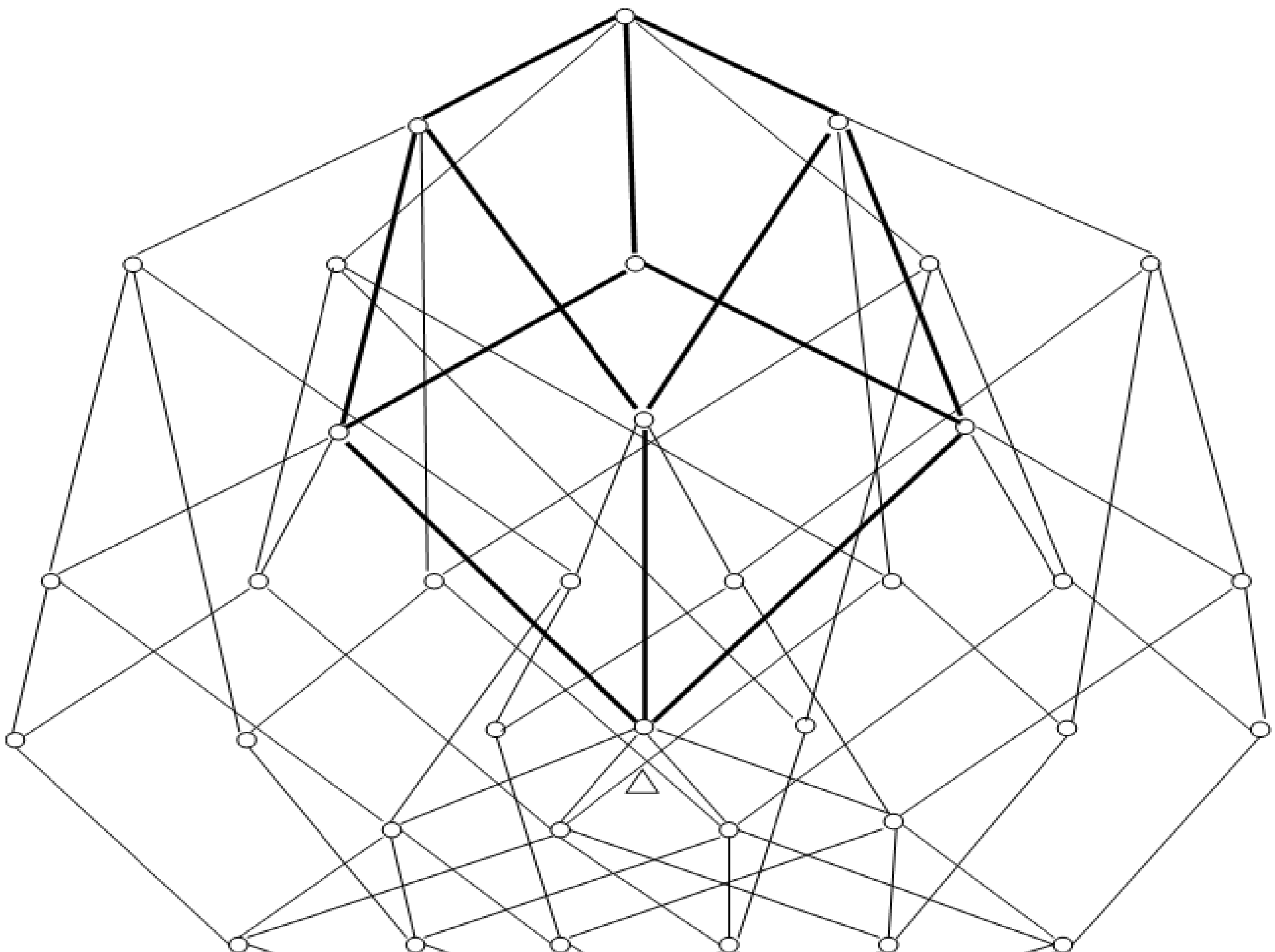
$C_w(L_2)$
Fig.1

2. The lattice of weak congruences $C_w(L_3)$ of a three-element chain L_3 is given in Fig. 2[17].



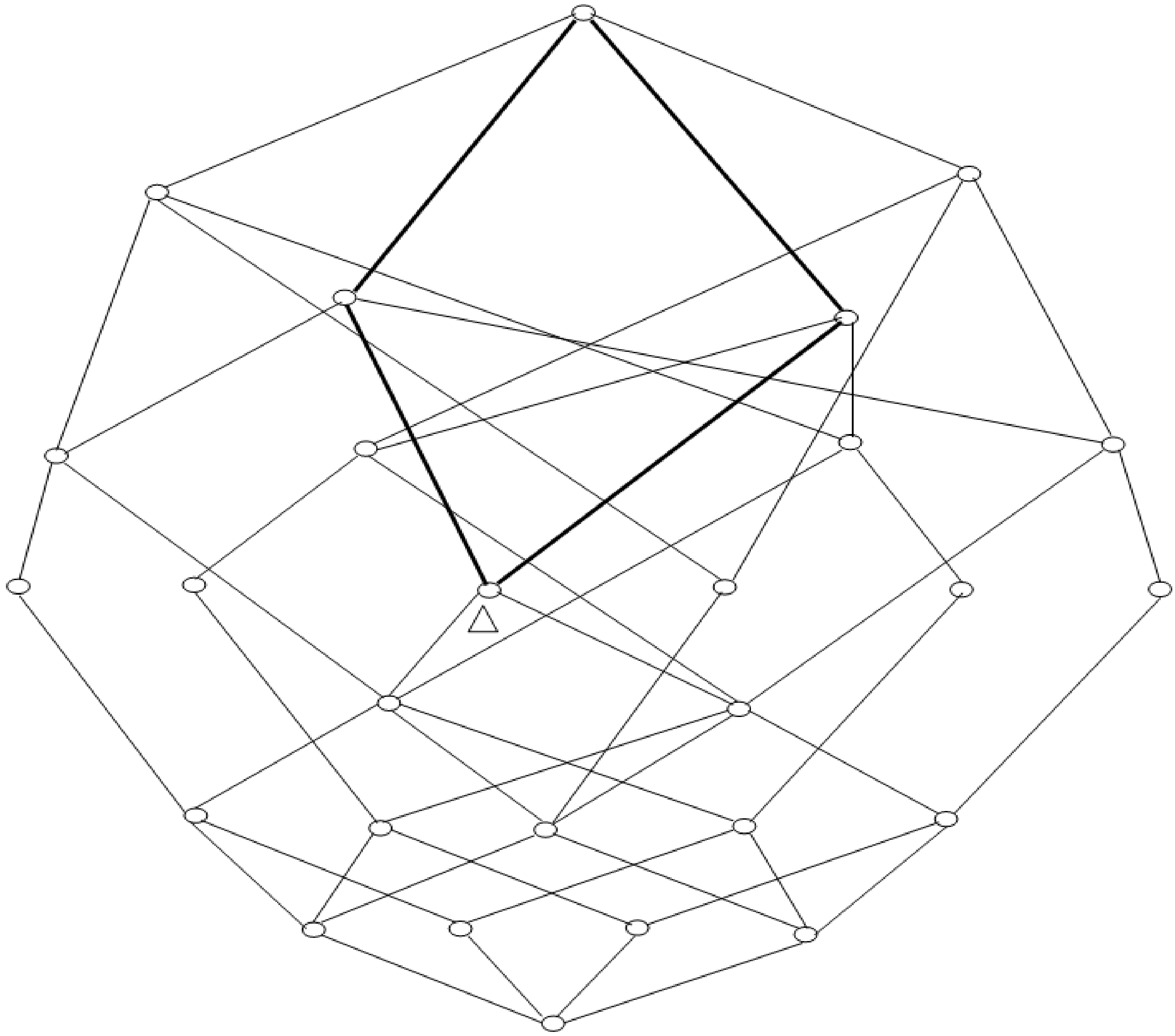
$C_w(L_3)$
Fig.2

3. The lattice of weak congruences $C_w(L_4)$ of a four-element chain L_4 is given in Fig. 3.



$C_w(L_4)$
Fig.3

4. The lattice of weak congruences $C_w(B_2)$ of a rank 2 Boolean lattice B_2 is given in Fig. 4.



$C_w(B_2)$
Fig. 4

Theorem 1: If L_n is a chain of n elements then $C_w(L_n)$ is supersolvable.

Proof: Consider the chain $\emptyset \prec \{(0,0)\} \prec \{(0,0),(1,1)\} \prec \{0,1\}^2 \prec \{0,x_1,1\}^2 \prec \{0,x_1,\ x_{2,}\ 1\}^2 \prec$

$\{0,x_1,\ x_{2,} \ldots\ldots x_{k,}1\}^2 \prec \tau$ in $C_w(L_n)$.

Since $\{(0, 0)\}$ and $\{(0, 0), (1, 1)\} \in Sub(L_n) \cong B_n$. Therefore, $\{(0, 0)\}$ and $\{(0,0), (1, 1)\}$ are modular in $C_w(L_n)$.

We claim that $\Theta = \{0, 1\}^2 = \{(0, 0), (0, 1), (1, 0), (1, 1)\}$ is modular.

Let $\Theta \leq Y$.

To prove that for all $X \in C_w(L_n)$, $(\Theta \vee X) \cap Y = \Theta \vee (X \cap Y)$:

Y is of the form $Y = \{0, x_1, x_2,\ldots,x_k, 1\}^2$ where k satisfies, $1 \leq k \leq n-1$.

Case (i) $X \cap Y = \emptyset$

RHS = $\Theta \vee (X \cap Y) = \Theta$
LHS = $(\Theta \vee X) \cap Y = \Theta \vee X$

Since $X \cap Y = \emptyset$, the components of elements of X and Y are distinct. $\Theta \vee X$ cannot contain elements with components equal to that of Y. For, if $(0, x_k) \in \Theta \vee X$ and x_k is a component of an element in X. Therefore, $(x_k, x_k) \in X$ also $(x_k, x_k) \in Y$ which implies $(x_k, x_k) \in X \vee Y = \emptyset$, which is contradiction. Therefore, $\Theta \vee X$ is comparable with Y.

Therefore, $(\Theta \vee X) \cap Y = \Theta$.

Case (ii) $X \cap Y = \Theta$

$\Theta \subset X$ and $\Theta \subset Y$.

Therefore, $\Theta \vee X = X$

RHS = $\Theta \vee (X \cap Y) = \Theta \vee \Theta = \Theta$

LHS = $(\Theta \vee X) \cap Y = X \cap Y = \Theta$.

Case (iii) $X \cap Y \neq \emptyset$

Y is of the form, $Y = \{0, x_1, x_2,\ldots,x_k, 1\}^2$ where k satisfies, $1 \leq k \leq n-1$.

Then $X \cap Y$ is in one of the following forms $\{(0, 0)\}$, $\{(x_1, x_1)\}$, $\{(x_2, x_2)\}$… $\{(x_k, x_k)\}$, $\{(1, 1)\}$ and their combinations, Y_1^2 for any subset Y_1 of A and $Y_1^2 \cup$ {some additional diagonal elements not in Y_1^2}

(1) Suppose that X will contain elements with components some of $x_1, x_2,\ldots,x_k$, say, , $x_1, x_2,\ldots,x_l$, $(l < k)$.

Therefore, $\Theta \vee X = \{0, x_1, x_2,\ldots,x_k, 1\}^2$.

LHS = $(\Theta \vee X) \cap Y = \{0, x_1, x_2,\ldots,x_k, 1\}^2$.

Now $X \cap Y = \{ x_1, x_2,\ldots,x_l\}$

RHS = $\Theta \vee (X \cap Y) = \{0, x_1, x_2,\ldots,x_l, 1\}^2$.

(2) X does not contain any element with components $x_1, x_2,\ldots,x_k$.

Therefore, $X \cap Y$ is either equal to $\{(0, 0)\}$ or $\{(1, 1)\}$ or $\{(0, 0), (1, 1)\}$ or Θ.

Therefore, RHS = $\Theta \vee (X \cap Y) = \Theta$.

LHS = $(\Theta \vee X) \cap Y = \Theta$.

Therefore, in both the cases, $(\Theta \vee X) \cap Y = \Theta \vee (X \cap Y)$.

Similarly, we can prove that any Θ which is of the form $\{0, x_1, x_2,\ldots,x_m, 1\}^2$ is modular.

Therefore, the maximal chain $\emptyset \prec \{(0,0)\} \prec \{(0,0),(1,1)\} \prec \{0,1\}^2 \prec \{0,x_1,1\}^2 \prec \{0,x_1,\ x_{2,}\ 1\}^2 \prec \{0,x_1,\ x_{2,} \ldots\ldots x_{k,} 1\}^2 \prec \tau$ is an M-chain in $C_w(L_n)$.

Therefore, $C_w(L_n)$ is supersolvable.

Corollary 2: $C_w(B_n)$, $C_w(M_n)$. $C_w(C_n)$. $C_w(N_5)$ are supersolvable.

Proof: The maximal chains containing 0 and 1 of B_n, M_n, C_n, and N_5 will generate M-chain in their corresponding lattice of weak congruences. Therefore, $C_w(B_n)$, $C_w(M_n)$. $C_w(C_n)$. $C_w(N_5)$ are supersolvable.

Conclusion: Since any bounded lattice will have such maximal chains, we conclude that $C_w(L)$ is supersolvable, for any bounded lattice.

Lemma 3: $C_w(L)$ is simplicial if and only if L is a two-element chain.

Proof: If L is a two-element chain L_2 then $C_w(L_2)$ is simplicial which can be easily seen from the figure 1

Conversely, if $C_w(L)$ is simplicial and $L \ncong L_2$ then $C_w(L)$ contains a join-irreducible element of the form $\{(x_1, x_1)\}^2$, where $x_1 \leq x_2$ and the interval $[\emptyset, \{(x_1, x_1)\}^2]$ is not Boolean.

Lemma 4: For any lattice L, $C_w(L)$ is not dual simplicial.

Proof: The atoms of $C_w(L)$ are of the form $\{(x, x)\}$, where $x \in L$. The upper interval $[\{(x, x)\}, \tau]$ contains the join-irreducible element $\{0, x\}^2$ which is of rank 2 in the upper interval which cannot be true in a Boolean algebra.

Therefore, $C_w(L)$ is not simplicial for any lattice L.

Lemma 5: For any lattice L, $C_w(L)$ is not Eulerian.

Proof: The atoms of $C_w(L)$ are of the form $\{(x, x)\}$, where $x \in L$. The upper interval $[\{(x, x)\}, \tau]$ contains the join-irreducible element $\{0, x\}^2$ which is of rank 2 in the upper interval which cannot be true in a Eulerian lattice since an Eulerian lattice is atomic. Therefore, $C_w(L)$ is not Eulerian for any lattice L.

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