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A SUFFICIENT CONDITION FOR A SEQUENCE IN A MENGER SPACE TO BE CAUCHY

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ABSTRACT

In this paper we obtain sufficient condition for a sequence in a Menger space to be Cauchy. Incidentally, we observe a fallacy in the argument of a result of B.K. Sharma and D.R. Sahu [4].

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1. INTRODUCTION:

Let us start with the following definitions.

Definition 1.1: (Schweizer and Sklar, [3]): A function $F: \mathbb{R} \to [0,1]$ is called a distribution function if

(i) F is non-decreasing,
(ii) F is left continuous,
(iii) $\inf_{x\in\mathbb{R}} F(x) = 0$ *and* $\sup_{x\in\mathbb{R}} F(x) = 1$

$\mathfrak{D}$ denotes for the class of all distribution functions.

Definition1.2: (Schweizer and Sklar, [3]): A triangular norm $* : [0,1] \, x \, [0,1] \to [0,1]$ is a function satisfying the following conditions:

(i) $\alpha * 1 = \alpha \quad \forall \alpha \in [0,1]$
(ii) $\alpha * \beta = \beta * \alpha \quad \forall \alpha, \beta \epsilon [0,1]$
(iii) $\gamma * \delta \geq \alpha * \beta \ \forall \alpha, \beta, \gamma, \delta \epsilon [0,1]$ *with* $\gamma \geq \alpha$ *and* $\delta \geq \beta$
(iv) $(\alpha * \beta) * \gamma = \alpha * (\beta * \gamma) \quad \forall \alpha, \beta, \gamma \in [0,1]$

Definition 1.3: (Schweizer and Sklar, [3]): Let X be a non-empty set and let $F: X \times X \to \mathfrak{D}$. For $p, q \in X$, we denote the image of the pair (p, q) by $F_{p,q}$ which is a distribution function so that $F_{p,q}(x) \in [0,1]$, for every real x. Suppose F satisfies:

(i) $F_{p,q}(x) = 1$ for all $x > 0$ if and only if $p = q$,
(ii) $F_{p,q}(0) = 0$,
(iii) $F_{p,q}(x) = F_{q,p}(x)$,
(iv) If $F_{p,q}(x) = 1$ and $F_{q,r}(y) = 1$ then $F_{p,r}(x + y) = 1$ where $p, q, r \in X$.

Then (X, F) is called a probabilistic metric space.

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Definition 1.4: (Schweizer and Sklar, [3]): Let X be a non empty set, $*$ a t-norm and $F : X \times X \to \mathfrak{D}$ satisfies:

(i) $F_{p,q}(0) = 0 \ \forall \ p,q \in X,$

(ii) $F_{p,q}(x) = 1 \ \forall \ x > 0$ if and only if $p = q$,

(iii) $F_{p,q}(x) = F_{q,p}(x) \ \forall \ p,q \in X,$

(iv) $F_{p,r}(x+y) \geq *(F_{p,q}(x), F_{q,r}(y)) \ \forall \ x,y \geq 0 \ and \ p,q,r \in X.$

Then the triplet $(X,F,*)$ is called a Menger space.

Definitions 1.5: (Schweizer and Sklar, [3]):

(i) Let $(X,F,*)$ be a Menger space and $p \in X$. For $\varepsilon > 0, 0 < \lambda < 1,$ the (ε,λ)- neighbourhood of p is defined as $U_p(\varepsilon,\lambda) = \{q \in X: F_{p,q}(\varepsilon) > 1 - \lambda\}$.
The topology induced by the family $\{U_p(\varepsilon,\lambda): p \in X, \varepsilon > 0, 0 < \lambda < 1\}$ is known as the (ε,λ) -topology. It may be observed that, if $*$ is continuous then the (ε,λ) -topology is Hausdorff.

(ii) A sequence $\{x_n\}$ in X is said to converge to $p \in X$ in the (ε,λ) -topology, if for any $\varepsilon > 0$ and $0 < \lambda < 1$ there exists a positive integer $N = N(\varepsilon,\lambda)$ such that $F_{x_n,p}(\varepsilon) > 1 - \lambda$ where $n > N$.

(iii) A sequence $\{x_n\}$ in X is said to be a Cauchy sequence in the (ε,λ) - topology, if for $\varepsilon > 0$ and $0 < \lambda < 1$ there exists a positive integer $N = N(\varepsilon,\lambda)$ such that
$F_{x_m,x_n}(\varepsilon) > 1 - \lambda$ for all $m,n > N$.

(iv) A Menger space$(X,F,*)$, where $*$ is continuous, is said to be complete if every Cauchy sequence in X is convergent in the (ε,λ) -topology.

Definition 1.6: (Hadzic, [1]): Let $*$ be a t-norm. For any $a \in [0,1]$, write $*_0(a) = 1$ and

$*_1(a) = *(*_0(a), a) = *(1,a) = a$

In general define $*_{n+1}(a) = *(*_n(a), a) \ for \ n = 0,1,2 \ldots..$

If the sequence $\{*_n\}$ is equicontinuous at 1, that is given $\varepsilon > 0$ *there exists* $\delta > 0$ *such that* $x > 1 - \delta$ *implies* $*_n(x) > 1 - \varepsilon \ \forall \ n \in \mathbb{N}$, then we say that $*$ is a Hadzic type t-norm.

We observe that 'min't-norm is of Hadzic type.

B. K. Sharma and D. R. Sahu [4] proved the following theorem.

Theorem 1.7: (Sharma and Sahu, [4], Theorem 3.1): Let $(X,F,*)$ be a Menger space and C, D be non empty subsets of X. Let$\{A_i\}_{i\in\mathbb{N}}: C \to D$. If there exists a function $\varphi: \mathbb{R}^+ \to \mathbb{R}^+$ which is non decreasing, $\lim_{n\to\infty} \varphi^n(t) = \infty \ \forall \ t > 0$ and mappings $S,T: C \to D$ such that for each $i,j \in \mathbb{N}$, $i \neq j$

(1.7.1) $A_i(C) \subseteq T(C)$ and $A_j(C) \subseteq S(C),$

(1.7.2) $F_{A_ix,A_jy}(a) \geq F_{Sx,Ty}(\varphi(a))$ for all x,y in C and for all $a > 0$,

(1.7.3) $T(C)$ is complete.

Then for each $i,j \in \mathbb{N}$, $i \neq j$, there exist $u,v \in C$ and $z \in D$ such that $A_iu = Su = z = A_jv = Tv$. If $C = D$ and for $i,j \in \mathbb{N}$, $i \neq j$, A_i and S, and A_j and T are 2- compatible, then z is the unique common fixed point of S,T and the family $\{A_i\}_{i\in\mathbb{N}}$.

In proving this theorem the sequence $\{y_n\}$ is constructed as follows:

Let $y_0 \in C$, the sequence $\{x_n\}$ is defined by $Tx_{2n+1} = A_ix_{2n} = y_{2n}$ and $Sx_{2n+2} = A_jx_{2n+1} = y_{2n+1} \ for \ n = 0,1,2 \ldots .$

Then it is observed that for all $a > 0 \ and \ n \in \mathbb{N}$

$F_{y_n,y_{n+1}}(a) \geq F_{y_{n-1},y_n}(\varphi(a)) \geq \ \ldots. \geq F_{y_0,y_1}(\varphi^n(a)).$

To show that $\{y_n\}$ is a Cauchy sequence, the authors proceeded like this:

$$F_{y_n,y_{n+m}}(a) \geq F_{y_n,y_{n+m}}\left(a\left(1-\frac{1}{m+1}\right)\right)$$

$$\geq \underbrace{* \ldots\ldots *}_{\text{m times}} \left(F_{y_n,y_{n+1}}\left(\frac{a}{1.2}\right), \ldots\ldots\ldots, F_{y_{n+m-1},y_{n+m}}\left(\frac{a}{m(m+1)}\right)\right)$$

$$\geq *\left(* \ldots * \left(F_{y_0,y_1}\left(\varphi^n\left(\frac{a}{1.2}\right)\right), F_{y_0,y_1}\left(\varphi^{n+1}\left(\frac{a}{2.3}\right)\right), \ldots, F_{y_0,y_1}\left(\varphi^{n+m-1}\left(\frac{a}{m(m+1)}\right)\right)\right)\right)$$

Finally the authors concluded that $\{y_n\}$ is a Cauchy sequence, since $\{*_n(x)\}$ is equicontinuous at $x = 1$ and $\lim_{n\to\infty} \varphi^{n+m-1}(a) = \infty\ \forall\ a > 0$.

However this argument is not sufficient to conclude this, since $\varphi^{n+m-1}\left(\frac{a}{m(m+1)}\right)$ may not go to ∞ $as\ n \to \infty\ for\ all\ m \in \mathbb{N}$.

This is shown in the following example:

Example 1.8: Define $\varphi: \mathbb{R}^+ \to \mathbb{R}^+$ by $\varphi(t) = \begin{cases} n+1 & if\ \ t \in [n, n+1), n \geq 1 \\ \frac{1}{m} & if\ \ t \in \left[\frac{1}{m+1}, \frac{1}{m}\right),\ m \geq 1 \\ 0 & if \qquad t = 0 \end{cases}$

For any $a > 0\ and\ m \in \mathbb{N}$, $\lim_{n\to\infty} \varphi^{n+m-1}\left(\frac{a}{m(m+1)}\right) \neq \infty$. In fact $\varphi^{n+m-1}\left(\frac{a}{m(m+1)}\right) = \frac{1}{n^2+n+2}$ for every $m \in \mathbb{N}$.

Thus the argument does not hold.

Consequently, in this paper we obtain a sufficient condition for a sequence in a Menger space to be Cauchy.

2. MAIN RESULTS:

In this section we obtain a sufficient condition for a sequence in a Menger space to be Cauchy. First we start with a definition.

Definition 2.1: (i) If $\varphi: \mathbb{R}^+ \to \mathbb{R}^+$ is such that

$(a)\varphi$ is increasing,
$(b)\varphi(t) > t\ \ \forall\ \ t > 0$,
$(c)\varphi(\varphi(t) - t) \geq \varphi^2(t) - \varphi(t)\ \ for\ every\ t > 0$,

then φ is called a contractive control function of type (A).

(ii) If $\varphi: \mathbb{R}^+ \to \mathbb{R}^+$ is such that a contractive control function which is strictly increasing, φ is onto and $\varphi\left(t - \varphi^{-1}(t)\right) \geq \varphi(t) - t\ for\ every\ t > 0$, then φ is called a contractive control function of type (AS).

Examples 2.2: If $\varphi: \mathbb{R}^+ \to \mathbb{R}^+$ is defined by $\varphi(t) = \begin{cases} n+1 & if\ \ t \in [n, n+1) \\ 1 & if\ \ \ t \in (0,1) \\ 0 & if \qquad t = 0 \end{cases}$

then φ is a contractive control function of type (A) but not of type (AS).

Examples 2.3: If $\varphi \mathbb{R}^+ \to \mathbb{R}^+$ is defined by $\varphi(t) = kt\ \ \ \forall\ \ t > 0\ and\ for\ some\ k > 0$, then φ is a contractive control function of type (AS).

We use the following lemma in our main result.

Lemma 2.4:

(i) If $\varphi: \mathbb{R}^+ \to \mathbb{R}^+$ is a contractive control function of type (A), then

$\varphi^n(\varphi(t) - t) \geq \varphi^{n+1}(t) - \varphi^n(t) for\ every\ n \in \mathbb{N}$.

(ii). If $\varphi:\mathbb{R}^+\to\mathbb{R}^+$ is a contractive control function of type (AS), then

$\varphi^n\big(\ t-\varphi^{-1}(t)\big)\geq\varphi^n(t)-\varphi^{n-1}(t)\ for\ every\ n\in\mathbb{N}$.

Proof: (i) Since $\varphi(\varphi(t)-t)\geq\varphi^2(t)-\varphi(t)\ for\ every\ t>0$, the result is true.

Now the proof follows by induction.

Assume that $\varphi^n(\varphi(t)-t)\geq\varphi^{n+1}(t)-\varphi^n(t)$

$$\begin{aligned}\Rightarrow\ \varphi^{n+1}(\varphi(t)-t)&\geq\varphi(\varphi^{n+1}(t)-\varphi^n(t))\\&=\varphi(\varphi(s)-s)\ \ where\ s=\ \varphi^n(t)\\&\geq\varphi^2(s)-\varphi(s)\\&=\varphi^2\big(\varphi^n(t)\big)-\varphi(\varphi^n(t))\\&=\varphi^{n+2}(t)-\varphi^{n+1}(t)\end{aligned}$$

Therefore by induction, $\varphi^n(\varphi(t)-t)\geq\varphi^{n+1}(t)-\varphi^n(t) for\ all\ n\in\mathbb{N}$.

(ii) Since $\varphi\big(t-\varphi^{-1}(t)\big)\geq\varphi(t)-t$, the result is true for n = 1.

The proof now follows by induction.

Assume that $\varphi^n\big(\ t-\varphi^{-1}(t)\big)\geq\varphi^n(t)-\varphi^{n-1}(t)$

$$\begin{aligned}\Rightarrow\varphi^{n+1}\big(\ t-\varphi^{-1}(t)\big)&\geq\varphi(\varphi^n(t)-\varphi^{n-1}(t))\\&=\varphi\big(s-\varphi^{-1}(s)\big)\ \ \text{where } s=\varphi^n(t)\\&\geq\varphi(s)-s\\&=\ \varphi\big(\varphi^n(t)\big)-\varphi^n(t)\\&\geq\varphi^{n+1}(t)-\varphi^n(t)\end{aligned}$$

Therefore by induction, $\varphi^n\big(\ t-\varphi^{-1}(t)\big)\geq\varphi^n(t)-\varphi^{n-1}(t)\ for\ every\ n\in\mathbb{N}$.

Notation: Let Φ be the class of contractive control functions of type (AS) such that $\varphi^n(t)-\varphi^{n-1}(t)\to\infty\ as\ n\to\infty$ for every $t>0$.

We observe that $\varphi\in\Phi\Rightarrow\varphi^n(t)\to\infty\ as\ n\to\infty$

Theorem 2.5: Let $(X,F,*)$ be a Menger space with Hadzic type t-norm * and $\varphi\in\Phi$. If $\{x_n\}$ is a sequence in X such that $F_{x_n,x_{n+1}}(t)\geq\ F_{x_{n-1},x_n}(\varphi(t))$ for every $t>0$, then $\{x_n\}$ is a Cauchy sequence in X.

Proof: By hypothesis,
$$\begin{aligned}F_{x_n,x_{n+1}}(t)\geq F_{x_{n-1},x_n}\big(\varphi(t)\big)\geq\ \ldots\ &\geq F_{x_0,x_1}\big(\varphi^n(t)\big)\qquad(2.5.1)\\&\geq F_{x_0,x_1}\big(\varphi^n(t)-\varphi^{n-1}(t)\big)\\&=\lambda_n(t)\end{aligned}$$

Since $\varphi\in\Phi,\lambda_n(\mathrm{t})\to1\ as\ n\to\infty$.

Now we show that $F_{x_n,x_{n+k}}(t)\geq *_k\ (\lambda_n(t))$

This is true for $k=1$ and any $n\in\mathbb{N}$ by (2.5.1)

Assume the truth for k.

$$F_{x_n,x_{n+k+1}}(t) \geq *\left(F_{x_n,x_{n+1}}(t-\varphi^{-1}(t)), F_{x_{n+1},x_{n+k+1}}\left(\varphi^{-1}(t)\right)\right)$$

$$\geq *\left(F_{x_0,x_1}\left(\varphi^n\left(t-\varphi^{-1}(t)\right)\right), *_k\left(F_{x_0,x_1}\left(\varphi^{n+1}(\varphi^{-1}(t)) - \varphi^n(\varphi^{-1}(t))\right)\right)\right)$$

$$\geq *\left(F_{x_0,x_1}\left(\varphi^n(t)-\varphi^{n-1}(t)\right), *_k\left(F_{x_0,x_1}\left(\varphi^n(t)-\varphi^{n-1}(t)\right)\right)\right), \text{ by Lemma 2.4}$$

$$= *_{k+1}\left(F_{x_0,x_1}\left(\varphi^n(t)-\varphi^{n-1}(t)\right)\right)$$

$$= *_{k+1}(\lambda_n(t)) \tag{2.5.2}$$

Let $\varepsilon > 0$, since $*$ is Hadzic type t-norm and $*$ is equicontinuous at 1, there exists $\eta \in (0,1)$ such that

$1 \geq s > 1-\eta \ implies \ *_{k+1}(s) > 1-\varepsilon.$

Since $\lambda_n(\text{t}) \to 1 \ as \ n \to \infty$, there exists N such that $n \geq N$ implies $\lambda_n(\text{t}) > 1-\eta$.

Hence by (2.5.2), we have $F_{x_n,x_{n+k+1}}(t) \geq *_{k+1}\left(\lambda_n(t)\right)$

$$> 1-\boldsymbol{\varepsilon} \ for\ all\ n \geq N.$$

Consequently $F_{x_n,x_m}(t) > \varepsilon$ whenever $m > n \geq N$.

Hence $\{x_n\}$ is a Cauchy sequence.

We conclude this paper with an open problem:

Open Problem 2.6: Is the above theorem still true if φ is not necessarily a member of Φ but is such that φ is increasing and $\varphi^n(t) \to \infty \ as \ n \to \infty \ \forall \ t > 0$?

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