

A SUFFICIENT CONDITION FOR A SEQUENCE
IN A MENGER SPACE TO BE CAUCHY

K. P. R. Sastry¹, G. A. Naidu², S. S. A. Sastri^{3*} and P. V. S. Prasad⁴

¹8-28-8/1, Tamil Street, Chinna Waltair, Visakhapatnam-530017, India
E-mail: kprsastry@hotmail.com

^{2,4}Department of Mathematics, Andhra University, Visakhapatnam-530 003, India
E-mail: drgolivean@yahoo.com, pvsprasad10@yahoo.in

³Department of Mathematics, GVP College of Engineering, Madhurawada, Visakhapatnam- 530048, India
E-mail: sambharasas@yahoo.co.in

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ABSTRACT

In this paper we obtain sufficient condition for a sequence in a Menger space to be Cauchy. Incidentally, we observe a fallacy in the argument of a result of B.K. Sharma and D.R. Sahu [4].

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1. INTRODUCTION:

Let us start with the following definitions.

Definition 1.1: (Schweizer and Sklar, [3]): A function $F: \mathbb{R} \rightarrow [0,1]$ is called a distribution function if

- (i) F is non-decreasing,
- (ii) F is left continuous,
- (iii) $\inf_{x \in \mathbb{R}} F(x) = 0$ and $\sup_{x \in \mathbb{R}} F(x) = 1$

\mathfrak{D} denotes for the class of all distribution functions.

Definition 1.2: (Schweizer and Sklar, [3]): A triangular norm $*: [0,1] \times [0,1] \rightarrow [0,1]$ is a function satisfying the following conditions:

- (i) $\alpha * 1 = \alpha \quad \forall \alpha \in [0,1]$
- (ii) $\alpha * \beta = \beta * \alpha \quad \forall \alpha, \beta \in [0,1]$
- (iii) $\gamma * \delta \geq \alpha * \beta \quad \forall \alpha, \beta, \gamma, \delta \in [0,1]$ with $\gamma \geq \alpha$ and $\delta \geq \beta$
- (iv) $(\alpha * \beta) * \gamma = \alpha * (\beta * \gamma) \quad \forall \alpha, \beta, \gamma \in [0,1]$

Definition 1.3: (Schweizer and Sklar, [3]): Let X be a non-empty set and let $F: X \times X \rightarrow \mathfrak{D}$. For $p, q \in X$, we denote the image of the pair (p, q) by $F_{p,q}$ which is a distribution function so that $F_{p,q}(x) \in [0,1]$, for every real x . Suppose F satisfies:

- (i) $F_{p,q}(x) = 1$ for all $x > 0$ if and only if $p = q$,
- (ii) $F_{p,q}(0) = 0$,
- (iii) $F_{p,q}(x) = F_{q,p}(x)$,
- (iv) If $F_{p,q}(x) = 1$ and $F_{q,r}(y) = 1$ then $F_{p,r}(x+y) = 1$ where $p, q, r \in X$.

Then (X, F) is called a probabilistic metric space.

Corresponding author: S. S. A. Sastri^{3}, *E-mail: sambharasas@yahoo.co.in

Definition 1.4: (Schweizer and Sklar, [3]): Let X be a non empty set, $*$ a t-norm and $F : X \times X \rightarrow \mathfrak{D}$ satisfies:

- (i) $F_{p,q}(0) = 0 \quad \forall p, q \in X,$
- (ii) $F_{p,q}(x) = 1 \quad \forall x > 0$ if and only if $p = q,$
- (iii) $F_{p,q}(x) = F_{q,p}(x) \quad \forall p, q \in X,$
- (iv) $F_{p,r}(x+y) \geq * (F_{p,q}(x), F_{q,r}(y)) \quad \forall x, y \geq 0$ and $p, q, r \in X.$

Then the triplet $(X, F, *)$ is called a Menger space.

Definitions 1.5: (Schweizer and Sklar, [3]):

- (i) Let $(X, F, *)$ be a Menger space and $p \in X$. For $\varepsilon > 0, 0 < \lambda < 1$, the (ε, λ) - neighbourhood of p is defined as $U_p(\varepsilon, \lambda) = \{q \in X: F_{p,q}(\varepsilon) > 1 - \lambda\}$.
The topology induced by the family $\{U_p(\varepsilon, \lambda): p \in X, \varepsilon > 0, 0 < \lambda < 1\}$ is known as the (ε, λ) -topology.
It may be observed that, if $*$ is continuous then the (ε, λ) -topology is Hausdorff.
- (ii) A sequence $\{x_n\}$ in X is said to converge to $p \in X$ in the (ε, λ) -topology, if for any $\varepsilon > 0$ and $0 < \lambda < 1$ there exists a positive integer $N = N(\varepsilon, \lambda)$ such that $F_{x_n,p}(\varepsilon) > 1 - \lambda$ where $n > N$.
- (iii) A sequence $\{x_n\}$ in X is said to be a Cauchy sequence in the (ε, λ) - topology, if for $\varepsilon > 0$ and $0 < \lambda < 1$ there exists a positive integer $N = N(\varepsilon, \lambda)$ such that $F_{x_m,x_n}(\varepsilon) > 1 - \lambda$ for all $m, n > N$.
- (iv) A Menger space $(X, F, *)$, where $*$ is continuous, is said to be complete if every Cauchy sequence in X is convergent in the (ε, λ) -topology.

Definition 1.6: (Hadzic, [1]): Let $*$ be a t-norm. For any $a \in [0,1]$, write $*_0(a) = 1$ and

$$*_1(a) = * (*_0(a), a) = * (1, a) = a$$

In general define $*_{n+1}(a) = * (*_n(a), a)$ for $n = 0, 1, 2 \dots$

If the sequence $\{*_n\}$ is equicontinuous at 1, that is given $\varepsilon > 0$ there exists $\delta > 0$ such that $x > 1 - \delta$ implies $*_n(x) > 1 - \varepsilon \quad \forall n \in \mathbb{N}$, then we say that $*$ is a Hadzic type t-norm.

We observe that 'min't-norm is of Hadzic type.

B. K. Sharma and D. R. Sahu [4] proved the following theorem.

Theorem 1.7: (Sharma and Sahu, [4], Theorem 3.1): Let $(X, F, *)$ be a Menger space and C, D be non empty subsets of X . Let $\{A_i\}_{i \in \mathbb{N}}: C \rightarrow D$. If there exists a function $\varphi: \mathbb{R}^+ \rightarrow \mathbb{R}^+$ which is non decreasing, $\lim_{n \rightarrow \infty} \varphi^n(t) = \infty \quad \forall t > 0$ and mappings $S, T: C \rightarrow D$ such that for each $i, j \in \mathbb{N}, i \neq j$

$$(1.7.1) \quad A_i(C) \subseteq T(C) \quad \text{and} \quad A_j(C) \subseteq S(C),$$

$$(1.7.2) \quad F_{A_i x, A_j y}(a) \geq F_{Sx, Ty}(\varphi(a)) \quad \text{for all } x, y \text{ in } C \text{ and for all } a > 0,$$

$$(1.7.3) \quad T(C) \text{ is complete.}$$

Then for each $i, j \in \mathbb{N}, i \neq j$, there exist $u, v \in C$ and $z \in D$ such that $A_i u = S u = z = A_j v = T v$. If $C = D$ and for $i, j \in \mathbb{N}, i \neq j$, A_i and S , and A_j and T are 2- compatible, then z is the unique common fixed point of S, T and the family $\{A_i\}_{i \in \mathbb{N}}$.

In proving this theorem the sequence $\{y_n\}$ is constructed as follows:

Let $y_0 \in C$, the sequence $\{x_n\}$ is defined by $T x_{2n+1} = A_i x_{2n} = y_{2n}$ and $S x_{2n+2} = A_j x_{2n+1} = y_{2n+1}$ for $n = 0, 1, 2 \dots$

Then it is observed that for all $a > 0$ and $n \in \mathbb{N}$

$$F_{y_n, y_{n+1}}(a) \geq F_{y_{n-1}, y_n}(\varphi(a)) \geq \dots \geq F_{y_0, y_1}(\varphi^n(a)).$$

To show that $\{y_n\}$ is a Cauchy sequence, the authors proceeded like this:

$$\begin{aligned}
 F_{y_n, y_{n+m}}(a) &\geq F_{y_n, y_{n+m}}\left(a\left(1 - \frac{1}{m+1}\right)\right) \\
 &\geq \underbrace{* \dots *}_{m \text{ times}} \left(F_{y_n, y_{n+1}}\left(\frac{a}{1.2}\right), \dots, F_{y_{n+m-1}, y_{n+m}}\left(\frac{a}{m(m+1)}\right) \right) \\
 &\geq * \left(* \dots * \left(F_{y_0, y_1}\left(\varphi^n\left(\frac{a}{1.2}\right)\right), F_{y_0, y_1}\left(\varphi^{n+1}\left(\frac{a}{2.3}\right)\right), \dots, F_{y_0, y_1}\left(\varphi^{n+m-1}\left(\frac{a}{m(m+1)}\right)\right) \right) \right)
 \end{aligned}$$

Finally the authors concluded that $\{y_n\}$ is a Cauchy sequence, since $\{*_n(x)\}$ is equicontinuous at $x = 1$ and $\lim_{n \rightarrow \infty} \varphi^{n+m-1}(a) = \infty \forall a > 0$.

However this argument is not sufficient to conclude this, since $\varphi^{n+m-1}\left(\frac{a}{m(m+1)}\right)$ may not go to ∞ as $n \rightarrow \infty$ for all $m \in \mathbb{N}$.

This is shown in the following example:

Example 1.8: Define $\varphi: \mathbb{R}^+ \rightarrow \mathbb{R}^+$ by $\varphi(t) = \begin{cases} n+1 & \text{if } t \in [n, n+1), n \geq 1 \\ \frac{1}{m} & \text{if } t \in \left[\frac{1}{m+1}, \frac{1}{m}\right), m \geq 1 \\ 0 & \text{if } t = 0 \end{cases}$

For any $a > 0$ and $m \in \mathbb{N}$, $\lim_{n \rightarrow \infty} \varphi^{n+m-1}\left(\frac{a}{m(m+1)}\right) \neq \infty$. In fact $\varphi^{n+m-1}\left(\frac{a}{m(m+1)}\right) = \frac{1}{n^2+n+2}$ for every $m \in \mathbb{N}$.

Thus the argument does not hold.

Consequently, in this paper we obtain a sufficient condition for a sequence in a Menger space to be Cauchy.

2. MAIN RESULTS:

In this section we obtain a sufficient condition for a sequence in a Menger space to be Cauchy. First we start with a definition.

Definition 2.1: (i) If $\varphi: \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is such that

- (a) φ is increasing,
- (b) $\varphi(t) > t \forall t > 0$,
- (c) $\varphi(\varphi(t) - t) \geq \varphi^2(t) - \varphi(t)$ for every $t > 0$,

then φ is called a contractive control function of type (A).

(ii) If $\varphi: \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is such that a contractive control function which is strictly increasing, φ is onto and $\varphi(t - \varphi^{-1}(t)) \geq \varphi(t) - t$ for every $t > 0$, then φ is called a contractive control function of type (AS).

Examples 2.2: If $\varphi: \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is defined by $\varphi(t) = \begin{cases} n+1 & \text{if } t \in [n, n+1) \\ 1 & \text{if } t \in (0,1) \\ 0 & \text{if } t = 0 \end{cases}$

then φ is a contractive control function of type (A) but not of type (AS).

Examples 2.3: If $\varphi: \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is defined by $\varphi(t) = kt \forall t > 0$ and for some $k > 0$, then φ is a contractive control function of type (AS).

We use the following lemma in our main result.

Lemma 2.4:

(i) If $\varphi: \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is a contractive control function of type (A), then

$$\varphi^n(\varphi(t) - t) \geq \varphi^{n+1}(t) - \varphi^n(t) \text{ for every } n \in \mathbb{N}.$$

(ii). If $\varphi: \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is a contractive control function of type (AS), then

$$\varphi^n(t - \varphi^{-1}(t)) \geq \varphi^n(t) - \varphi^{n-1}(t) \text{ for every } n \in \mathbb{N}.$$

Proof: (i) Since $\varphi(\varphi(t) - t) \geq \varphi^2(t) - \varphi(t)$ for every $t > 0$, the result is true.

Now the proof follows by induction.

Assume that $\varphi^n(\varphi(t) - t) \geq \varphi^{n+1}(t) - \varphi^n(t)$

$$\begin{aligned} \Rightarrow \varphi^{n+1}(\varphi(t) - t) &\geq \varphi(\varphi^{n+1}(t) - \varphi^n(t)) \\ &= \varphi(\varphi(s) - s) \text{ where } s = \varphi^n(t) \\ &\geq \varphi^2(s) - \varphi(s) \\ &= \varphi^2(\varphi^n(t)) - \varphi(\varphi^n(t)) \\ &= \varphi^{n+2}(t) - \varphi^{n+1}(t) \end{aligned}$$

Therefore by induction, $\varphi^n(\varphi(t) - t) \geq \varphi^{n+1}(t) - \varphi^n(t)$ for all $n \in \mathbb{N}$.

(ii) Since $\varphi(t - \varphi^{-1}(t)) \geq \varphi(t) - t$, the result is true for $n = 1$.

The proof now follows by induction.

Assume that $\varphi^n(t - \varphi^{-1}(t)) \geq \varphi^n(t) - \varphi^{n-1}(t)$

$$\begin{aligned} \Rightarrow \varphi^{n+1}(t - \varphi^{-1}(t)) &\geq \varphi(\varphi^n(t) - \varphi^{n-1}(t)) \\ &= \varphi(s - \varphi^{-1}(s)) \text{ where } s = \varphi^n(t) \\ &\geq \varphi(s) - s \\ &= \varphi(\varphi^n(t)) - \varphi^n(t) \\ &\geq \varphi^{n+1}(t) - \varphi^n(t) \end{aligned}$$

Therefore by induction, $\varphi^n(t - \varphi^{-1}(t)) \geq \varphi^n(t) - \varphi^{n-1}(t)$ for every $n \in \mathbb{N}$.

Notation: Let Φ be the class of contractive control functions of type (AS) such that $\varphi^n(t) - \varphi^{n-1}(t) \rightarrow \infty$ as $n \rightarrow \infty$ for every $t > 0$.

We observe that $\varphi \in \Phi \Rightarrow \varphi^n(t) \rightarrow \infty$ as $n \rightarrow \infty$

Theorem 2.5: Let $(X, F, *)$ be a Menger space with Hadzic type t-norm $*$ and $\varphi \in \Phi$. If $\{x_n\}$ is a sequence in X such that $F_{x_n, x_{n+1}}(t) \geq F_{x_{n-1}, x_n}(\varphi(t))$ for every $t > 0$, then $\{x_n\}$ is a Cauchy sequence in X .

Proof: By hypothesis, $F_{x_n, x_{n+1}}(t) \geq F_{x_{n-1}, x_n}(\varphi(t)) \geq \dots \geq F_{x_0, x_1}(\varphi^n(t))$ (2.5.1)

$$\begin{aligned} &\geq F_{x_0, x_1}(\varphi^n(t) - \varphi^{n-1}(t)) \\ &= \lambda_n(t) \end{aligned}$$

Since $\varphi \in \Phi$, $\lambda_n(t) \rightarrow 1$ as $n \rightarrow \infty$.

Now we show that $F_{x_n, x_{n+k}}(t) \geq *_k(\lambda_n(t))$

This is true for $k = 1$ and any $n \in \mathbb{N}$ by (2.5.1)

Assume the truth for k .

$$\begin{aligned}
 F_{x_n, x_{n+k+1}}(t) &\geq * \left(F_{x_n, x_{n+1}}(t - \varphi^{-1}(t)), F_{x_{n+1}, x_{n+k+1}}(\varphi^{-1}(t)) \right) \\
 &\geq * \left(F_{x_0, x_1}(\varphi^n(t - \varphi^{-1}(t))), *_{k+1} \left(F_{x_0, x_1}(\varphi^{n+1}(\varphi^{-1}(t)) - \varphi^n(\varphi^{-1}(t))) \right) \right) \\
 &\geq * \left(F_{x_0, x_1}(\varphi^n(t) - \varphi^{n-1}(t)), *_{k+1} \left(F_{x_0, x_1}(\varphi^n(t) - \varphi^{n-1}(t)) \right) \right), \text{ by Lemma 2.4} \\
 &= *_{k+1} \left(F_{x_0, x_1}(\varphi^n(t) - \varphi^{n-1}(t)) \right) \\
 &= *_{k+1} (\lambda_n(t))
 \end{aligned} \tag{2.5.2}$$

Let $\varepsilon > 0$, since $*$ is Hadzic type t-norm and $*$ is equicontinuous at 1, there exists $\eta \in (0,1)$ such that

$$1 \geq s > 1 - \eta \text{ implies } *_{k+1}(s) > 1 - \varepsilon.$$

Since $\lambda_n(t) \rightarrow 1$ as $n \rightarrow \infty$, there exists N such that $n \geq N$ implies $\lambda_n(t) > 1 - \eta$.

$$\begin{aligned}
 \text{Hence by (2.5.2), we have } F_{x_n, x_{n+k+1}}(t) &\geq *_{k+1}(\lambda_n(t)) \\
 &> 1 - \varepsilon \text{ for all } n \geq N.
 \end{aligned}$$

Consequently $F_{x_n, x_m}(t) > \varepsilon$ whenever $m > n \geq N$.

Hence $\{x_n\}$ is a Cauchy sequence.

We conclude this paper with an open problem:

Open Problem 2.6: Is the above theorem still true if φ is not necessarily a member of Φ but is such that φ is increasing and $\varphi^n(t) \rightarrow \infty$ as $n \rightarrow \infty \forall t > 0$?

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