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DUAL PSUEDO-COMPLEMENTED ALMOST DISTRIBUTIVE LATTICES

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ABSTRACT

The concept of a dual pseudo-complemented Almost Distributive Lattice is introduced. Necessary and sufficient conditions for an Almost Distributive Lattice to become a dual pseudo-complemented Almost Distributive Lattice are derived. It is proved that a dual pseudo-complemented Almost Distributive Lattice is equationally definable. A one to one correspondence between the set of all dual pseudo-complementations on an ADL A *and the set of all maximal elements of* A *is obtained. Also proved that the set* $A_* = \{x_* \wedge m \mid x \in A\}$ *is a Boolean algebra.*

Keywords: *Almost Distributive Lattice; Maximal element; Principal ideal; Dual pseudo-complementation.*

AMS Subject Classification: *06D15, 06D99.*

INTRODUCTION

A pseudo-complemented lattice is a lattice A such that to each $x \in A$, there exists an element $x^* \in A$ such that $x \wedge y = 0$ for any $y \in A$ if and only if $y \le x^*$. In [6], Swamy, U.M., Rao, G.C. and Nanaji Rao, G. introduced the concept of Pseudo-Complementation on Almost Distributive Lattice and studied its properties. They observed that an Almost Distributive Lattice (ADL) A can have several pseudo-complementations and they discussed the relation between the maximal elements and the pseudo-complementations on A. Unlike in lattices, the dual of an ADL is not an ADL, in general. For this reason, in this paper, the concept of a dual pseudo-complementation in an ADL is introduced and important properties of a dual pseudo-complementation in an ADL are derived. It is proved that dual pseudo-complementation on an ADL is equationally definable. A number of characterizations for an ADL to become a dual pseudo-complementated ADL are obtained.

1. PRELIMINARIES

In this section, we give the necessary definitions and important properties of an ADL taken from [5]

Definition 1.1: [5] An algebra $(A, V, \Lambda, 0)$ of type (2, 2, 0) is called an Almost Distributive Lattice (ADL) if it satisfies the following axioms:
(i) $x \vee 0 = x$
(ii) $0 \wedge x = 0$
(iii) $(x \vee y) \wedge z = (x \wedge z) \vee (y \wedge z)$
(iv) $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$
(v) $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$
(vi) $(x \vee y) \wedge y = y$ for all $x, y, z \in A$.

The binary relation $\le$ defined on an ADL A by $x \le y$ if and only if $x \wedge y = x$ or equivalently $x \vee y = y$, is a partial ordering on A. A non-empty subset I of an ADL A is called an ideal of A if $x \vee y \in I$ and $x \wedge a \in I$ for any $x, y \in I$ and $a \in A$. The principal ideal of A generated by x is denoted by $(x]$. The set $PI(A)$ of all

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principal ideals of A forms a distributive lattice under the operations $\vee$, $\wedge$ defined by $(x]\vee(y]=(x\vee y]$ and $(x]\wedge(y]=(x\wedge y]$ in which $(0]$ is the least element. If A has a maximal element m, then $(m]$ is the greatest element of $PI(A)$.

Theorem 1.2: [5] Let A be an ADL and $x, y \in A$. Then the following are equivalent:
(i) $(x]\subseteq(y]$
(ii) $y\wedge x = x$
(iii) $y \vee x = y$
(iv) $[y)\subseteq[x)$

For other properties of an ADL, we refer to [5].

2. DUAL PSEUDO-COMPLEMENTATION ON ADLS

We begin with the following definition of a dually pseudo-complementation in an ADL.

Definition 2.1: Let $(A, V, \Lambda, 0)$ be an ADL. Then a unary operation $*$ on A is called a dual pseudo-complementation on A if, for any $x, y \in A$, it satisfies the following conditions:
d$_1$: if $x\vee y = m$, then $(x_* \vee y)\wedge m = y\wedge m$.
d$_2$: $x\vee x_*$ is a maximal element of A.
d$_3$: $(x\wedge y)_* = x_* \vee y_*$.

An ADL A with a dual pseudo-complementation is called a dually Pseudo-Complemented Almost Distributive Lattice (or simply dual PCADL). Here afterwards, A stands for a dual PCADL $(A, V, \Lambda, *, 0, m)$ with a maximal element m.

In the following theorem, some important fundamental properties of $*$ which will be frequently used are given and they can be proved directly from the definition.

Theorem 2.2: For any $x, y \in A$, we have the following:
(i) $m_* = 0$.
(ii) If x is maximal, then $x_* = 0$.
(iii) $(x\wedge m)_* = x_*$.
(iv) $(x\wedge y)_* = (y\wedge x)_*$ and $(x\vee y)_* = (y\vee x)_*$.
(v) $0_* \wedge m = m$.
(vi) $m_{**}\wedge m = m$.
(vii) $x_{**}\wedge m \le x\wedge m$ and $x\wedge x_{**} = x_{**}$.

Theorem 2.3: For any $x, y \in A$, we have the following:
(i) If $x\le y$, then $y_* \le x_*$ and $x_{**}\le y_{**}$.
(ii) $x_* = x_{***}$.
(iii) $x_* = 0 \Leftrightarrow x_{**}\wedge m = m$.
(iv) $x\wedge m = m \Leftrightarrow x_{**}\wedge m = m$.

Proof: Suppose $x \le y$. Then $x = x \wedge y$. Thus $x_* = (x \wedge y)_* = (y \wedge x)_* = y_* \vee x_*$ and hence $y_* \le x_*$. Similarly, we get $x_{**} \le y_{**}$. Thus we get (i). Since $x_{**} = x \wedge x_{**}$, we get $x_{***} = (x \wedge x_{**})_* = x_* \vee x_{***} = x_*$ (by Theorem 2.2 (vii)). We get (iii), by using the facts that $0_* \wedge m = m, m_* = 0$ and $x_* = x_{***}$. Now we prove (iv). Now if $x \wedge m = m$, then $0 = m_* = (x \wedge m)_* = x_*$ and hence $x_{**} \wedge m = m$. Converse follows from Theorem 2.2 (vii).

Corollary 2.4: For any $x, y \in A$, we have the following:

(i) $x_{**} = (x \wedge y)_{**} \vee x_{**}$.

(ii) $x_{**} = x_{**} \wedge (x \vee y)_{**}$.

In a distributive lattice, the dual pseudo-complementation is unique (if it exists). But, in an ADL, there can be several dual pseudo-complementations. Now we prove the following.

Lemma 2.5: Let A be an ADL and $*$ and $\perp$ be dual pseudo-complementations on A. Then, for any $x, y \in A$, we have the following:

(i) $x_\perp \wedge x_* = x_*$ and $x_\perp \vee x_* = x_\perp$.

(ii) $x_{*\perp} = x_{\perp\perp}$.

(iii) $x_* = y_* \Leftrightarrow x_\perp = y_\perp$.

(iv) x_* is maximal $\Leftrightarrow$ $x_\perp$ is maximal.

(v) $x_\perp = x_* \wedge 0_\perp$.

(vi) $x_* \wedge x_{**} = 0 \Leftrightarrow x_\perp \wedge x_{\perp\perp} = 0$.

Proof: Since $(x \vee x_\perp) \wedge m = m$, we get $(x_* \vee x_\perp) \wedge m = x_\perp \wedge m.$ Now $x_\perp \wedge x_* = x_\perp \wedge m \wedge x_* = (x_* \vee x_\perp) \wedge m \wedge x_* = (x_\perp \vee x_*) \wedge x_* = x_*$ and hence $x_\perp \vee x_* = x_\perp$. Thus we get (i).

Now $x_{*\perp} = (x_\perp \wedge x_*)_\perp = (x_* \wedge x_\perp)_\perp = x_{\perp\perp}$. To prove (iii), suppose $x_* = y_*$.

Then $x_\perp = x_{\perp\perp\perp} = x_{*\perp\perp} = y_{*\perp\perp} = y_{\perp\perp\perp} = y_\perp$. By symmetry, we get the converse. (iv) follows from (i). Now $x_* \wedge 0_\perp = x_\perp \wedge x_* \wedge 0_\perp = x_\perp \wedge 0_\perp = x_\perp$ (since $x_\perp \le 0_\perp$). Thus we get (v). Suppose $x_* \wedge x_{**} = 0$.

Then $x_\perp \wedge x_{\perp\perp} = x_\perp \wedge x_{*\perp} = (x_* \wedge 0_\perp) \wedge (x_{**} \wedge 0_\perp) = 0.$

Hence $x_\perp \wedge x_{\perp\perp} = 0.$ By symmetry, we get the converse.

If (A, V, Λ) is a finite distributive lattice and if we define, for any $x \in A$, $x_* = \Lambda\{y \in A \mid x \vee y = 1\}$. Then $*$ is a dual pseudo-complementation on A. Using this, we prove the following.

Theorem 2.6: If A is a finite ADL, then A is a dual PCADL.

Proof: Let A be a finite ADL and m be a maximal element of A with respective $\le$. Then $([0,m], \vee, \wedge)$ is a distributive lattice and hence a dual pseudo-complemented lattices. For any $x \in A$, define $x_\perp = (x \wedge m)_*$ where $(x \wedge m)_*$ is the dual pseudo-complement of $x \wedge m$ in $[0,m]$. Let $x, y \in A$.

Now

$(x \vee x_\perp) \wedge m = (x \wedge m) \vee (x_\perp \wedge m)$

$$=(x\wedge m)\vee((x\wedge m)_*\wedge m)$$
$$=(x\wedge m)\vee(x\wedge m)_*$$
$$=m.$$

Suppose $x\vee y=m.$ Then $(x_\perp\vee y)\wedge m=((x\wedge m)_*\wedge m)\vee(y\wedge m)$
$$=(x\wedge m)_*\vee(y\wedge m)$$
$$=(y\wedge m) \text{ (since } (x\wedge m)\vee(y\wedge m)=m\text{).}$$

Finally, $(x\wedge y)_\perp=((x\wedge y)\wedge m)_*=(x\wedge m)_*\vee(y\wedge m)_*=x_\perp\vee y_\perp$. Hence A is a dual PCADL.

Theorem 2.7: For any $x,y\in A$, the following are equivalent:

(i) $(x\vee y)\wedge m=m.$

(ii) $(x_{**}\vee y)\wedge m=m.$

(iii) $(x_{**}\vee y_{**})\wedge m=m.$

(iv) $(x\vee y_{**})\wedge m=m.$

Proof: (i) $\Longrightarrow$ (ii): Suppose $(x\vee y)\wedge m=m$. Then $(x\wedge m)\vee(y\wedge m)=m$ and hence, by definition $((x\wedge m)_*\vee(y\wedge m))\wedge m=y\wedge m$. Thus, we get $(x_*\vee y)\wedge m=y\wedge m$.
Now $(x_{**}\vee y)\wedge m=(x_{**}\wedge m)\vee[(x_*\vee y)\wedge m]=(x_{**}\vee x_*\vee y)\wedge m=m.$ (ii) $\Longrightarrow$ (iii) follows from (i) $\Longrightarrow$ (ii). Since $x_{**}\wedge m\le x\wedge m$, we get (iii) $\Longrightarrow$ (iv). Similarly, since $y_{**}\wedge m\le y\wedge m$, we get (iv) $\Longrightarrow$ (i).

Theorem 2.8: For any $x,y\in A, (x\vee y)_{**}=x_{**}\vee y_{**}.$

Proof: From theorem 2.3 (i), we get $x_{**}\vee y_{**}\le(x\vee y)_{**}$. Since for any $x,y\in A,$
$((x\vee y)\vee(x\vee y)_*)\wedge m=m,$ by Theorem 2.7, we get $[x_{**}\vee y_{**}\vee(x\vee y)_*]\wedge m=m.$
Thus $((x\vee y)_{**}\vee x_{**}\vee y_{**})\wedge m=(x_{**}\vee y_{**})\wedge m$ and hence $(x\vee y)_{**}=x_{**}\vee y_{**}.$

In the following two theorems we prove that dual PCADL is equationally definable.

Theorem 2.9: Let A be an ADL with a maximal element m. A unary operation $*$ on A is a dual pseudo-complementation on A if and only if, for any $x,y\in A$, the following conditions hold:

(i) $(x\vee x_*)\wedge m=m.$

(ii) $(x_{**}\vee x)\wedge m=x\wedge m.$

(iii) $(x\vee y)_{**}=x_{**}\vee y_{**}.$

(iv) $(m_*\vee x)\wedge m=x\wedge m.$

(v) $(x\wedge y)_*=x_*\vee y_*.$

Proof: Suppose A satisfies conditions (i) to (v). It is enough to prove $\mathbf{d_1}$ of definition 2.1. For this, first we prove that

(a). $(x\wedge m)_*\wedge m=(x_*\vee m_*)\wedge m=(m_*\vee x_*)\wedge m=x_*\wedge m$ (by (iv)).

(b). $x_{***}\wedge m=(x_{***}\vee m_*)\wedge m$ (by (iv))
$$=(x_{**}\wedge m)_*\wedge m \text{ (by (v))}$$

$$= (x_{**} \wedge x \wedge m)_* \wedge m \text{ (by (ii))}$$
$$= (x_{***} \vee x_* \vee m_*) \wedge m = x_* \wedge m \text{ (by (a) and (iv)).}$$

Let $x, y \in A$ such that $x \vee y = m$.

Then $y \wedge m = (m_* \vee y) \wedge m$ (by (iv))
$$= \{[(x_* \vee x_{**}) \wedge m]_* \vee y\} \wedge m \text{ (by (i))}$$
$$= [((x_* \vee x_{**}) \wedge m)_* \wedge m] \vee (y \wedge m)$$
$$= [(x_* \vee x_{**})_* \wedge m] \vee ((y_{**} \vee y) \wedge m) \text{ (by (a) and (ii))}$$
$$= ((x \wedge x_*)_{**} \vee y_{**} \vee y) \wedge m \text{ (by (v))}$$
$$= [((x \wedge x_*) \vee y)_{**} \vee y] \wedge m \text{ (by (iii))}$$
$$= [((x \vee y) \wedge (x_* \vee y))_{**} \vee y] \wedge m$$
$$= [(m \wedge (x_* \vee y))_{**} \vee y] \wedge m$$
$$= [(x_* \vee y)_{**} \vee y] \wedge m$$
$$= (x_{***} \vee y_{**} \vee y) \wedge m \text{ (by (iii))}$$
$$= (x_* \vee y) \wedge m \text{ (by (b) and (ii)).}$$

Hence A is a dual PCADL. Conversely, if A is a dual PCADL, then conditions (i) to (v) are already proved.

Theorem 2.10: Let A be an ADL with a maximal element m. A unary operation $*$ on A is a dual pseudo-complementation if and only if, for any $x, y \in A$, it satisfies the following conditions:

(i) $(x_* \vee y) \wedge m = ((x \vee y)_* \vee y) \wedge m.$

(ii) $(m_* \vee x) \wedge m = x \wedge m.$

(iii) $m_{**} \wedge m = m.$

(iv) $(x \wedge y)_* = x_* \vee y_*.$

Proof: Suppose $*$ is a dual pseudo-complementation on A. Clearly, we have (ii), (iii) and (iv). Since $(x \vee y \vee (x \vee y)_*) \wedge m = m$, we get $(y \vee (x \vee y)_*) \wedge m = (x_* \vee y \vee (x \vee y)_*) \wedge m = (x_* \vee y) \wedge m$.

Conversely, suppose that $*$ satisfies conditions (i) to (iv). Let $x, y \in A$ such that $x \vee y = m$.

Then $(x \vee y)_* \wedge m = [(x \vee y)_* \vee y] \wedge m$ (by (i))
$$= (m_* \vee y) \wedge m = y \wedge m$$ (by (ii)).Thus we get $\mathbf{d_1}$ of def 2.1.

Now $(x_* \vee x) \wedge m = ((x_* \wedge m) \vee x) \wedge m$
$$= \{[(m_* \vee x) \wedge m]_* \vee x\} \wedge m \text{ (by (ii))}$$
$$= ((m_* \vee x)_* \vee x) \wedge m \text{ (since } (x \wedge m)_* \wedge m = x_* \wedge m \text{)}$$
$$= (m_{**} \vee x) \wedge m \text{ (by (i))}$$
$$= m.$$

Hence A is a dual PCADL.

Theorem 2.11: Let A be an ADL with a maximal element m. Then the following are equivalent:
(i) A is a dual PCADL.
(ii) $[a, a \vee m]$ is a dual pseudo-complemented lattice for all $a \in A$.
(iii) $[0, m]$ is a dual pseudo-complemented lattice.

Proof: (i) $\Longrightarrow$ (ii): Suppose A is a dual PCADL and $a \in A$. For $x \in A$, define $x_\perp = a \vee (x_* \wedge m)$. Let $x, y \in [a, a \vee m]$ such that $x \vee y = a \vee m$. Then $(x \vee y) \wedge m = m$ and hence $(x_* \vee y) \wedge m = y \wedge m$. This implies $(x_* \vee y) \wedge m_1 = y \wedge m_1 = y$ where $m_1 = a \vee m$. Therefore $x_* \wedge m_1 \leq y$ and

hence $x_\perp = a \vee (x_* \wedge m) \leq y$. Conversely, suppose $x_\perp \leq y$.

Then $x \vee x_\perp = (x \vee x_\perp) \wedge m_1 = [x \vee a \vee (x_* \wedge m)] \wedge m_1 = m_1 = a \vee m$. Hence $[a, a \vee m]$ is a dual pseudo-complemented lattice. (ii) $\Longrightarrow$ (iii) is trivial. Now we show that (iii) $\Longrightarrow$ (i). Suppose $[0, m]$ is a dual pseudo-complemented lattice under the unary operation $\perp$. For $x \in A$, define $x_* = (x \wedge m)_\perp$.

Then $(x \vee x_*) \wedge m = [(x \wedge m) \vee (x_* \wedge m)] = (x \wedge m) \vee (x \wedge m)_\perp = m$. Suppose $x, y \in A$ such that $x \vee y = m$. Then $(x \wedge m) \vee (y \wedge m) = m$.

Thus $(x \wedge m)_\perp \leq y \wedge m$ and hence $(x_* \vee y) \wedge m = (x_* \wedge m) \vee (y \wedge m) = (x \wedge m)_\perp \vee (y \wedge m) = y \wedge m$. Finally, $(x \wedge y)_* = (x \wedge y \wedge m)_\perp = [(x \wedge m) \wedge (y \wedge m)]_\perp = (x \wedge m)_\perp \vee (y \wedge m)_\perp = x_* \vee y_*$. Hence A is a dual PCADL. If A is an ADL, then the set $PI(A)$ of all principal ideals of A forms a distributive lattice [5]. Now, we prove the following.

Theorem 2.12: Let A be an ADL. Then A is a dual PCADL if and only if $PI(A)$ is a dual pseudo-complemented lattice.

Proof: Suppose $(A, \vee, \wedge, *, 0, m)$ is a dual PCADL. For any $a \in A$, define $(x]_+ = (x_*]$. Let $x, y \in A$ such that $(x] \vee (y] = A$.. Then $m = (x \vee y) \wedge m$, so that $y \wedge m = (x_* \wedge m) \vee (y \wedge m)$ and hence $(x_*] \subseteq (y]$. Also $(x] \vee (x_*] = (x \vee x_*] = ((x \vee x_*) \wedge m] = (m] = A$. Hence $PI(A)$ is a dual pseudo-complemented lattice.

Conversely, suppose $(PI(A), \vee, \wedge, +)$ is a dual pseudo-complemented lattice. For $x \in A$, define $x_* = a \wedge m$ where $(x]_+ = (a]$. Since $(a] = (b]$ if and only if $a \wedge m = b \wedge m$, we get that $*$ is well defined. We also get that $(x]_+ = (x_*]$. Let $x, y \in A$ such that $x \vee y = m$. Then $(x] \vee (y] = A$ and hence $(x]_+ \subseteq (y]$. Therefore $x_* \wedge m \leq y \wedge m$.. Now, $(m] = (x] \vee (x]_+ = (x \vee x_*]$ and hence $(x \vee x_*) \wedge m = m$. Therefore $x \vee x_*$ is a maximal. Finally, let $x, y \in A$ suppose $(x]_+ = (a]$ and $(y]_+ = (b]$.

Then $(x \cap y]_+ = ((x] \cap (y])_+ = (x]_+ \vee (y]_+ = (x_*] \vee (y_*] = (x_* \vee y_*]$.
Hence by definition, $(x \wedge y)_* = x_* \vee y_*$.
Thus A is a dual PCADL.

Theorem 2.13: Let N be the set of all maximal elements in A and $DPC(A)$ be the set of all dual pseudo-complementations on A. For any $n \in N$, define $*_n : A \to A$ by $x_{*_n} = x_* \wedge n$ for all $x \in A$. Then $(A, \vee, \wedge, *_n, 0, n)$ is a dual PCADL and the map $\varphi : N \to DPC(A)$ defined by $\phi(x) = x_{*_n}$ is a bijection.

Proof: First we prove that φ is well-defined. Let $x, y \in A$ such that $x \vee y$ is maximal.

Then $(x_{*_n} \vee y) \wedge n = ((x_* \wedge n) \vee y) \wedge m \wedge n = y \wedge m \wedge n = y \wedge n$
and $(x \vee x_{*_n}) \wedge n = (x \wedge n) \vee (x_* \wedge n) = (x \vee x_*) \wedge n = m \wedge n = n$.

Therefore $x \vee x_{*_n}$ is maximal.

Now $(x \wedge y)_{*_n} = (x \wedge y)_* \wedge n = (x_* \vee y_*) \wedge n = (x_* \wedge n) \vee (y_* \wedge n) = x_{*_n} \wedge y_{*_n}$. Therefore $*_n$ is a dual pseudo-complementation on A. Let n_1 and n_2 be two maximal elements such that $*_{n_1} = *_{n_2}$.

Then $n_1 = 0_* \wedge n_1 = 0_{*_{n_1}} = 0_{*_{n_2}} = 0_* \wedge n_2 = n_2$.

Finally, we prove φ is onto. Let $\perp \in DPC(A)$.

Then $n_0 = 0_\perp$ and for any $x \in A$, $x_{*_{n_0}} = x_* \wedge 0_\perp = x_\perp$ (by lemma 2.5 (v)). Thus φ is a bijection.

Let A be an ADL. It may be recalled that $(A, \leq)$ is a partial ordered set if we define $x \leq y$ if and only if $x \wedge y = x$ or equivalently $x \vee y = y$, for any $x, y \in A$. Now we prove the following.

Theorem 2.14: $(A_*, \leq)$ is a Boolean algebra, where $A_* = \{a_* \wedge m \mid a \in A\}$.

Proof: For any $x \in A$, we have $0 \leq x$ and hence $x_* \leq 0_*$. So that, for any $x, y \in A$, $(x \wedge y)_* = x_* \vee y_* = y_* \vee x_*$, and hence $(x_* \vee y_*) \wedge m$ is the l.u.b of $x_* \wedge m, y_* \wedge m$ in $(A_*, \leq)$. Also, since $x \wedge m \leq (x \vee y) \wedge m$ and $y \wedge m \leq (x \vee y) \wedge m$, we get $x_* \geq (x \vee y)_*$ and $y_* \geq (x \vee y)_*$. Thus $(x \vee y)_* \wedge m$ is lower bound of $x_* \wedge m, y_* \wedge m$ in $(A_*, \leq)$. Suppose $t_* \wedge m \in A_*$ such that $t_* \wedge m \leq x_* \wedge m$ and $t_* \wedge m \leq y_* \wedge m$.

Then $x_{**} \leq t_{**}$, $y_{**} \leq t_{**}$ and hence $x_{**} \vee y_{**} \leq t_{**}$. Thus $t_* \wedge m \leq (x_{**} \vee y_{**})_* \wedge m$. If we write $(x_* \wedge m) \overline{\Lambda} (y_* \wedge m) = (x_{**} \vee y_{**})_* \wedge m$, then $(x_* \wedge m) \overline{\Lambda} (y_* \wedge m)$ is the g.l.b of $x_{**} \wedge m, y_{**} \wedge m$ in the poset $(A_*, \leq)$.

Hence $(A_*, V, \overline{\Lambda}, 0, 0_* \wedge m)$ is a bounded lattice. For any $x \in A_*$, $(x_* \vee x_{**}) \wedge m = m = 0_* \wedge m$ and $(x_* \wedge m) \overline{\Lambda} (x_{**} \wedge m) = (x_{**} \vee x_{***})_* \wedge m = 0$. Thus $(A_*, V, \overline{\Lambda}, 0, 0_* \wedge m)$ is a complemented lattice.

Finally, we prove the distributivity. Let $x_* \wedge m, y_* \wedge m, z_* \wedge m \in A_*$. Then

$$
\begin{aligned}
\left((x_* \wedge m) \overline{\Lambda} (y_* \wedge m)\right) \vee \left((x_* \wedge m) \overline{\Lambda} (z_* \wedge m)\right) &= \left((x_{**} \vee y_{**})_* \wedge m\right) \vee \left((x_{**} \vee z_{**})_* \wedge m\right) \\
&= \left[(x_{**} \vee y_{**}) \wedge (x_{**} \vee z_{**})\right]_* \wedge m \\
&= [x_{**} \vee (y_{**} \wedge z_{**})]_{***} \wedge m \\
&= \left[x_{****} \vee (y_{**} \wedge z_{**})_{**}\right]_* \wedge m \\
&= \left[x_{**} \vee (y_{***} \vee z_{***})_*\right]_* \wedge m \\
&= \left[x_* \vee (y_* \vee z_*)_*\right]_* \wedge m \\
&= (x_* \wedge m) \overline{\Lambda} \left((y_* \wedge m) \vee (z_* \wedge m)\right).
\end{aligned}
$$

Hence $(A_*, V, \overline{\Lambda}, 0, 0_* \wedge m)$ is a Boolean algebra.

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