International Journal of Mathematical Archive-3(2), 2012, Page: 556-565
Available online through www.ijma.info ISSN 2229 – 5046

$g^{\#}$ semi–closed sets in Bitopological spaces

Mrs. Veronica Vijayan* & K. Reena**

Department of mathematics, Nirmala College for women, Coimbatore, India

*****Department of mathematics, Sree Narayana Guru College, Coimbatore, India***

E-mail:*reenamaths1@gmail.com

(Received on: 03-02-12; Accepted on: 25-02-12)

ABSTRACT

In this paper, we introduce $g^{\#}$ semi –closed sets in bitopological spaces. Properties of these sets are investigated and we introduce four new bitopological spaces namely, (i,j)-${}_{\alpha}T_b^{\#}$, (i,j)-$T_b^{\#}$, (i,j)-$T_b^{\#\#}$, (i,j)-${}^{\#}T_b$ spaces as applications. Further, we introduce and study (i,j)-$g^{\#}$s –continuous and (i,j)-$g^{\#}$s-irresolute maps.

***Keywords**: (i,j)- $g^{\#}$ s–closed sets, (i,j)-${}_{\alpha}T_b^{\#}$, (i,j)-$T_b^{\#}$, (i,j)-$T_b^{\#\#}$, (i,j)-${}^{\#}T_b$ spaces,(i,j)-$g^{\#}$s –continuous and (i,j)-$g^{\#}$s-irresolute maps.*

---

1. Introduction:

A triple (X, τ_1, τ_2) where X is a nonempty set and τ_1 and τ_2 are topologies on X is called a bitopological space and Kelly initiated the study of such spaces. Levine introduced and studied semi-open sets[1] [11] and generalized closed sets[2][10] in 1963 and 1970 respectively. S.P. Arya and T.Nour[3] [3] defined generalized semi-closed sets (briefly gs-closed sets) in 1990 for obtaining some characterizations of s-normal spaces. Njåstad[4] [16] and Abd El-Monsef et. al [5][1] introduced α-sets (called as α-closed sets) and semi-preopen sets respectively. Semi-preopen sets are also known as β-sets[6] [2]. Maki et.al. Introduced generalized α-closed sets (briefly gα-closed sets) [7][13] and α-generalized closed sets (briefly αg-closed sets)[8][12] in 1993 and 1994 respectively. M.K.R.S. Veera Kumar introduced and studied $g^{\#}$-semi-closed sets, ${}^{\#}T_b$ space, $T_b^{\#}$ space, $T_b^{\#\#}$ space and ${}_{\alpha}T_b^{\#}$ space, $g^{\#}$s-continuous and $g^{\#}$s-irresolute maps for general topology. The class of $g^{\#}$-semi-closed sets is independent from the classes of g-closed sets, gα-closed sets, αg-closed sets and pre closed sets.

The purpose of this paper is to introduce the concepts of $g^{\#}$-semi-closed sets, ${}^{\#}T_b$ space, $T_b^{\#}$ space, $T_b^{\#\#}$ space and ${}_{\alpha}T_b^{\#}$ space, $g^{\#}$s-continuous and $g^{\#}$s-irresolute maps for bitopological spaces and investigate some of their properties.

2. Prerequisites:

Throughout this paper (X,τ_1, τ_2), (Y, σ_1, σ_2) and (Z, η_1, η_2) represent non-empty bitopological spaces on which no separation axioms are assumed unless otherwise mentioned. If A is a subset of X with topology τ then cl(A), int(A) and C(A) denote the closure of A, the interior of A and the complement of A in X respectively. We recall the following definitions, which will be used often throughout this paper.

Definition 2.1: A subset A of a space (X, τ) is called

(1) a preopen set if $A \subseteq int(cl(A))$ and a preclosed set if $cl(int(A)) \subseteq A$.
(2) a semi-open set if $A \subseteq cl(int(A))$ and a semi-closed set if $int(cl(A)) \subseteq A$.
(3) an α-open set if $A \subseteq int(cl(int(A)))$ and a α-closed set if $cl(int(cl(A))) \subseteq A$.
(4) a semi-preopen set (=β-open) if $A \subseteq cl(int(cl(A)))$ and a semi-preclosed set (=β-closed) if $int(cl(int(A))) \subseteq A$.

The semi-closure (resp. α-closure) of a subset A of (X,τ) is denoted by scl(A) (resp. αcl(A) and spcl(A))and is the intersection of all semi-closed (resp. α-closed and semi-preclosed) sets containing **A.**

Corresponding author: K. Reena**, *E-mail: reenamaths1@gmail.com

Definition 2.2: A subset A of a space (X,τ) is called

(1) a generalized closed (briefly g-closed) set[2][10] if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is open in (X, τ).
(2) a generalized semi-closed (briefly gs-closed) set[3][3] if $scl(A) \subseteq U$ whenever $A \subseteq U$ and U is open in (X, τ).
(3) a generalized semi-preclosed (briefly gsp-closed) set[12][9] if $spcl(A) \subseteq U$ whenever $A \subseteq U$ and U is open in (X, τ).
(4) an α-generalized closed (briefly αg-closed) set[8][12] if $\alpha cl(A) \subseteq U$ whenever $A \subseteq U$ and U is open in (X, τ).
(5) a generalized α-closed (briefly gα-closed) set[7][13] if $\alpha cl(A) \subseteq U$ whenever $A \subseteq U$ and U is α-open in (X, τ).

Definition 2.3: A function $f: (X, \tau) \to (Y, \sigma)$ is called

(1) ***semi-continuous*[1]** [**11**] if $f^{-1}(V)$ is semi-open in (X, τ) for every open set V of (Y, σ).
(2) ***pre-continuous*[11]** [**14**] if $f^{-1}(V)$ is pre-closed in (X, τ) for every closed set V of (Y, σ).
(3) ***α-continuous*[12]** [**15**] if $f^{-1}(V)$ is α-closed in (X, τ) for every closed set V of (Y, σ).
(4) ***β-continuous*[5]** [**1**] if $f^{-1}(V)$ is semi-preopen in (X, τ) for every open set V of (Y, σ).
(5) ***g-continuous*[13]** [**4**] if $f^{-1}(V)$ is g-closed in (X, τ) for every closed set V of (Y, σ).
(6) ***gs-continuous*[14]** [**7**] if $f^{-1}(V)$ is gs-closed in (X, τ) for every closed set V of (Y, σ).
(7) ***αg-continuous*[2]** [**10**] if $f^{-1}(V)$ is αg-closed in (X, τ) for every closed set V of (Y, σ).
(8) ***gα-continuous*[7]** [**13**] if $f^{-1}(V)$ is gα-closed in (X, τ) for every closed set V of (Y, σ).
(9) ***gsp-continuous*[16]** [**9**] if $f^{-1}(V)$ is gsp-closed in (X, τ) for every closed set V of (Y, σ).
(10) ***αg-irresolute*[10]** [**6**] if $f^{-1}(V)$ is αg-closed in (X, τ) for every αg-closed set V of (Y, σ).
(11) ***pre-semi-open*[15]** [**5**] if f(U) is semi-open in (Y, σ) for every semi-open set U in (X, τ).

Definition 2.4: A topological space (X, τ) is said to be

(1) a $T_{1/2}$ space if every g-closed set in it is closed.
(2) a T_b space if every gs-closed set in it is closed.
(3) an ${}_{\alpha}T_b$ space if every αg-closed set in it is closed.

Definition 2.5: A subset A of a bitopological space (X, τ_1, τ_2) is called:

(1) (i,j)-g–closed if τ_j–$cl(A)\subseteq U$ whenever $A\subseteq U$ and U is open in τ_i
(2) (i,j)-g*–closed if τ_j–$cl(A)\subseteq U$ whenever $A\subseteq U$ and U is g-open in τ_i
(3) (i,j)-rg–closed if τ_j–$cl(A)\subseteq U$ whenever $A\subseteq U$ and U is regular open in τ_i
(4) (i,j)-gpr–closed if τ_j–$pcl(A)\subseteq U$ whenever $A\subseteq U$ and U is regular open in τ_i

The family of all (i,j)-g–closed sets (resp. (i,j)-g*–closed, (i,j)-rg–closed, (i,j)-gpr–closed) subsets of a bitopological space (X,τ_1, τ_2) is denoted by D(i, j) (resp. D*(i, j), $D_r(i, j)$,$\xi(i, j)$).

Definition 2.6: A subset A of a bitopological space (X, τ_1, τ_2) is called:

(1) (i,j)-$T_{1/2}$ space if every (i,j)-g-closed sets is τ_j- closed.
(2) (i,j)-T_b space if every (i,j)-gs-closed set is τ_j - closed.
(3) (i,j)-${}_{\alpha}T_b$ space if every (i,j)-αg-closed set is τ_j - closed**.**

Definition 2.7: A function $f : (X,\tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is called

(1) τ_j- *semi-continuous*[1] [**11**] if $f^{-1}(V)$ is semi-open in (X, τ_1, τ_2) for every open set V of (Y, σ_1, σ_2).
(2) τ_j- *α-continuous*[12][**15**] if $f^{-1}(V)$ is α-closed in (X, τ_1, τ_2) for every closed set V of (Y, σ_1, σ_2).
(3) τ_j- σ_k– continuous if $f^{-1}(V) \in \tau_j$, for every $V \in \sigma_k$.
(4) (i,j)-*gs-continuous*[14][**7**] if $f^{-1}(V)$ is gs-closed in $(X,\tau_1, \tau_2,)$ for every closed set V of (Y, σ_1, σ_2).
(5) (i,j)-*gsp-continuous*[14][**7**] if $f^{-1}(V)$ is gsp-closed in $(X,\tau_1, \tau_2,)$ for every closed set V of (Y, σ_1, σ_2).

3. $g^{\#}$ semi –closed sets in Bitopological spaces:

In this section we introduce the concept of $g^{\#}$ semi –closed sets in bitopological spaces and discuss the related properties**.**

Definition 3.1:A Subset A of a space (X,τ_i ,τ_j) is called a (i,j)-$g^{\#}$ semi –closed set(written (i,j)-$g^{\#}$s -closed) if τ_j–$scl(A)\subseteq U$ whenever $A\subseteq U$ and U is αg-open in τ_i

Remark 3.2: By setting $\tau_i = \tau_j$ in Definition 3.1, a (i,j)-$g^{\#}$ semi –closed set is a $g^{\#}$ semi –closed set[17].

Theorem 3.3: Every τ_j –semi-closed (resp. τ_j .α-closed, (i,j)-$g^{\#}$s -closed)set is a (i,j)-$g^{\#}$s –closed(resp. (i,j)-$g^{\#}$s –closed, (i,j)-gs –closed)set. The converses are not true.

Proof: Follows from the definitions.

The following examples show that a (1, 2)-$g^{\#}$s –closed (resp. (1, 2)-gs-closed) set need not beτ_2 –semi-closed, τ_2 -α-closed.

Example 3.4: Let X={a, b, c} , τ_1 = {ϕ,X,{a, b}}, τ_2 = {ϕ,X,{a}} then {a, c} is (1,2)-$g^{\#}$s –closed but not τ_2 –semi-closed.

Example 3.5: Let X={a, b, c} , τ_1 = {ϕ,X,{a}}, τ_2 = {ϕ,X,{a},{b, c}} then {b} is (1,2)-gs –closed but not (1,2)-$g^{\#}$s –closed**.**

Example 3.6: Let X={a, b, c} , τ_1 = {ϕ,X,{a, b}}, τ_2 = {ϕ,X,{a}} then{a, c} is (1,2)-$g^{\#}$s -closed but not τ_2 -α-closed Thus the class of (i,j)-$g^{\#}$s -closed sets properly contains the class of τ_j –closed sets, the class of τ_j -α-closed sets and the class of τ_j –semi-closed sets.

Theorem 3.7: Every τ_j –closed (resp. (i, j)-$g^{\#}$s -closed) set is (i, j)-$g^{\#}$s –closed (resp. (i, j)-gsp -closed) set. But the converses are not true.

Proof: Proof follows from the definitions.

Example 3.8: Let X={a, b, c}, τ_1 = {ϕ,X,{a}}, τ_2 ={ϕ,X,{a},{b, c}} then {b} is (1,2)-gsp-closed but not (1,2)-$g^{\#}$ s-closed**.**

Example 3.9: Let X={a, b, c}, τ_1 ={ϕ, X, {a, b}}, τ_2 ={ϕ, X,{a}} then {b} is (1,2)-$g^{\#}$ s-closed but not τ_2 –closed. So the class of (i,j)-$g^{\#}$ s-closed sets(resp. (i, j)-gsp-closed) sets properly contains the class of τ_j-closed (resp. (i, j)-$g^{\#}$ s-closed)sets. The class of (i, j)-$g^{\#}$ s-closed sets properly contains the class of τ_j-α closed sets in view of the above theorem since every τ_j-α closed set is a τ_j-semi- closed set.

The following examples shows that (i, j)-$g^{\#}$ s-closedness is independent from (i, j)-g-closedness, (i, j)-gα-closedness, (i, j)-αg-closedness , (i, j)-gs-closedn**ess.**

Example 3.10: Let X={a, b, c}, τ_1 ={ϕ,X,{c},{a, b}}, τ_2 ={ϕ,X,{a}} then the set {a, c} is (1,2)-g-closed set, (1,2)-gα-closed set, (1,2)-αg-closed set, (1,2)-gs-closed set but not (1,2)-$g^{\#}$ s-closed.

Proposition 3.11: If A is (i,j)-$g^{\#}$ s-closed set such that $A \subseteq B \subseteq \tau_j$-Scl(A) then B is also (i,j)-$g^{\#}$ s-closed.

Proof: Let U be τ_i-αg-open set such that $B \subseteq U$ then $A \subseteq B \subseteq U$. Since A is (i,j)-$g^{\#}$ s-closed, τ_j-Scl(A) $\subseteq$ U.

Now $B \subseteq \tau_j$-Scl(A) implies τ_j-Scl(B)$\subseteq$ τ_j-Scl(τ_j-Scl(A)) = τ_j-Scl(A) $\subseteq$ U $\Rightarrow \tau_j$-Scl(B) $\subseteq$ U Hence B is also (i,j)-$g^{\#}$ s-closed.

Proposition 3.12: If A is (i,j)-$g^{\#}$ s-closed then τ_j-Scl(A) – A contains no non-empty τ_i-αg-closed set.

Proof: Let A be an (i,j)-$g^{\#}$ s-closed set and F be a non-empty τ_i -αg-closed subset such that $F \subseteq \tau_j$-Scl(A) – A = τ_j-Scl(A)$\cap A^c$ $\therefore$ $F \subseteq \tau_j$-Scl(A) and $F \subseteq A^c$ Since F^c is τ_i - αg-open and A is (i,j)-$g^{\#}$ s-closed we have, τ_j-Scl(A) $\subseteq F^c$ i.e $F \subseteq (\tau_j\text{-Scl}(A))^c$ Hence $F \subseteq \tau_j$-Scl(A) $\cap$ $(\tau_j\text{-Scl}(A))^c = \phi$ $\therefore$ τ_j-Scl(A) – A contains no non-empty τ_i -αg-closed set

Corollary 3.13: If A is (i,j)-$g^{\#}$ s-closed set in (X, τ_i, τ_j), then A is τ_j –semi-closed iff

τ_j-Scl(A) – A is τ_i -αg-closed.

Proof: Necessity: If A is τ_j-semi-closed then τ_j-Scl(A)=A i.e τ_j-Scl(A) – A = ϕ and hence τ_j-Scl(A) – A is τ_i -αg-closed. [by prop.3.12]

Sufficiency: If τ_j-Scl(A)–A is τ_i -αg-closed then by proposition 3.12 we have,τ_j-Scl(A) – A = ϕ [since A is (i,j)-$g^{\#}$ s-closed] $\therefore$ τ_j-Scl(A) = A. Hence A is τ_j – semi-closed.

Proposition 3.14: For each element x of (X, τ_i, τ_j), {x} is τ_i -αg-closed (or) $\{x\}^c$ is (i,j)-$g^{\#}$ s-closed.

Proof: If {x} is not τ_i -αg-closed then the only τ_i -αg-open set containing X-{x} is X Thus X-{x} is (i,j)-$g^{\#}$ s-closed. i.e $\{x\}^c$ is (i,j)-$g^{\#}$ s-closed. Hence Proved**.**

Proposition 3.15: If A is an τ_i -αg-open and (i,j)-$g^{\#}$ s-closed set of (X, τ_i, τ_j) then A is τ_j–semi-closed.

Proof: Let A be τ_i -αg-open and (i,j)-$g^{\#}$ s-closed . Since A is (i,j)-$g^{\#}$ s-closed , we have τ_j–scl(A)$\subseteq$ U whenever A$\subseteq$U and U is τ_i -αg-open $\Rightarrow$ τ_j–scl(A) = A $\Rightarrow$A is τ_j–semi-closed.

Remark 3.16: An (i,j)-$g^{\#}$ s-closed set need not be (j,i)-$g^{\#}$ s-closed.

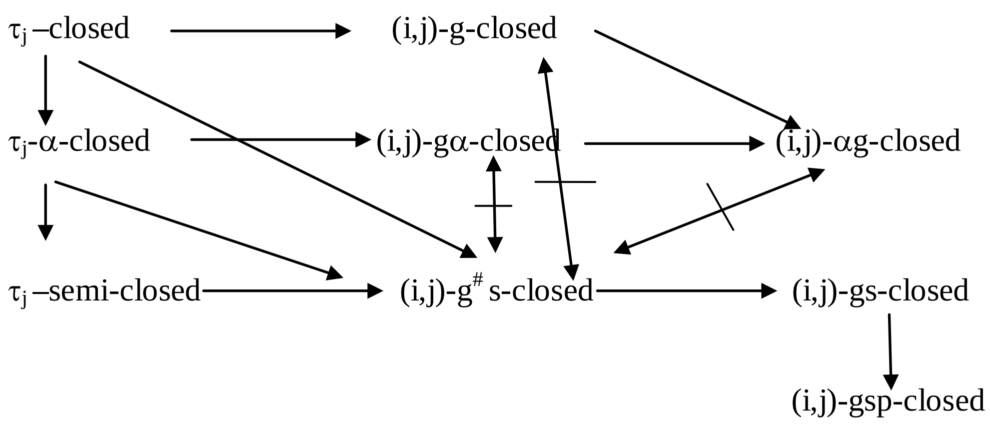
Proof: Consider the Example Let X= {a, b, c}, τ_1 = {ϕ, X, {a, b}}, τ_2 = {ϕ, X, {a}} then {b} is (1, 2)-$g^{\#}$s –closed but not (2, 1)-$g^{\#}$s –closed

Definition 3.17: The family of all (i,j)-$g^{\#}$ s-closed set in (X, τ_i, τ_j) is defined as $D^{\#}$ S(i,j)

Proposition 3.18: If A, B$\in$ $D^{\#}$ S (i,j) then A$\cup$B$\in$ $D^{\#}$ S(i,j)

Proof: Let A, B $\in$ $D^{\#}$ S(i,j). Let us assume that A$\cup$B$\subseteq$U where U is τ_i -αg-open. Implies A$\subseteq$U and U is τ_i -αg-open $\Rightarrow$ τ_j–scl(A)$\subseteq$U and B$\subseteq$U and U is τ_i -αg-open $\Rightarrow$ τ_j–scl(B)$\subseteq$U Hence τ_j–scl(A)$\cup$ τ_j–scl(B)$\subseteq$U but τ_j–scl(A$\cup$B) $\subseteq$ τ_j –scl(A)$\cup$ τ_j –scl(B)$\subseteq$U $\Rightarrow$$\tau_j$ –scl(A$\cup$B)$\subseteq$U . Therefore A$\cup$B is (i,j)-$g^{\#}$ s-closed hence A$\cup$B $\in$ $D^{\#}$ S(i,j).Hence proved.

The following figure shows the relationships of (i,j)-$g^{\#}$ s-closed sets with other sets



where A$\longrightarrow$B (resp.A $\leftarrow\!\!\!\!+\!\!\!\!\rightarrow$ B) represents A implies B but not conversely (resp.A and B are independent).

4. Applications of (i, j)-$g^{\#}$ s-closed Set:

In this chapter we introduce four new spaces namely (i,j)-${}_{\alpha}T_b^{\#}$ space, (i,j)-$T_b^{\#}$ space, (i,j)-$T_b^{\#\#}$ space, (i,j)-${}^{\#}T_b$ space. We now introduce a new space (i,j)-${}_{\alpha}T_b^{\#}$ space .

Definition 4.1: A space (X, τ_i, τ_j) is called an (i,j)-${}_{\alpha}T_b^{\#}$ space if every (i,j)-$g^{\#}$ s-closed set is τ_j–semi-closed.

Proposition 4.2: Every (i,j)-$T_{1/2}$ space is an (i,j)-${}_{\alpha}T_b^{\#}$ space but not conversely.

Proof: Let (X, τ_i, τ_j) be (i,j)-$T_{1/2}$–space and A be a (i,j)-$g^{\#}$ s-closed set of (X, τ_i, τ_j) .$\therefore$ A is (i,j)-gs-closed. Since (X, τ_i, τ_j) is (i,j)-$T_{1/2}$ space , A is τ_j–semi-closed set of (X, τ_i, τ_j)

$\therefore$(X, τ_i, τ_j) is an (i,j)-${}_{\alpha}T_b^{\#}$ space.

The converse of above proposition need not be true which is shown by the following example.

Example 4.3: Consider the example X={a,b,c},τ_1={ϕ,X,{a}},τ_2 ={ϕ,X,{a},{b,c}} then (X,τ_1 ,τ_2) is (1,2)-${}_{\alpha}T_b^{\#}$ space but not (1,2)-$T_{1/2}$–space.

Theorem 4.4: Every (i,j)-T_b space is an (i,j)-${}_{\alpha}T_b^{\#}$ space but not conversely.

Proof: Let (X, τ_i , τ_j) be a (i,j)-T_b space implies every (i,j)-gs-closed set is τ_j-closed .Let A be a (i,j)-$g^{\#}$ s-closed set. We know that every (i,j)-$g^{\#}$ s-closed set is (i,j)-gs-closed hence A is (i,j)-gs-closed . Therefore A is τ_j-closed. But every τ_j –closed set is τ_j –semi-closed. Implies A is τ_j –semi-closed. hence (X, τ_i , τ_j) is a (i,j)-${}_{\alpha}T_b^{\#}$ space The converse is not true which is shown by the following example**.**

Example 4.5: Let X={a, b, c} , τ_1 ={ϕ,X,{a}}, τ_2 ={ϕ,X,{a},{b, c}} then (X, τ_1 , τ_2) is (1,2)-${}_{\alpha}T_b^{\#}$ space but not (1,2)-T_b space

Remark 4.6: A (1, 2)-${}_{\alpha}T_b^{\#}$ space need not be a (1, 2)-${}_{\alpha}T_b$ space

Proof: Let X={a, b, c}, τ_1 ={ϕ,X,{a}}, τ_2 ={ϕ,X,{a},{b, c}} then (X, τ_1 , τ_2) is (1,2)-${}_{\alpha}T_b^{\#}$ space but not (1,2)-${}_{\alpha}T_b$ space

Characterization of (i,j)-${}_{\alpha}T_b^{\#}$ space

Theorem 4.7: For a space (X, τ_i , τ_j) the following conditions are equivalent.
1. (X, τ_i , τ_j) is an (i, j)-${}_{\alpha}T_b^{\#}$ space
2. Every singleton of X is either τ_i -αg-closed or τ_j - semi-open

Proof: To Prove (1) $\Rightarrow$(2) Let x$\in$X and suppose that {x} is not τ_i -αg-closed. Then X-{x} is (i,j)-$g^{\#}$ s-closed set since X is the only τ_i -αg-open set containing X-{x}. By propositions 3.13 and 3.14 we have, X-{x} is τ_j – semi-closed.(i.e) {x} is τ_j - semi-open

To Prove (2)$\Rightarrow$(1) Let A be a (i,j)-$g^{\#}$-closed set of (X, τ_i , τ_j).Clearly A$\subseteq\tau_j$ – scl(A)**.**Let x$\in$X. by (2) {x} is either τ_i -αg-closed or τ_j - semi-open

Case (i) Suppose {x} is τ_i -αg-closed.If x$\notin$A, then τ_j – scl (A)-A contains the τ_i -αg-closed set {x} and A is (i,j)-$g^{\#}$ s-closed set. By theorem 3.13 ,we arrive at a contradiction. Thus x$\in$A.

Case (ii) Suppose that {x} is τ_j - semi-open. Since x$\in\tau_j$ – scl(A), then {x}$\cap$A $\neq$ ϕ . So x$\in$A. Thus in any case x$\in$A. So τ_j – scl(A)$\subseteq$A $\therefore$ A=τ_j – scl(A) (or) equivalently A is τ_j - semi-closed. Thus (X, τ_i , τ_j) is an (i,j)-${}_{\alpha}T_b^{\#}$ space.

We now introduce a new space (i,j)-$T_b^{\#}$

Definition 4.8: A space (X, τ_i , τ_j) is called (i,j)-$T_b^{\#}$ space if every (i,j)-$g^{\#}$s-closed set is τ_j –closed.

Theorem 4.9: Every (i,j)-$T_b^{\#}$ space is an (i,j)-${}_{\alpha}T_b^{\#}$ space but not conversely.

Proof: Let (X, τ_i , τ_j) be (i,j)-$T_b^{\#}$ space.$\Rightarrow$Every (i,j)-$g^{\#}$s-closed set is τ_j – closed Therefore every (i,j)-$g^{\#}$s-closed set is τ_j – semi closed**.** Hence (X, τ_i , τ_j) is (i,j)-${}_{\alpha}T_b^{\#}$ space. Hence proved.

Example 4.10: Let X={a,b,c}, τ_1 ={ϕ,X,{c},{a,b}}, τ_2 ={ϕ,X,{a}} then (X, τ_1, τ_2) is (1,2)-${}_{\alpha}T_b^{\#}$ space but not (1,2)-$T_b^{\#}$ space**.**

Theorem 4.11: Every (i,j)-T_b space is (i,j)-$T_b^{\#}$ space but not conversely.

Proof: Let (X, τ_i , τ_j) be (i,j)-T_b space and A be a (i,j)-$g^{\#}$s-closed set

Then by theorem 3.3 A is (i,j)-gs-closed .Since (X, τ_i , τ_j) is a (i,j)-T_b space, then A is τ_j –closed.

$\therefore$ (X, τ_i , τ_j) is a (i,j)-$T_b^{\#}$ space. Hence proved.

Example 4.12: Let X={a, b, c}, τ_1 ={ϕ,X,{a}}, τ_2 ={ϕ,X,{a},{b, c}}. Then (X, τ_1 , τ_2) is (1,2)-$T_b^{\#}$ space but not (1,2)-T_b space

Theorem 4.13: If (X, τ_i , τ_j) is a (i,j)-$T_b^{\#}$ space, then every singeleton of X is either τ_i -αg-closed or τ_j – open.

Proof: Let $x \in X$ and suppose that $\{x\}$ is not τ_i -αg-closed. Then X-$\{x\}$ is not τ_i -αg-open. Then X is the only τ_i -αg-open set containing X-$\{x\}$. So X-$\{x\}$ is (i,j)-$g^{\#}$s-closed Since (X, τ_i, τ_j) is a (i,j)-$T_b^{\#}$ space, then X-$\{x\}$ is τ_j–closed or equivalently $\{x\}$ is τ_j–open**.**

Remark 4.14: (i,j)-$T_b^{\#}$ ness is independent from (i,j)-$_{\alpha}T_b$ ness and (i,j)-$T_{1/2}$ ness

Proof: Consider the space (X,τ_1,τ_2),X =$\{a, b, c\}$,τ_1 =$\{\phi,X,\{a\}\}$,τ_2 =$\{\phi,X,\{a\},\{b, c\}$Then (X,τ_1,τ_2) is (1,2)-$T_b^{\#}$ space but not (1,2)-$T_{1/2}$–space and (1,2)-$_{\alpha}T_b$ space.

We introduce another new space (i,j)-$T_b^{\#\#}$ space

Definition 4.15: A space (X, τ_i, τ_j) is called (i,j)-$T_b^{\#\#}$ space if every (i,j)-$g^{\#}$s-closed set is τ_j–α-closed

Theorem 4.16: Every (i,j)-$T_b^{\#}$ space ((i,j)-T_b) space is a (i,j)-$T_b^{\#\#}$ space but not conversely.

Proof: Let (X,τ_i,τ_j) be a (i,j)-$T_b^{\#}$ space and A be a (i,j)-$g^{\#}$s-closed set of (X,τ_i,τ_j)Since(X, τ_i, τ_j) is a (i,j)-$T_b^{\#}$ space, A is τ_j–closed. Since every τ_j–closed set is τ_j–α-closed set. Implies A is τ_j–α-closed$\therefore$ (X, τ_i, τ_j) is a (i,j)-$T_b^{\#\#}$ space. Let (X, τ_i, τ_j) be (i,j)-T_b space.$\Rightarrow$ every (i,j)-gs-closed set is τ_j-closed. Let A be (i,j)-$g^{\#}$s-closed We know that every (i,j)-$g^{\#}$s-closed set is (i,j)-gs-closed implies A is (i,j)-gs-closed. Since (X, τ_i, τ_j) is (i,j)-T_b space $\Rightarrow$A is τ_j -closed But τ_j–closed is τ_j–α-closed.

$\therefore$ A is τ_j–α-closed. Hence every (i,j)-$g^{\#}$s-closed set is τ_j–α-closed $\Rightarrow$ (X, τ_i, τ_j) is (i,j)-$T_b^{\#\#}$ space**.**

Example 4.17: Let X=$\{a, b, c\}$, τ_1 =$\{\phi,X,\{c\},\{a, b\}\}$, τ_2 =$\{\phi,X,\{a\}\}$ then (X, τ_1, τ_2) is (1,2)-$T_b^{\#\#}$ space but not (1,2)-$T_b^{\#}$ ((1,2)-T_b) space.

Thus the class of (i,j)-$T_b^{\#\#}$ spaces properly contains the class of (i,j)-$T_b^{\#}$ spaces and hence the class of (i,j)-T_b spaces.

Theorem 4.18: If (X, τ_i, τ_j) is a (i,j)-$T_b^{\#\#}$ space, then every singleton of X is either τ_i–αg-closed or τ_j–α-open.

Proof: Suppose that (X, τ_i, τ_j) is a (i,j)-$T_b^{\#\#}$ space. Suppose that $\{x\}$ is not τ_i–αg-closed for some $x \in X$. Then X-$\{x\}$ is not τ_i–αg-open Then X is the only τ_i–αg-open set containing X-$\{x\}$ So X-$\{x\}$ is a (i,j)-$g^{\#}$s-closed. Since (X, τ_i, τ_j) is a (i,j)-$T_b^{\#\#}$ space, X-$\{x\}$ is τ_j–α-closed or equivalently $\{x\}$ is τ_j–α-open

We now introduce a new space (i,j)-$^{\#}T_b$ space

Definition 4.19: A space (X, τ_i, τ_j) is called a (i,j)-$^{\#}T_b$ space if every (i,j)-gs-closed set is (i,j)-$g^{\#}$s-closed

Theorem 4.20: Every (i,j)-$T_{1/2}$ space is a (i,j)-$^{\#}T_b$ space but not conversely.

Proof: Let (X, τ_i, τ_j) be a (i,j)-$T_{1/2}$ space .Let A be a (i,j)-gs-closed set. Since (X, τ_i, τ_j) is (i,j)-$T_{1/2}$ space, A is τ_j–semi-closed. Therefore A is (i,j)-$g^{\#}$s-closed . hence (X, τ_i, τ_j) is a (i,j)-$^{\#}T_b$ space. Hence proved.

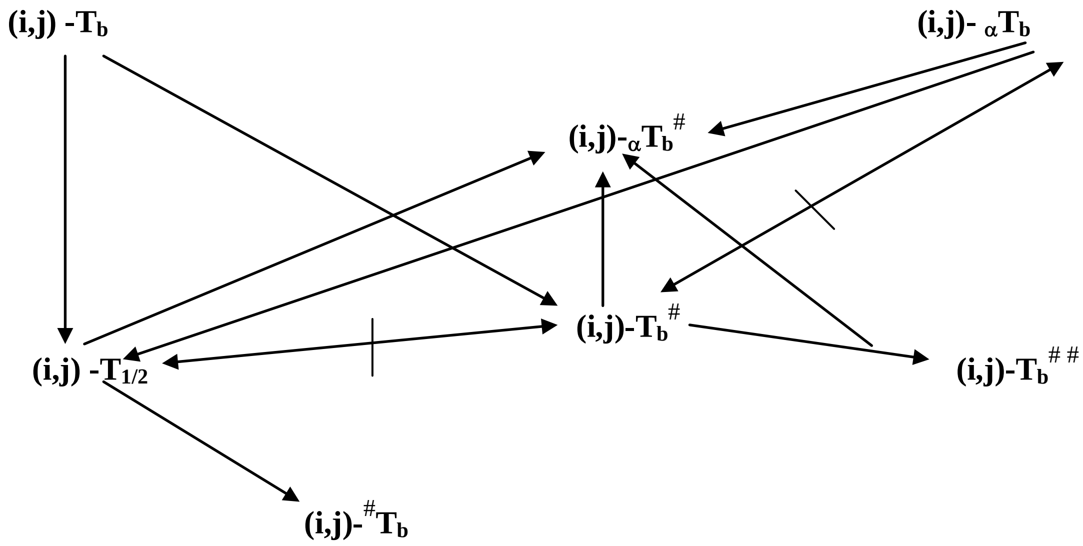
Example 4.21: Let X=$\{a, b, c\}$, τ_1 =$\{\phi,X,\{a, c\},\{c\}\}$, τ_2 =$\{\phi,X,\{a\}\}$ then (X,τ_1,τ_2) is (1,2)-$^{\#}T_b$ space but not (1,2)-$T_{1/2}$ space**.**

Remark 4.22:A (i,j)-$^{\#}T_b$ space need not be a (i,j)-T_b ((i,j)-$_{\alpha}T_b$)space.

Proof: In example 4.21 we have (X, τ_1, τ_2) is (1,2)-$^{\#}T_b$ space but not (1,2)-T_b space and (1,2)-$_{\alpha}T_b$ space

Theorem 4.23: A space (X, τ_i, τ_j) is (i,j)-$T_{1/2}$–space if and only if (X, τ_i, τ_j) is (i,j)-$^{\#}T_b$ and (i,j)-$_{\alpha}T_b^{\#}$ space .

Proof: Let (X, τ_i, τ_j) be a (i,j)-$T_{1/2}$ space. By proposition 3.3, every (i,j)-$T_{1/2}$ is an (i,j)-$_{\alpha}T_b^{\#}$ space. But every (i,j)-$T_{1/2}$ space is a (i,j)-$^{\#}T_b$ space.$\therefore$ (X, τ_i, τ_j) is a (i,j)-$^{\#}T_b$ and (i,j)-$_{\alpha}T_b^{\#}$ space. Conversely, suppose that (X, τ_i, τ_j) is a (i,j)-$^{\#}T_b$ and (i,j)-$_{\alpha}T_b^{\#}$ space**.** Let A be (i,j)-g-closed set. Since every (i,j)-g-closed set is (i,j)-gs-closed $\Rightarrow$ A is (i,j)-gs-closed Since (X, τ_i, τ_j) is a (i,j)-$^{\#}T_b$ space, A is (i,j)-$g^{\#}$s-closed set. Since (X, τ_i, τ_j) is an (i,j)-$_{\alpha}T_b^{\#}$ space , A is τ_j–semi-closed$\therefore$ by theorem 6.5[8], (X, τ_i, τ_j) is a (i,j)-$T_{1/2}$ space. Hence proved**.** The following diagram shows the inter relationships between the separation axioms discussed in this section.



where A ⟶B (resp. A ⟷B) represents A implies B but B need not imply A (resp. A and B are independent).

5. $g^{\#}$s –continuous maps in bitopological spaces:

We introduce the following definition.

Definition 5.1: A function f : (X, τ_1, τ_2) $\rightarrow$ (Y, σ_1, σ_2) is called (i ,j)- $g^{\#}$s –continuous if f^{-1} (V) is (i ,j)- $g^{\#}$s – closed set of (X, τ_1, τ_2) for every closed set V of (Y, σ_1, σ_2).

Proposition 5.2: If f: (X, τ_1, τ_2) $\rightarrow$ (Y, σ_1, σ_2) is τ_j -σ_k–continuous then it is (i ,j)- $g^{\#}$s –continuous but not conversely.

Proof: follows from the definitions.

Example 5.3: Let X = {a, b, c}, τ_1 = {ϕ, X,{a, b}}, τ_2 ={ϕ, X, {a}} and Y = {p, q}, σ_1 = {ϕ, Y, {p}}, σ_2 = {ϕ, Y, {q}}. Define a map f: (X, τ_1, τ_2) $\rightarrow$ (Y, σ_1, σ_2) by f(a) = q, f(b) = f(c) = p. then f is (1 ,2)- $g^{\#}$s –continuous but not τ_1 -σ_2 –continuous.

Proposition 5.4: If f : (X, τ_1, τ_2) $\rightarrow$ (Y, σ_1, σ_2) is (i ,j)- $g^{\#}$s –continuous, then it is (i ,j)- gs –continuous and (i ,j)- gsp –continous but not conversely.

Proof: follows from the definitions.

The converses are not true which is shown by the following examples.

Example 5.5: Let X = {a, b, c}, τ_1 = {ϕ, X,{a}}, τ_2 ={ϕ, X, {a},{b, c}} and Y = {p, q}, σ_1 = {ϕ, Y, {p}}, σ_2 = {ϕ, Y, {q}}. Define a map f : (X, τ_1, τ_2) $\rightarrow$ (Y, σ_1, σ_2) by f(a) = f(c) =q, f(b) = p. then f is (1 ,2)- gs –continuous but not (1 ,2)- $g^{\#}$s –continuous.

Example 5.6: Let X = {a,b,c}, τ_1 = {ϕ, X,{a}}, τ_2 ={ϕ, X, {a},{b,c}} and Y = {p, q}, σ_1 = {ϕ, Y, {p}}, σ_2 = {ϕ, Y, {q}}. Define a map f : (X, τ_1, τ_2) $\rightarrow$ (Y, σ_1, σ_2) by f(a) = f(c) = q, f(b) = p. then f is (1 ,2)- gsp –continuous but not (1 ,2)- $g^{\#}$s –continuous.

Remark 5.7: (I, j) - $g^{\#}$s –continuous and (i ,j)- g-continuous are independent which are shown by the following example.

Let X = {a, b, c}, τ_1 = {ϕ, X,{c},{a,b}}, τ_2 ={ϕ, X, {a}} and Y = {p, q}, σ_1 = {ϕ, Y, {p}}, σ_2 = {ϕ, Y, {q}}. Define a map f : (X, τ_1, τ_2) $\rightarrow$ (Y, σ_1, σ_2) by f(a) = f(c) =q, f(b) = p. then f is (1 ,2)- g –continuous but not (1 ,2)- $g^{\#}$s –continuous.

Theorem 5.8: Let f : (X, τ_1, τ_2) $\rightarrow$ (Y, σ_1, σ_2) be a map.

1. If (X, τ_1, τ_2) is an (i,j)-$T_{1/2}$ space then f is (i ,j)- g –continuous if it is (i,j)- $g^{\#}$s –continuous.
2. If (X, τ_1, τ_2) is an (i,j)-$T_b^{\#}$ space then f is τ_j -σ_k–continuous. Iff it is (i ,j)- $g^{\#}$s –continuous.

Proof:

1. Let V be a σ_k - closed set Since f is (i ,j)- g–continuous, f^{-1} (V) is (i ,j)- g –closed But (X, τ_1, τ_2) is an (i,j)-$T_{1/2}$ space

We have every (i ,j)- g –closed set is τ_j -closed.

We know that every τ_j – closed set is (i ,j)- $g^{\#}s$ –closed.

$\therefore f^{-1}$ (V) is (i ,j)- $g^{\#}s$ –closed. Hence f is (i ,j)- $g^{\#}s$ –continuous.

2. Obviously, f is (i,j)-$g^{\#}s$-continuous. Conversely, suppose that f is (i ,j)-$g^{\#}s$- continuous. Let V be a σ_k - closed set.

Since f is (i ,j)- $g^{\#}s$ –continuous

we have f^{-1} (V) is (i ,j)- $g^{\#}s$ –closed. But (X, τ_1, τ_2) is an (i, j)-$T_b^{\#}$ space we have

f^{-1} (V) is τ_j – closed $\therefore$ f is τ_j -σ_k –continuous Hence proved.

Theorem 5.9: Every τ_j -σ_k –semi-continuous map is (i ,j)- $g^{\#}s$ –continuous but not conversely.

Proof: obvious.

The following example supports that the converse of the above theorem is not true in general.

Example 5.10: Let X = {a, b, c}, τ_1 = {ϕ, X,{a, b}}, τ_2 ={ϕ, X, {a}} and Y = {a, b, c}, σ_1 = {ϕ, Y, {a}}, σ_2 = {ϕ, Y, {b,c}}. Define a map f: (X, τ_1, τ_2) $\rightarrow$ (Y, σ_1, σ_2) by f(a) = b, f (b) = a, f(c) = c. then f is (1 ,2)- $g^{\#}s$ –continuous but not τ_1 -σ_2 –semi-continuous.

Theorem 5.11: Every τ_j -σ_k -α-continuous map is (i ,j)- $g^{\#}s$-continuous. But not conversely.

Proof: follows from definitions.

The converse of the above theorem is not true which is shown by the following example.

Example 5.12: Let X = {a, b, c}, τ_1 = {ϕ, X, {a, b}}, τ_2 = {ϕ, X, {a}} and Y = {a, b, c}, σ_1 = {ϕ, Y, {a}}, σ_2 = {ϕ, Y, {b, c}}. Define a map f: (X, τ_1, τ_2) $\rightarrow$ (Y, σ_1, σ_2) by f(a) = b, f (b) = a, f(c) = c. then f is (1 ,2)- $g^{\#}s$ –continuous but not τ_1 -σ_2 -α –continuous.

Thus the class of (i ,j)- $g^{\#}s$-continuous maps properly contains the class of τ_j -σ_k -continuous maps, the class of τ_j -σ_k -α-continuous maps, the class of τ_j -σ_k -semi-continuous maps. And also the class of (i ,j)- $g^{\#}s$-continuous maps is properly contained in the class of (i ,j)- gs- continuous maps and hence in the class of (i ,j)- gsp-continuous maps.

Theorem 5.13: Let f : (X,τ_1,τ_2) $\rightarrow$ (Y,σ_1,σ_2) be (i ,j)-$g^{\#}s$-continuous map. If (X, τ_1, τ_2), the domain of f is an (i,j)-${}_{\alpha}T_b^{\#}$ space, then f is τ_j -σ_k–semi-continuous.

Proof: Let V be σ_k –closed set in (Y,σ_1,σ_2) . Then f^{-1} (V) is (i ,j)- $g^{\#}s$ –closed, since f is (i,j)- $g^{\#}s$ –continuous. Since (X, τ_1, τ_2) is an (i,j)-${}_{\alpha}T_b^{\#}$ space, every (i ,j)- $g^{\#}s$-closed set is τ_j -semi-closed $\Rightarrow f^{-1}$ (V) is τ_j -semi-closed hence f is τ_j -σ_k–semi-continuous. Hence proved.

Theorem 5.14: Let f : (X, τ_1, τ_2) $\rightarrow$ (Y, σ_1, σ_2) be (i ,j)- $g^{\#}s$-continuous map. If (X, τ_1, τ_2), the domain of f is (i,j)-$T_b^{\#\#}$ space, then f is τ_j -σ_k–α-continuous.

Proof: Let V be σ_k –closed set in (Y,σ_1,σ_2). Then f^{-1} (V) is (i, j) - $g^{\#}s$ –closed, since f is (i, j)- $g^{\#}s$ –continuous. Since (X, τ_1, τ_2) is an (i,j)-$T_b^{\#\#}$ space, every (i ,j)- $g^{\#}s$-closed set is τ_j -α-closed $\Rightarrow f^{-1}$ (V) is τ_j -α-closed hence f is τ_j -σ_k–α-continuous. Hence proved.

Theorem 5.15: Let f: (X, τ_1, τ_2) $\rightarrow$ (Y, σ_1, σ_2) be (i ,j)- $g^{\#}s$-continuous map. If (X, τ_1, τ_2), the domain of f is (i,j)-$T_b^{\#}$ space, then f is τ_j -σ_k-continuous.

Proof: Let V be σ_k –closed set in (Y,σ_1,σ_2). Then f^{-1}(V) is (i ,j)- $g^{\#}s$ –closed, since f is (i ,j)- $g^{\#}s$ –continuous. Since (X, τ_1, τ_2) is an (i,j)-$T_b^{\#}$ space, every (i ,j)- $g^{\#}s$-closed set is τ_j –closed $\Rightarrow f^{-1}$ (V) is τ_j –closed hence f is τ_j -σ_k-continuous. Hence proved.

Theorem 5.16: Let f: $(X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be (i ,j)- gs-continuous map. If (X, τ_1, τ_2), the domain of f is (i,j)-$^{\#}T_b$ space, then f is (i ,j)- $g^{\#}s$ –continuous.

Proof: Let V be σ_k–closed set in (Y,σ_1,σ_2).

Then $f^{-1}(V)$ is (i ,j)- gs –closed, since f is (i ,j)- gs –continuous. Since (X, τ_1, τ_2) is an (i,j)-$^{\#}T_b$ space we have every (i ,j)- gs-closed set is (i ,j)- $g^{\#}s$ –closed.$\Rightarrow$ $f^{-1}(V)$ is (i ,j)- $g^{\#}s$ –closed hence f is (i ,j)- $g^{\#}s$ –continuous. Hence proved.

WE INTRODUCE THE FOLLOWING DEFINITION

Definition 5.17: A function f : $(X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is called (i ,j)- $g^{\#}s$ –irresolute map if $f^{-1}(V)$ is (i ,j)- $g^{\#}s$ –closed set of (X, τ_1, τ_2) for every (i ,j)- $g^{\#}s$ – closed set of (Y, σ_1, σ_2).

Theorem 5.18: Every (i ,j)- $g^{\#}s$ –irresolute map is (i ,j)- $g^{\#}s$ –continuous but not conversely.

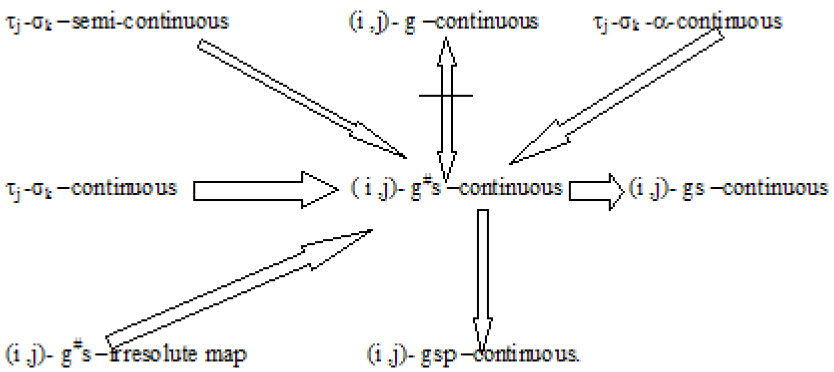
Proof: Let f is (i ,j)- $g^{\#}s$ –irresolute Let V be σ_k–closed set. Then $f^{-1}(V)$ is (i ,j)- $g^{\#}s$ –closed, since f is (i ,j)- $g^{\#}s$ –irresolute. hence f is (i ,j)- $g^{\#}s$ –continuous. Hence proved.

The converse of the above theorem is not true which is shown by the following example. Consider example 5.10

From example 5.10 we have, f is (1, 2) - $g^{\#}s$ –continuous.

But f is not (1,2)- $g^{\#}s$ –irresolute because{b} is (1 ,2)- $g^{\#}s$ –closed and $f^{-1}(\{b\}) = \{a\}$ and {a} is not (1 ,2)- $g^{\#}s$ –closed. Hence proved.

Remark 5.19: The following diagram summarizes the above discussions.



where A ⟶ B (resp. A ⟷ B) represents A implies B but B need not imply A (resp. A and B are independent).

REFERENCES:

[1] M.E.ABD EL-MONSEF, S.N.EL-DEEB AND MAHMOUD, β-open sets and β-continuous mapping, Bull Fac. Sci. Assiut Univ.12(1983) 77-90

[2] D.ANDRUEVIC, Semi-preopen sets, Mat.Vesnik38 (1) (1986)24-32.

[3] S.P.ARYA AND T.NOUR, Characterizations of s-normal spaces, Indian J. Pure. Appl. Math. 12(8) (1990)717-719.

[4] K.BALACHANDRAN, P.SUNDARAM AND H.MAKI, On generalized continuous maps in topological spaces, Mem. Fac. Sci. Kochi Univ. Ser.A. Math. 12(1991)5-13.

[5] S.G. CROSSLEY AND S.K.HILDEBRAND, semi-topological properties, Fund. Math. 74(1972)233-254.

[6] R.DEVI,K.BALACHANDRAN AND H.MAKI, Generalized α-closed maps and α-generalized closed maps, Indian J.Pure.Appl.Math.29(1)(1998)37-49.

[7] R.DEVI, K.BALACHANDRAN AND H.MAKI, Semi-generalized homeomorphisms and generalized semi-homeomorphisms in topological spaces, Indian. J.Pure.Appl.Math.26 (3) (1995)271-284.

[8] R.DEVI, H.MAKI AND K.BALACHANDRAN, Semi-generalized closed maps and generalized semi-closed maps Mem. Fac. Sci. Kochi. Univ. Ser. A. Math. 14 (1993) 41-54.

[9] J.DONTCHEV, On generalizing semi-pre open sets, Mem. Fac. Sci. Kochi Univ.Ser.A.Math.16 (1995)35-48.

[10] N.LEVINE, generalized closed sets in topology, Rend. Circ. Mat. Palermo 19(2) (1970)

[11] N.LEVINE, Semi-open sets and semi-continuity in topological spaces, Amer. Math. Monthly, 70(1963)36-41.

[12] H.MAKI, R.DEVI AND K.BALACHANDRAN, Associated topologies of generalized α-closed sets and α-generalized closed sets, Mem. Fac. Sci. Kochi Univ.Ser.A.Math.15(1994)51-63.

[13] H.MAKI, R.DEVI AND K.BALACHANDRAN, Generalized α-closed sets in topology, Bull. Fukuoka Univ. Ed .part III 42(1993)13-21.

[14] A.S.MASHHOUR, M.E.ABD EL-MONSEF AND S.N.EL-DEEB, On pre-continuous and weak pre-continuous mappings, Proc. Math. And Phys. Soc. Egypt 53(1982)47-53.

[15] A.S.MASHHOUR, I.A.HASANEIN AND S.N.EL.DEEB, α-continuous and α-open mappings, Acta ath.Hung.41 (3-4) (1983)213-218.

[15] O.NJASTAD, On some classes of nearly open sets, Pacific J.Math.15 (1965)961-870.

[16] M.K.R.S.VEERA KUMAR, $g^{\#}$ semi closed sets in topological spaces, Indian Journal of mathematics Vol.44, No.1, 2002, 73-87.

****