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A COMMON FIXED POINT THEOREM FOR SIX MAPPINGS IN PROBABILISTIC METRIC SPACES SATISFYING IMPLICIT RELATION AND VARIANTS OF R-WEAK COMMUTATIVITY

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ABSTRACT

*The aim of the present paper is to prove a common fixed point theorem for six mappings via **variants** of R-weakly commuting mapping in **probabilistic** metric spaces satisfying implicit relations.*

Mathematies subject classification: *47H10, 54H25.*

Keywords: *Probabilistie metric spaces, reciprocally continuous mappings, compatible mappings, R-weakly commuting mappings of type (i), (ii) and (iii), implicit relation, common fixed point*

1. INTRODUCTION

In 1942, K. Menger [6] introduced the notion of probabilistic metric space (briefly, PM-space) as a generalization of metric space. The development of fixed point theory in PM- spaces was due to Schweizer and Sklar [11. Sehgal [12] initiated the study of contraction mapping theorems in PM- spaces.

The notions of improving commutativity of mappings have been extended to PM- spaces by various mathematicians. Singh and Pant [14] extended the notion of weak commutativity (introduced by Sessa [13] in metric space). Mishra [8] extended the notion of compatibility (introduced by Jungck [2] in metric space). Ciric and Milovanovic Arandjelovic [1] extended the notion of pointwise R-weak commutativity (introduced by Pant [9] in metric space. In 2007, Kohli and Vashishtha [3] extended the notion of R-weak commutativity and its variants to probabilistic metric spaces. Our result open up for common fixed point theorems for six mappings via above variants.

2. PRELIMINARIES

In this section, we give definitions which are used in the paper. Let R^+ denotes the set of all non –negative real numbers.

Definition: 2.1 ([11]). A mapping F: $R \rightarrow R^+$ is called a distribution function if it is non – decreasing and left continuous with $\inf_{t \in R} F(t)=0$ and $\sup_{t \in R} F(t)=1$

The Heaviside function H is a distribution function defined by

$$\mathbf{H(t)} = \begin{cases} 0 & if\ t \le 0 \\ 1 & if\ t > 0 \end{cases}$$

Definition: 2.2 ([11]) Let X be a non – empty set and let L denote the set of all distribution functions. An orderd pair (X, F) is called probabilistic metric spaces if F is a mapping from X x X to L satisfying the following conditions.

(1) $F_{x,y}(t) = H(t)$ if and only if x=y
(2) $F_{x,y}(0) = 0$
(3) $\mathbf{F_{x,y}(t) = F_{y,x}(t)}$
(4) If $F_{x,y}(t) = 1$ and $F_{y,z}(s) = 1$, then $F_{x,z}(t+s) = 1$ for all x, y, z € X and t, s ≥ 0

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Definition: 2.3 ([11]). A t-norm is a function T: [0, 1] x [0, 1] → [0, 1] satisfying following conditions:

(1) T(a,1)=a for all a € [0,1] , T (0,0) =0
(2) T(a, b) = T(b, a)
(3) T (c, d) ≥ T (a, b) for c ≥a , d ≥b
(4) T (T(a, b),c) =T(a, T(b, c)) for all a, b, c € [0,1]

Example of t-norms are T(a, b) =min {a, b}, T(a, b) =ab and T(a, b) =max {a+b-1,0}

Definition: 2.4 ([11]) A Menger probabilistic metric space (X, F, T) is an ordered triad, where T is a t-norm and (X, F) is a probabilistic métric space satisfying.

$F_{x,z}$ (t+s) ≥ T($F_{x,y}$ (t), $F_{y,z}$ (s)) for all x, y, z € X and t, s ≥ 0

Definition: 2.5 A sequence {x_n} in (X, F, T) is said to converge to a point x € X if for every ε >0, λ>0, there exists a positive integer N (ε, λ) such that

$F_{Xn,x}$ (ε) > 1-λ for all n ≥ N (ε,λ)

Definition: 2.6 ([11]) A sequence {x_n} in (X, F,T) is said to be a cauchy sequence if for every ε > 0 and λ > 0, there exists a positive integer N (ε,λ) such that F_{XnXm} (ε) >1-λ for all n,m ≥ N (ε, λ)

Definition: 2.7 ([11]). A Menger space (X, F, T) with continuous t-norm is said to be complete if every Cauchy sequence in X converges to a point in X.

Definition: 2.8 Two self mappings f and g of a probabilistic metric space (X, F) are said to be weakly commuting if $F_{\mathbf{fgx,gfx}}$ (t) ≥ $F_{\mathbf{fx,gx}}$ (t) for each x € X and t >0

Definition: 2.9 ([8]). Two self mappings f and g of a probabilistic metric space (X, F) will be called compatible if and only if $F_{\mathbf{fgxn\ gfxn}}$ **(t) → 1** for all t > 0 whenever {x_n} is a sequence is such that f_{xn},g_{xn} → z for some zεx.

Definition: 2.10 ([1]) Two self mappings f and g of a probabilistic metric space (X, F) are said to be pointwise R-weakly commuting if given x€X, there exists R>0 such that $F_{\mathbf{fgx,gfx}}$ (t) ≥ $F_{\mathbf{fx,gx}}$ (t/R) for (t)≥0

Note: It is easy to see that weakly commuting mappings are pointwise R-weakly commuting with R=1.

Definition: 2.11 Two self mappings f and g of a probabilistic metric space(X, F) are said to be reciprocally continuous if fgx_n→fz and gfx_n →gz, whenever {x_n} is a sequence such that f_{xn},g_{xn}→z for some zεX.

Definition: 2.12 ([3]) Two self mappings f and g of a probabilistic metric space(X, F) are said to be

(i) R-weakly commuting of type (i) if there exists a positive real number R such that $F_{\mathbf{ffx,gfx}}$ (t) ≥ $F_{\mathbf{fx,gx}}$ (t/R) for each x in and each t≥0.

(ii) R-weakly commuting of type (ii) if there exists a positive real number R such that $F_{\mathbf{fgx,ggx}}$ (t) ≥ $F_{\mathbf{fx,gx}}$ (t/R) for each x in and each t≥0.

(iii) R-weakly commuting of type (iii) if there exists a positive real number R such that $F_{\mathbf{ffx,ggx}}$ (t) ≥ $F_{\mathbf{fx,gx}}$ (t/R) for each x in X and each t>0.

Remark: See Example 3.4, Ex 3.5, Ex 3.6, Ex 3.7, Ex 3.8, in ([3])

3. IMPLICIT RELATIONS

In [7] Mihet established a fixed point theorem concerning probabilistic contraction satisfying an implicit relation. In [10], Popa used the family F_4 of implicit-real functions to find the fixed point of two pairs of semi-compatible mappings in a d-compatible topological space. Here F_4 deuotes the family of all real continuous function f: $(R^+)^4 \rightarrow R$ satisfying the following properties**:**

(F_k) There exists k≥1 such that every u≥0 v≥0 with f (u, v, u, v,) ≥ 0 or f (u, v, v, u) ≥0, we have u≥kv

(F_u) F (u, u, 0, 0) <0 for all u>0.

In our result, we deal with the class ϕ of all real continuous functions $\phi: (R_+)^4 \rightarrow R$, non – decreasing in first argument and satisfying

(3.I) for u, v≥0, $\phi(u, v, u, v) \geq 0$ or $\phi(u, v, v, u) \geq 0$ implies $u \geq v$

(3.II) $\phi(u, u, 1, 1) \geq 0$ for all $u \geq 1$

4 Lemma: Let (X, F, T) be a complete Menger space where T denotes a continuous t-norm. Further, let (A,SH) and (B,TR) be R-weakly commuting mappings of type (i) (respectively type (ii) respectively type (iii) of X satisfying A(X)

C TR (X) , B(X) C SH(X) (4.1)

$\phi(f_{Ax,By}(kt), F_{SHx,TRy}(t), F_{Ax,SHx}(t), F_{By,TRy}(t), (kt) \geq 0$ (4.2)

$\phi(f_{Ax,By}(kt), F_{SHx,TRy}(t), F_{Ax,SHx}(kt), F_{By,TRy}(t), (t) \geq 0$ (4.3)

for all x,y ε X, t>0 ,kε(0,1) and for some φ € **ϕ**. Then the continuity of one of mappings in the compatible pairs (A, SH) or (B, TR) on (X, F, T) implies their reciprocal continuity.

Remark: For proof see {(Lemma 4.2([4])}

5. COMMON FIXED POINT THEOREM:

Theorem: Let (X, F, T) be a complete Menger space whenever T denotes a continuous t-norm. Further, let (A, SH) and (B, TR) be R-weakly commuting mappings of type

(i) (respectively type

(ii) respectively type

(iii) of X satisfying (4.1), (4.2) and (4.3) for all x,y ε X, t > 0 , k ε (0.1) and for some φεϕ.

If one of the mapping in compatible pair (A, SH) or (B, TR) is continuous implies their reciprocal continuity then A, B.S, T.H and R have a common fixed point in X.

Proof: Let u_0εX,By (4.1), we define the sequences $\{u_n\}$ and $\{v_n\}$ in x such that for all n= 0,1,2,------

Now, $V_{2n+1} = Au_{2n} = TRu_{2n+1}$

$V_{2n+2} = Bu_{2n+1} = SHu_{2n+2}$

By (4.2)

$\varphi(F_{Au2n, Bu2n+1}(kt), F_{SHu2n, TRu2n+1}(t), F_{Au2n, SHu2n}(t), F_{Bu2n+1, TRu2n+1}(kt) \geq 0$

$\varphi(F_{v2n+1, v2n+2}(kt), (F_{v2n, v2n+1}(t), (F_{v2n+1, v2n}(t), (F_{v2n+2, v2n+1}(kt) \geq 0$

Using (3.I), we get

$F_{v2n+1}, v_{2n+2}(kt) \geq F_{v2n, v2n+1}(t)$

Similarly, by (4.3) and using (3.II), we get

$F_{v2n+2}, v_{2n+3}(kt) \geq f_{v2n+1}, v_{2n+2}(t)$

Thus, for any n and t we have

$F_{vn, vn+1}(kt) \geq F_{vn-1, vn}(t)$

Hence, by Lemma [see lemma 2.12 of ([4])]

$\{v_n\}$ is a cauchy sequence in X, since X is complete,

$\{v_n\}$ converges to z.

Its subsequenees $\{A_{u2n}\}$,$\{Bu_{2n+1}\}$,

$\{SH_{U2n}\}$ and $\{TR\mathbf{u}_{2n+1}\}$ also converge to z**.**

Now, suppose that (A, SH) is a compatible pair and SH is continuous. Then, by Lemma 4, A and SH are reciprocally continuous, then $ASHu_{2n} \rightarrow Az$ and $SHA_{u2n} \rightarrow SHz$. Compatibility of A and SH gives

$F_{ASHu2n,\ SHAu2n}(t) \rightarrow 1$, i.e $F_{AZ,SHZ}(t) \rightarrow 1$ as $n \rightarrow \infty$, Hence, Az = SHz

Since, A(X) **c** TR (X), there exists a point u in X such that Az = TRu

By (4.2),

$\varphi(F_{Az,Bu}(kt), F_{SHz,TRu}(t), F_{Az,SHz}\ \mathbf{(t)}, F_{Bu,TRu}(kt)) \geq 0$

That is, $\varphi(F_{Az,Bu}(kt),1,1, F_{Bu,Az}(kt)) \geq 0$

Using (3.I),we get $(F_{Az,Bu}(kt) \geq 1$ for all t>0

which gives $F_{AZ,Bu}(kt)= 1$

Hence, AZ=Bu

Thus, AZ=SHz=Bu=TRu

Since, A and SH are R – weakly commuting mappings of type (i), there exists R > 0 such that

$F_{AAz,SHAz}(t) \geq F_{Az,SHz}(t/R) = 1$

That is ASHz=SHAz

and AAZ=ASHZ=SHAZ=SHSHZ.

Similarly, since B and TR are R-weakly commuting mapping of type (i), we have

BBu = BTRu=TRBu=TRTRu

Again, by (4.2), putting x =Az and y=u, we get

$\varphi(F_{AAz,Bu}(kt), F_{SHAz,\ TRu}(t), F_{AAz,SHAz,}(t), F_{Bu,TRu}(kt)) \geq 0$

That is $\varphi(F_{AAz,Az}(kt), F_{AAz,\ Az}(t),1,1) \geq 0$

As φ is non-decreasing in the first argument, we have

$\varphi(F_{AAz,Az}(t), F_{AAz,\ Az}(t),1,1) \geq o$

Using (3.II), we have

$F_{AAz,Az}(t) \geq 1$ for all $t \geq 0$. This gives $(F_{AAz,Az}(t)=1$,

implying AAz = Az and Az = AAz =SHAz,

Thus , Az is a common fixed point of A,and SH.

Now, we show that Az is a common fixed point of A,S and H. Putting x=Hz and y=u in(4.2), we have

$\varphi(F_{AHz,Bu}(kt), F_{SHHz,\ TRu}(t), F_{AHz,SHHz,}(t), F_{Bu,TRu}(kt)) \geq 0$

$\varphi(F_{HAz,Az}(kt), F_{HSHz,Az}(t), F_{HAz,HSHz,}(t), F_{Az,Az}(kt)) \geq 0$

$\varphi(F_{HAz,Az}(kt), F_{HAz,Az}(t), F_{HAz,HAz}(t), F_{Az,Az}(kt)) \geq o$

$\varphi(F_{HAz,Az}(kt), F_{HAz,Az}(t), 1, 1) \geq 0$

As φ is non-decreasing in the first argument

$\varphi(F_{HAz,Az}(t), F_{HAz,Az}(t), 1, 1) \geq 0$

$F_{HAz,Az}(t) \geq 1$ for all t >0

Thus, HAZ=AZ

AZ is the common fixed point of A, S and H .

Similarly, we can show that Az is the common fixed point of B, T and R. Thus, Az is the common fixed point of A, S, H, B, T and R. Again, we can prove the above theorem by taking other variants of R-weak commutativity.

That is R-weak commutativity of type **(ii)** and respectively of type **(iii),**

Example: 5.1 Let x = [0,1] with the metric d defined by d (x,y)= |x-y|. For each t > 0, let $f_{x,y}$ (t)= H (t-d(x,y)), for all x,y in X and let t denote a continuous t-norm. It is easily verified that (X,F,T) is a complete Menger space.

Let, fx =1 and gx = $\begin{cases} 1 & if\ x\ is\ rational \\ 0 & if\ x\ is\ irrational \end{cases}$

for each x € X. Clearly, f(x) **C** g(x), f is continuous and g is discontinuous on [0,1]. Let r: [0, 1] → [0, 1] be defined by $r(t) = (t)^{1/2}$ for o < t ≤ |. Then r (t) > t for o < t < | and $F_{fx,fy}(t) \geq r(F_{gx,gy}(t))$ for all x, y in X. Moreover, f and g are R-weakly commuting mappings. Thus, 1 is the unique common fixed point of f and g. Furthermore, f and g are R- weakly commuting mappings of type (i) as well as type **(ii)** and type **(iii).**

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