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INEQUALITIES FOR A POLYNOMIAL AND ITS DERIVATIVE

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ABSTRACT

In this paper we discuss some refinements of the Bernstein's Inequality for polynomials and prove some related results which among other things also generalize some known results on the inequalities for a polynomial and its derivative.

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1. INTRODUCTION AND STATEMENT OF RESULTS

Let P (z) be a polynomial of degree n and P' (z) its derivative.Concerning the maximum modulus of P' (z) on the unit circle $|z|=1$, we have the following inequality, known as Bernstein's Inequality(for reference see[10]).

$$\max_{|z|=1}|P'(z)| \le n\max_{|z|=1}|P(z)| \tag{1}$$

Concerning the maximum modulus of P (z) on a larger circle $|z|=R>1$, we have

$$\max_{|z|=R>1}|P(z)| \le R^n\max_{|z|=1}|P(z)| \tag{2}$$

Inequality (2) is a simple consequence of the maximum modulus principle (for reference see [8], [9]).

In both (1) and (2) equality holds for the polynomial $P(z)=\alpha z^n, |\alpha|\ne 0$, i.e. if and only if P (z) has all its zeros at the origin. It was proved by Frappier, Rahman and Ruscheweyh [4, Theorem8] that if P(z) is a polynomial of degree n, then

$$\max_{|z|=1}|P'(z)| \le n\max_{1\le k\le 2n}\left|P(e^{\frac{ik\pi}{n}})\right| \tag{3}$$

Since the maximum of $|P(z)|$ on $|z|=1$ may be larger than the maximum of $|P(z)|$ taken over the (2n)th roots of unity ,(3) represents a refinement of (1). Consider, for example, the polynomial $P(z)=z^n+ia, a>0.$

A. Aziz [1] proved the following interesting refinement of (3) and hence of Bernstein's Inequality (1) as well.

Theorem A: If P(z) is a polynomial of degree n, then for every given real α ,

$$\max_{|z|=1}|P'(z)| \le \frac{n}{2}(M_\alpha+M_{\alpha+\pi}), \tag{4}$$

where ,

$$M_\alpha=\max_{1\le k\le n}\left|P(e^{\frac{i(\alpha+2k\pi)}{n}})\right| \tag{5}$$

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and $M_{\alpha+\pi}$ is obtained from (5) by replacing α by $\alpha+\pi$ .The result is best possible and equality in (4) holds for $P(z)=z^n+re^{i\alpha},-1\le r\le 1.$

As an application of Theorem A, A. Aziz [1] proved the following result, which constitutes the corresponding refinement of (2).

Theorem B: If P(z) is a polynomial of degree n, then for all real α and R>1,

$$\max_{|z|=1}\left|P(Rz-P(z))\right|\le(\frac{R^n-1}{2})(M_\alpha+M_{\alpha+\pi}), \tag{6}$$

where M_α is defined by (5), $M_{\alpha+\pi}$ is obtained from M_α

by replacing α by $\alpha+\pi$ The result is best possible and equality in (6) holds for the polynomial

$$P(z)=z^n+re^{i\alpha},-1\le r\le 1.$$

If we restrict ourselves to the class of polynomials having no zero in $|z|<1$, inequality (1) is sharpened. In fact P. Erdos conjectured and later P.D.Lax [5] (see also [3]) verified that

$$\max_{|z|=1}\left|P'(z)\right|\le\frac{n}{2}\max\left|P(z)\right| \tag{7}$$

In this connection, A. Aziz [1] proved the following improvement of (4).

Theorem C: If P (z) is a polynomial of degree n having no zero in $|z|<1$, then for every real α

$$\max_{|z|=1}\left|P'(z)\right|\le\frac{n}{2}\left\{M_\alpha{}^2+M_{\alpha+\pi}{}^2\right\}^{\frac{1}{2}}, \tag{8}$$

where M_α is defined by (5) for all real α .The result is best possible and equality in (8) holds for $P(z)=z^n+e^{i\alpha}$.

As an application of Theorem C , A. Aziz [1] proved the following improvement of (6)

Theorem D: If P(z) is a polynomial of degree n having no zero in $|z|<1$,then for every given real α and R>1,

$$\max_{|z|=1}\left|P(Rz)-P(z)\right|\le(\frac{R^n-1}{2})(M_\alpha{}^2+M_{\alpha+\pi}{}^2)^{\frac{1}{2}}, \tag{9}$$

where M_α is defined by (5).The result is sharp and equality in (9) holds for $P(z)=z^n+e^{i\alpha}$.

In this paper we present generalizations of Theorems C and D .we shall also find the Corresponding inequalities if the zeos of the polynomial lie inside or outside a disk of radius less than or equal to 1.In fact ,we prove

Theorem 1: If P(z) is a polynomial of degree n having all its zeros in $|z|\ge k\ge 1$, then

$$\max_{|z|=1}\left|P'(z)\right|^2\le\frac{n^2}{2(1+k^2)}\left\{M_\alpha{}^2+M_{\alpha+\pi}{}^2\right\},$$

where M_α is defined by (5).

Remark 1: Taking k=1,Theorem 1 reduces to Theorem C.

Theorem 2: If P (z) is a polynomial of degree n having all its zeros in $|z|\ge k\ge 1$, then for all real α and R>1,

$$\left|P(Rz)-P(z)\right|\le\frac{R^n-1}{\sqrt{2(1+k^2)}}\left\{M_\alpha{}^2+M_{\alpha+\pi}{}^2\right\}^{\frac{1}{2}},$$

where M_α and $M_{\alpha+\pi}$ are defined as in Theorem A.

Remark 2: For k=1, Theorem 2 reduces to Theorem D.

Remark3: Applying Theorem 2 to the polynomial $Q(z) = z^n \overline{P(\frac{1}{\bar{z}})}$ and noting that $|P(z)| = |Q(z)|$, for $|z| = 1$, we get the following result.

Corollary 1: If P (z) is a polynomial of degree n, then for all real α and $r \le 1$,

$$\max_{|z|=1} |P(rz) - r^n P(z)| \le \frac{1-r^n}{\sqrt{2(1+k^2)}} \{M_\alpha{}^2 + M^2{}_{\alpha+\pi}\}^{\frac{1}{2}}$$

Theorem 3: If P (z) is a polynomial of degree n having all its zeros in $|z| < k, k \le 1$, then

$$\max_{|z|=1} |P'(z)| \le \frac{n}{\sqrt{2(1+k^2)}} \{M_\alpha{}^2 + M_{\alpha+\pi}{}^2\}^{\frac{1}{2}}$$

Theorem 4: If P(z) is a polynomial of degree n having all its zeros in $|z| \le k, k \le 1,$ then for all real α and R >1,

$$|P(Rz) - P(z)| \le \frac{R^n - 1}{\sqrt{2(1+k^{2n})}} \{M_\alpha + M^2{}_{\alpha+\pi}\}^{\frac{1}{2}}$$

Theorem 5: If P (z) is a self-inversive polynomial of degree n, then

$\max_{|z|=1} |P'(z)| \le \frac{n}{2}\sqrt{M_\alpha{}^2 + M_{\alpha+\pi}{}^2}$, where M_α is defined by (5).

PROOFS OF THEOREMS

For the proofs of the above theorems we need the following results:

Lemma 1: If P (z) is a polynomial of degree n, then for $|z| = 1$ and for every real α,

$$|P'(z)|^2 + |nP(z) - zP'(z)|^2 \le \frac{n^2}{2}\{M_\alpha{}^2 + M_{\alpha+\pi}{}^2\}, \tag{10}$$

where M_α and $M_{\alpha+\pi}$ are defined as in Theorem 1.

Lemma 2: If P (z) is a polynomial of degree n having all its zeros in $|z| \ge k \ge 1$, then

$$k^s |P(e^{i\theta})| \le |Q^s(e^{i\theta})|, 0 \le \theta \le 2\pi, \tag{11}$$

Where $Q(z) = z^n \overline{P(\frac{1}{\bar{z}})}$.

Lemma 3: If P (z) is a polynomial of degree n having all its zeros in $|z| < k, k \le 1,$ then

$$k^n \max_{|z|=1} |P'(z)| \le \max_{|z|=1} |Q'(z)|, \tag{12}$$

where Q(z) is as in Lemma 2.

Lemma 1 is due to A.Aziz (for reference see [1], lemma 3).

Lemmas 2 and 3 are due to N. K. Govil and Q. I. Rahman (for reference see[5], [6])

Proof of Theorem 1: Let $Q(z) = z^n \overline{P(\frac{1}{\bar{z}})}$. Then

$$|Q'(z)| = |nP(z) - zP'(z)| \text{, for } |z| = 1.$$

Using in (10), we get

$$|P'(z)| + |Q'(z)| \le \frac{n^2}{2}\{M_\alpha{}^2 + M_{\alpha+\pi}{}^2\} \tag{13}$$

From (11) with s=1, we have,

$$k\left|P'(z)\right| \le \left|Q'(z)\right|, \text{ for } \left|z\right| = 1.$$

Hence

$$\begin{aligned}(1+k^2)\left|P'(z)\right|^2 &= \left|P'(z)\right|^2 + k^2\left|P'(z)\right|^2 \\ &\le \left|P'(z)\right|^2 + \left|Q'(z)\right|^2 \\ &\le \frac{n^2}{2}\left\{M^2{}_{\alpha} + M^2{}_{\alpha+\pi}\right\} \qquad \text{(by (13))}\end{aligned}$$

This gives

$$\left|P'(z)\right|^2 \le \frac{n^2}{2(1+k^2)}\left\{M_{\alpha}{}^2 + M_{\alpha+\pi}{}^2\right\},$$

There by proving Theorem 1.

Proof of Theorem 2. we have for all $t \ge 1$ and $0 \le \theta \le 2\pi$,

$$\left|P'(te^{i\theta})\right| \le t^{n-1} \max_{|z|=1}\left|P'(z)\right| \quad \text{(by using (2) to P'(z))}$$

Applying Theorem 1 to the polynomial P' (z), which is of degree n-1, we get

$$\left|P'(te^{i\theta})\right| \le t^{n-1}.\frac{n}{\sqrt{2(1+k^2)}}\left\{M_{\alpha}{}^2 + M_{\alpha+\pi}{}^2\right\}^{\frac{1}{2}}$$

Hence for each $\theta, 0 \le \theta \le 2\pi$ and R>1, we have

$$\begin{aligned}\left|P(\mathrm{Re}^{i\theta}) - P(e^{i\theta})\right| &= \left|\int_1^R e^{i\theta}P'(te^{i\theta})dt\right| \\ &\le \int_1^R \left|P'(te^{i\theta})dt\right| \\ &\le \frac{n}{\sqrt{2(1+k^2)}}\left\{M_{\alpha}{}^2 + M_{\alpha+\pi}{}^2\right\}^{\frac{1}{2}}\int_1^R t^{n-1}dt \\ &= \frac{R^n - 1}{\sqrt{2(1+k^2)}}\left\{M_{\alpha}{}^2 + M_{\alpha+\pi}{}^2\right\}^{\frac{1}{2}}\end{aligned}$$

This implies

$$\left|P(Rz) - P(z)\right| \le \frac{R^n - 1}{\sqrt{2(1+k^2)}}\left\{M_{\alpha}{}^2 + M_{\alpha+\pi}{}^2\right\}^{\frac{1}{2}},$$

for $\left|z\right| = 1$ and R>1, which is the desired result.

Proof of Theorem 3: We have by Lemma 1,

$$\left|P'(z)\right|^2 + \left|nP(z) - zP'(z)\right|^2 \le \frac{n^2}{2}\left\{M_{\alpha}{}^2 + M_{\alpha+\pi}{}^2\right\},$$

Now $$\begin{aligned}(1+k^{2n})\max\left|P'(z)\right|^2 &= \left|P'(z)\right|^2 + k^{2n}\left|P'(z)\right|^2 \\ &= \left|P'(z)\right|^2 + \left|k^n P'(z)\right|^2\end{aligned}$$

Therefore

$$(1+k^{2n})\max_{|z|=1}|P'(z)|^2 \le \max_{|z|=1}\left\{|P'(z)|^2+|Q'(z)|^2\right\},$$

(This is true only when P' (z) and Q' (z) attain the maximum moduli at the same points).

$$=\max_{|z|=1}\left\{|P'(z)|^2+|nP(z)-zP'(z)|^2\right\}$$

$$\le \frac{n^2}{2}\left\{M_\alpha{}^2+M_{\alpha+\pi}{}^2\right\} \text{ (by lemma 1)}$$

$$\Rightarrow \max_{|z|=1}|P'(z)| \le \frac{n}{\sqrt{2(1+k^{2n})}}\left\{M^2{}_\alpha+M^2{}_{\alpha+\pi}\right\}^{\frac{1}{2}},$$

and the proof of Theorem 3 is complete.

Proof of Theorem 4: follows on the same lines as Theorem 2.

Proof of Theorem 5: Since $P(z)=z^n\overline{P(\frac{1}{\bar{z}})}$,

we have,

$$P'(z)=nz^{n-1}\overline{P(\frac{1}{\bar{z}})}-z^{n-2}\overline{P(\frac{1}{\bar{z}})}$$

$$\Rightarrow \quad zP'(z)=nz^n\overline{P(\frac{1}{\bar{z}})}-z^{n-1}\overline{P(\frac{1}{\bar{z}})}$$

$$\Rightarrow \quad zP'(z)=nP(z)-z^{n-1}\overline{P(\frac{1}{\bar{z}})}$$

$$\Rightarrow |nP(z)-zP'(z)|=|P'(z)|, \text{ for } |z|=1.$$

Using this, we have, by Lemma 1,

$$2|P'(z)|^2=|P'(z)|^2+|nP(z)-zP'(z)|^2$$

$$\Rightarrow |P'(z)|^2 \le \frac{n^2}{4}\left\{M_\alpha{}^2+M_{\alpha+\pi}{}^2\right\},$$

$$\Rightarrow |P'(z)| \le \frac{n}{2}\left\{M_\alpha{}^2+M_{\alpha+\pi}{}^2\right\}^{\frac{1}{2}},$$

thereby proving the result.

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