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A NON-LOCAL BOUNDARY VALUE PROBLEM WITH INTEGRAL CONDITIONS FOR A FOURTH ORDER PSEUDOHYPERBOLIC EQUATION

Azizbayov E.I.* and Y. T. Mehraliyev

Mechanics-mathematics faculty, Baku State University, Baku, Azerbaijan

E-mail: *azel_azerbaijan@mail.ru*

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ABSTRACT

In the paper, the classic solution of one-dimensional boundary value problem for a pseudohyperbolic equation with non-classic boundary conditions is investigated. For that the stated problem is reduced to the not-self-adjoint boundary value problem with equivalent boundary condition. Then, using the method of separation of variables, by means of the known spectral problem the given not self-adjoint boundary value problem is reduced to an integral equation. The existence and uniqueness of the integral equation is proved by means of the contraction mappings principle and it is shown that this solution is a unique solution for a not-adjoint boundary value problem. Finally, using the equivalence, the theorem on the existence and uniqueness of a non-local boundary value problem with integral condition is proved.

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1. INTRODUCTION:

Contemporary problems of natural sciences make necessary to state and investigate qualitative new problems, the striking example of which is the class of non-local problems for partial differential equations. Among non-local problems we can distinguish a class of problems with integral conditions. Such conditions appear by mathematical simulation of phenomena related to physical plasma [1], distribution of the heat [2] process of moisture transfer in capillary-simple environments [3], with the problems of demography and mathematical biology.

2. THE PROBLEM STATEMENT AND ITS REDUCTION TO THE EQUIVALENT PROBLEM:

Consider the equation [4]

$$u_{tt}(x,t)-u_{ttxx}(x,t)-u_{xx}(x,t)=q(t)u(x,t)+f(x,t) \tag{1}$$

in the domain $D_T=\{(x,t):0\le x\le 1, 0\le t\le T\}$ and state for it a problem with initial conditions

$$u(x,0)=\varphi(x),\quad u_t(x,0)=\psi(x)\ (0\le x\le 1) \tag{2}$$

and non-local conditions

$$u(0,t)=\beta u(1,t)\quad (0\le t\le T), \tag{3}$$

$$\int_0^1 u(x,t)dx=0\quad (0\le t\le T), \tag{4}$$

where $\beta\ne\pm1$ is a given number, $q(t)$, $f(t,x)$, $\varphi(x)$, $\psi(x)$ are the given functions, $u(x,t)$ is a sought function.
Earlier, the boundary value problems with non-local integral equations were considered in the papers [1], [2] and [8].
Here, for $\beta=0$ we have an Ionkin type boundary condition [3].

***Corresponding author: Azizbayov E.I.*, *E-mail: azel_azerbaijan@mail.ru**

Definition: Under the classic solution of problem (1)-(4) we understand the function $u(x,t)$ continuous in a closed domain D_T together with all its derivatives contained in equation (1), and satisfying all conditions (1)-(4) in the ordinary sense.

The following lemma is proved similarly [7].

Lemma 1: Let $q(t)\in C[0,T]$, $f(x,t)\in C(D_T)$, $\varphi(x),\ \psi(x)\in C[0,1]$, $\int_0^1 f(x,t)dx=0 \quad (0\le t\le T)$ and the following agreement conditions be fulfilled:

$$\varphi(0)-\beta\varphi(1)=0\,,\quad \int_0^1\varphi(x)dx=0\,,\ \ \varphi'(1)=\varphi'(0)\,,$$

$$\psi(0)-\beta\psi(1)=0\,,\ \int_0^1\psi(x)dx=0\,,\quad \psi'(1)=\psi'(0)\,. \tag{5}$$

Then the problem on finding the classic solution of problem (1)-(4) is equivalent to the problem on defining of the function $u(x,t)$ from (1)-(3) and

$$u_x(0,t)=u_x(1,t)\quad (0\le t\le T)\,. \tag{6}$$

Proof: Let $u(x,t)$ be the solution of problem (1)-(4). Integrating equation (1) with respect to x from 0 to 1, we have:

$$\frac{d^2}{dt^2}\int_0^1 u(x,t)dx-(u_{ttx}(1,t)-u_{ttx}(0,t))-(u_x(1,t)-u_x(0,t))=q(t)\int_0^1 u(x,t)dx+\int_0^1 f(x,t)dx\ \ (0\le t\le T)\,. \tag{7}$$

Assuming that $\int_0^1 f(x,t)dx=0\ \ (0\le t\le T)$, $\varphi'(1)=\varphi'(0)$, $\psi'(1)=\psi'(0)$ and allowing for (2), we find:

$$(u_{ttx}(1,t)-u_{ttx}(0,t))+(u_x(1,t)-u_x(0,t))=0,$$

$$u_x(1,0)-u_x(0,0)=0,\quad u_{tx}(1,0)-u_{tx}(0,0)=0\,.$$

Hence we arrive at fulfillment of (6).

Now, assume that $u(x,t)$ is the solution of problem (1)-(3), (6). Then allowing for (6), from (8) we find:

$$\frac{d^2}{dt^2}\int_0^1 u(x,t)dx-q(t)\int_0^1 u(x,t)dx=0\ \ (0\le t\le T). \tag{8}$$

From (2) and $\int_0^1\varphi(x)dx=0$, $\int_0^1\psi(x)dx=0,$ it is obvious that

$$\int_0^1 u(x,0)dx=\int_0^1\varphi(x)dx=0\,,\quad \int_0^1 u_t(x,0)dx=\int_0^1\psi(x)dx=0\,. \tag{9}$$

Since problem (8), (9) has only a trivial solution, then $\int_0^1 u(x,t)dx=0\ \ (0\le t\le T),$ i.e. condition (4) is satisfied. The lemma is proved.

3. AUXILIARY FACTS:

Now, in order to investigate problem (1)-(3), (6) we cite some known facts.

Consider the following spectral problem [3] and [5]:

$$X''(x)+\lambda X(x)=0\quad (0\le x\le 1)\,, \tag{10}$$

$$X(0)=\beta X(1),\ X'(0)=X'(1)\quad (\beta\ne\pm1)\,. \tag{11}$$

Boundary value problem (10), (11) is not self-adjoint. The problem

$$Y''(x)+\lambda Y(x)=0 \ (0\le x\le 1), \tag{12}$$

$$Y(0)=Y(1), \ \ Y'(1)=\beta Y'(0), \tag{13}$$

will be a conjugated problem.

We denote the system of eigen and adjoint functions of problem (10), (11) in the following way [5]:

$$X_0(x)=ax+b,..., X_{2k-1}(x)=(ax+b)\cos\lambda_k x, \ X_{2k}(x)=\sin\lambda_k x,..., \tag{14}$$

where

$$\lambda_k=2k\pi \ (k=0,1,2,...), \ \ a=(1-\beta)/(1+\beta)\neq 0, \ \ b=\beta/(1+\beta). \tag{15}$$

We choose the system of eigen and adjoint functions of the conjugated problem as follows [5]:

$$Y_0(x)=2,..., Y_{2k-1}(x)=4\cos\lambda_k x, \ Y_{2k}(x)=4(1-b-ax)\sin\lambda_k x,.... \tag{16}$$

It is directly verified that the biorthogonality conditions

$$(X_i,Y_j)=\int_0^1 X_i(x)Y_j(x)dx=\delta_{ij}$$

are fulfilled.

Here, δ_{ij} is Kronecker's symbol.

The following theorem is valid.

Theorem 1 [7]**:** The system of functions (14) forms a Riesz basis in the space $L_2(0,1)$ and the estimates

$$r\left\|g(x)\right\|_{L_2(0,1)} \le \sum_{k=0}^{\infty} g_k^2 \le R\left\|g(x)\right\|_{L_2(0,1)}, \tag{17}$$

where

$$g_k=(g(x),Y_k(x))=\int_0^1 g(x)Y_k(x)dx, \ (k=0,1,...)$$

$$r=\left\{\frac{1}{3}\left(\left(a+\frac{3}{2}b\right)^2+\frac{3}{4}b^2\right)+\frac{1}{2}\left(1+\left\|(ax+b)^2\right\|_{c[0,1]}\right)\right\}^{-1},$$

$R=8\left(1+\left\|(1-b-ax)^2\right\|_{C[0,1]}\right)$ are valid for any function $g(x)\in L_2(0,1)$.

Under the assumptions

$$g(x)\in C^{2i-1}[0,1], \ \ g^{(2i)}(x)\in L_2(0,1)$$

$$g^{(2s)}(0)=\beta g^{(2s)}(1), \ g^{(2s+1)}(0)=g^{(2s+1)}(1) \quad (s=0,\overline{i-1}; \ i\ge 1)$$

we establish the validity of the estimates:

$$\left(\sum_{k=1}^{\infty}(\lambda_k^{2i}g_{2k-1})^2\right)^{\frac{1}{2}} \le 2\sqrt{2}\left\|g^{(2i)}(x)\right\|_{L_2(0,1)}, \tag{18}$$

$$\left(\sum_{k=1}^{\infty}(\lambda_k^{2i}g_{2k})^2\right)^{\frac{1}{2}} \le 2\sqrt{2}\left\|g^{(2i)}(x)(1-b-ax)-2aig^{(2i-1)}(x)\right\|_{L_2(0,1)}. \tag{19}$$

Further, under the assumptions

$$g(x) \in C^{2i}[0,1], \quad g^{(2i+1)}(x) \in L_2(0,1),$$

$$g^{(2s)}(0) = \beta g^{(2s)}(1), \quad g^{(2s-1)}(0) = g^{(2s-1)}(1) \ (i \ge 1; \ s = \overline{0,i}).$$

we prove the validity of the estimates:

$$\left(\sum_{k=1}^{\infty} (\lambda_k^{2i+1} g_{2k-1})^2 \right)^{\frac{1}{2}} \le 2\sqrt{2} \left\| g^{(2i+1)}(x) \right\|_{L_2(0,1)}, \tag{20}$$

$$\left(\sum_{k=1}^{\infty} (\lambda_k^{2i+1} g_{2k})^2 \right)^{\frac{1}{2}} \le 2\sqrt{2} \left\| g^{(2i+1)}(x)(1-b-ax) - a(2i+1) g^{(2i)}(x) \right\|_{L_2(0,1)}. \tag{21}$$

Now, denote by $B_{2,T}^{\alpha}$ [6] an aggregate of all the functions of the form

$$u(x,t) = \sum_{k=0}^{\infty} u_k(t) X_k(x)$$

Considered in D_T, where each of the functions from $u_k(t)$ $(k = 0,1,2...)$ is continuous on $[0,T]$ and

$$J(u) \equiv \left\| u_0(t) \right\|_{C[0,T]} + \left(\sum_{k=1}^{\infty} (\lambda_k^{\alpha} \left\| u_{2k-1}(t) \right\|_{C[0,T]})^2 \right)^{\frac{1}{2}} + \left(\sum_{k=1}^{\infty} (\lambda_k^{\alpha} \left\| u_{2k}(t) \right\|_{C[0,T]})^2 \right)^{\frac{1}{2}} < +\infty,$$

where $\alpha \ge 0$. The norm in this set is defined as follows:

$$\left\| u(x,t) \right\|_{B_{2,T}^{\alpha}} = J(u).$$

It is known that $B_{2,T}^{\alpha}$ is a Banach space.

3. EXISTENCE AND UNIQUENESS OF THE SOLITION OF THE BOUNDARY VALUE PROBLEM:

Since the system (14) forms a Riesz basis in $L_2(0,1)$ and systems (14), (16) form a system of functions biorthogonal in $L_2(0,1)$, each solution of problem (1)-(3), (6) has the form:

$$u(x,t) = \sum_{k=0}^{\infty} u_k(t) X_k(x), \tag{22}$$

where

$$u_k(t) = \int_0^1 u(x,t) Y_k(x) dx \ (k = 0,1,...), \tag{23}$$

Moreover, $X_k(x)$ and $Y_k(x)$ are defined by relations (14) and (16) respectively.

Applying the method of separation of variables for determining the sought functions $u_k(t)$ $(k = 0,1,...)$, from (1), (2) we have:

$$u_0''(t) = q(t) u_0(t) + f_0(t), \quad (0 \le t \le T) \tag{24}$$

$$(1+\lambda_k^2) u_{2k-1}''(t) + \lambda_k^2 u_{2k-1}(t) = q(t) u_{2k-1}(t) + f_{2k-1}(t) \ (k = 1,2,...; \ 0 \le t \le T), \tag{25}$$

$$(1+\lambda_k^2) u_{2k}''(t) + \lambda_k^2 u_{2k}(t) = q(t) u_{2k}(t) + f_{2k}(t) - 2a\lambda_k (u_{2k-1}''(t) + u_{2k-1}(t)) \ (k = 1,2,...; \ 0 \le t \le T), \tag{26}$$

$$u_k(0) = \varphi_k, \ u_k'(0) = \psi_k \ (k = 0,1,...), \tag{27}$$

where

$$f_k(t) = \int_0^1 f(x,t) Y_k(x) dx, \quad \varphi_k = \int_0^1 \varphi(x) Y_k(x) dx, \quad \psi_k = \int_0^1 \psi(x) Y_k(x) dx \quad (k = 0,1,...).$$

Solving problem (21)-(24), we have:

$$u_0(t)=\varphi_0+t\psi_0+\int_0^t (t-\tau)F_0(\tau;u)d\tau \quad (0\le t\le T), \tag{28}$$

$$u_{2k-1}(t)=\varphi_{2k-1}\cos\beta_k t+\frac{1}{\beta_k}\psi_{2k-1}\sin\beta_k t+\frac{1}{\beta_k(1+\lambda_k^2)}\int_0^t F_{2k-1}(\tau;u)\sin\beta_k(t-\tau)d\tau \ (k=1,2,...;\ 0\le t\le T), \tag{29}$$

$$\begin{aligned}u_{2k}(t)&=\varphi_{2k}\cos\beta_k t+\frac{1}{\beta_k}\psi_{2k}\sin\beta_k t+\frac{1}{\beta_k(1+\lambda_k^2)}\int_0^t F_{2k}(\tau;u)\sin\beta_k(t-\tau)d\tau\\
&-\frac{a\lambda_k(1-\beta_k^2)}{\beta_k(1+\lambda_k^2)}\left[t\varphi_{2k-1}\sin\beta_k t+\left(\frac{1}{\beta_k}\sin\beta_k t-t\cos\beta_k t\right)\frac{1}{\beta_k}\psi_{2k-1}\right]\\
&-\frac{2a\lambda_k(1-\beta_k^2)}{\beta_k^2(1+\lambda_k^2)^2}\int_0^t\left(\int_0^\tau F_{2k-1}(\xi;u)\sin\beta_k(t-\xi)d\xi\right)\sin\beta_k(t-\tau)d\tau\\
&-\frac{2a\lambda_k}{\beta_k(1+\lambda_k^2)^2}\int_0^t F_{2k}(\tau;u)\sin\lambda_k(t-\tau)d\tau, \qquad (k=1,2,...;\ 0\le t\le T)\end{aligned} \tag{30}$$

where

$$F_k(t;u)=f_k(t)+q(t)u_k(t) \quad (k=0,1,2,...).$$

After substitution of expressions $u_0(t)$, $u_{2k-1}(t)$, $u_{2k}(t)$ of (28), (29), (30), respectively in (22) we have:

$$\begin{aligned}u(x,t)&=\left(\varphi_0+t\psi_0+\int_0^t(t-\tau)F_0(\tau;u)d\tau\right)X_0(x)\\
&+\sum_{k=1}^{\infty}\left(\varphi_{2k-1}\cos\beta_k t+\frac{1}{\beta_k}\psi_{2k-1}\sin\beta_k t+\frac{1}{\beta_k(1+\lambda_k^2)}\int_0^t F_{2k-1}(\tau;u)\sin\beta_k(t-\tau)d\tau\right)X_{2k-1}(x)\\
&+\sum_{k=1}^{\infty}\left(\varphi_{2k}\cos\beta_k t+\frac{1}{\beta_k}\psi_{2k}\sin\beta_k t+\frac{1}{\beta_k(1+\lambda_k^2)}\int_0^t F_{2k}(\tau;u)\sin\beta_k(t-\tau)d\tau\right.\\
&-\frac{a\lambda_k(1-\beta_k^2)}{\beta_k(1+\lambda_k^2)}\left[t\varphi_{2k-1}\sin\beta_k t+\left(\frac{1}{\beta_k}\sin\beta_k t-t\cos\beta_k t\right)\frac{1}{\beta_k}\psi_{2k-1}\right]\\
&\left.-\frac{2a\lambda_k(1-\beta_k^2)}{\beta_k^2(1+\lambda_k^2)^2}\int_0^t\left(\int_0^\tau F_{2k-1}(\xi;u)\sin\beta_k(t-\xi)d\xi\right)\sin\beta_k(t-\tau)d\tau-\frac{2a\lambda_k}{\beta_k(1+\lambda_k^2)^2}\int_0^t F_{2k}(\tau;u)\sin\lambda_k(t-\tau)d\tau,\right)X_{2k}(x)\ .\end{aligned} \tag{31}$$

Now, proceeding from definition of the solution of problem (1)-(3), (6) similar to [6], the following lemma is proved.

Lemma 2: If $u(x,t)=\sum_{k=0}^{\infty}u_k(t)X_k(x)$ is any solution of problem (1)-(3), (6), the functions $u_k(t)$ $(k=0,1,2...)$ satisfy system (28)-(30).

Theorem 2: Let

1. $q(t)\in C[0,T]$, $\beta\ne\pm1$;
2. $\varphi(x)\in C^2[0,1],\ \varphi'''(x)\in L_2(0,1),\ \varphi(0)=\beta\varphi(1),\ \varphi'(0)=\varphi'(1),\ \varphi''(0)=\beta\varphi''(1)$;
3. $\psi(x)\in C^2[0,1],\ \psi'''(x)\in L_2(0,1),\ \psi(0)=\beta\psi(1),\ \psi'(0)=\psi'(1),\ \psi''(0)=\beta\psi''(1)$;
4. $f(x,t)\in C(D_T),\ f_x(x,t)\in L_2(D_T),\ f(0,t)=\beta f(1,t)\ \ (0\le t\le T)$.

Then problem (1)-(3), (6) under small values of T has a unique classic solution.

Proof: Denoting

$$\mathrm{P}\, u = \sum_{k=0}^{\infty} \mathrm{P}_k(t;u) X_k(x),$$

where $\mathrm{P}_0(t;u)$, $\mathrm{P}_{2k-1}(t;u)$, $\mathrm{P}_{2k}(t;u)$ equal the right hand sides of (28), (29), (30), respectively and we write equation (31) in the form:

$$u = \mathrm{P}\, u. \tag{32}$$

We'll study equation (32) in the space $B_{2,T}^3$.

It is easy to see that

$1/\sqrt{2} < \beta_k < 1$, $0 < 1 - \beta_k^2 < \dfrac{1}{2}$,

Taking into account these relations, we have:

$$\|\mathrm{P}_0(t;u)\|_{C[0,T]} \le |\varphi_0| + T|\psi_0| + T\sqrt{T}\left(\int_0^T |f_0(\tau)|^2 d\tau + T^2 \|q(t)\|_{C[0,T]} \|u_0(t)\|_{C[0,T]}\right).$$

$$\left(\sum_{k=1}^{\infty}(\lambda_k^3 \|\mathrm{P}_{2k-1}(t;u)\|_{C[0,T]})^2\right)^{\frac{1}{2}} \le 2\left(\sum_{k=1}^{\infty}(\lambda_k^3|\varphi_{2k-1}|)^2\right)^{\frac{1}{2}} + 2\sqrt{2}\left(\sum_{k=1}^{\infty}(\lambda_k^3|\psi_{2k-1}|)^2\right)^{\frac{1}{2}}$$
$$+ 2\sqrt{2T}\left(\int_0^T \sum_{k=1}^{\infty}(\lambda_k |f_{2k-1}(\tau)|)^2 d\tau\right)^{\frac{1}{2}} + 2\sqrt{2}T\|q(t)\|_{C[0,T]}\left(\sum_{k=1}^{\infty}(\lambda_k^3 \|u_{2k-1}(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}},$$

$$\left(\sum_{k=1}^{\infty}(\lambda_k^3 \|\mathrm{P}_{2k}(t;u)\|_{C[0,T]})^2\right)^{\frac{1}{2}} \le \sqrt{10}\left(\sum_{k=1}^{\infty}(\lambda_k^3|\varphi_{2k}|)^2\right)^{\frac{1}{2}} + \sqrt{20}\left(\sum_{k=1}^{\infty}(\lambda_k^3|\psi_{2k}|)^2\right)^{\frac{1}{2}}$$
$$+ \sqrt{20T}\left(\int_0^T \sum_{k=1}^{\infty}(\lambda_k |f_{2k}(\tau)|)^2 d\tau\right)^{\frac{1}{2}} + \sqrt{20}T\|q(t)\|_{C[0,T]}\left(\sum_{k=1}^{\infty}(\lambda_k^3 \|u_{2k}(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}}$$
$$+ \sqrt{10}aT\left(\sum_{k=1}^{\infty}(\lambda_k^3|\varphi_{2k-1}|)^2\right)^{\frac{1}{2}} + \sqrt{10}a(1+T)\left(\sum_{k=1}^{\infty}(\lambda_k^3|\psi_{2k-1}|)^2\right)^{\frac{1}{2}}$$
$$+ \sqrt{10}aT\left(\sum_{k=1}^{\infty}(\lambda_k^3|\varphi_{2k-1}|)^2\right)^{\frac{1}{2}} + \sqrt{10}a(1+T)\left(\sum_{k=1}^{\infty}(\lambda_k^3|\psi_{2k-1}|)^2\right)^{\frac{1}{2}}$$
$$+ 2\sqrt{10}aT\sqrt{T}\left(\int_0^T \sum_{k=1}^{\infty}(\lambda_k |f_{2k-1}(\tau)|)^2 d\tau\right)^{\frac{1}{2}}.$$
$$+ 2\sqrt{10}aT^2\|q(t)\|_{C[0,T]}\left(\sum_{k=1}^{\infty}(\lambda_k^3 \|u_{2k-1}(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}}$$

Here, allowing for (18)-(21), we have:

$$\|\mathrm{P}_0(t;u)\|_{C[0,T]} \le a\|\varphi(x)\|_{L_2(0,1)} + aT\|\psi(x)\|_{L_2(0,1)} + T\sqrt{T}a\|f(x,t)\|_{L_2(D_T)} + T^2\|q(t)\|_{C[0,T]}\|u(x,t)\|_{B_{2,T}^3}, \tag{33}$$

$$\left(\sum_{k=1}^{\infty}(\lambda_k^3\|\mathrm{P}_{2k-1}(t;u)\|_{C[0,T]})^2\right)^{\frac{1}{2}} \le 4\sqrt{2}\|\varphi'''(x)\|_{L_2(0,1)} + 4\sqrt{2}\|\psi'''(x)\|_{L_2(0,1)} + 4\sqrt{2T}\|f_x(x,t)\|_{L_2(D_T)}, \\ + 2T\|q(t)\|_{C[0,T]}\|u(x,t)\|_{B^3_{2,T}} \tag{34}$$

$$\left(\sum_{k=1}^{\infty}(\lambda_k^3\|\mathrm{P}_{2k}(t;u)\|_{C[0,T]})^2\right)^{\frac{1}{2}} \le 8\|\varphi'''(x)(1-b-ax)-3a\varphi''(x)\|_{L_2(0,1)} + 8\|\psi'''(x)(1-b-ax)-3a\psi''(x)\|_{L_2(0,1)} \\ + 8\sqrt{T}\|f_x(x,t)(1-b-ax)-af(x,t)\|_{L_2(D_T)} + 8aT\|\varphi'''(x)\|_{L_2(0,1)} \\ + 8a(1+T)\|\psi'''(x)\|_{L_2(0,1)} \\ + 8aT\sqrt{T}\|f_x(x,t)\|_{L_2(D_T)} + 2\sqrt{2}T(1+aT)\|q(t)\|_{C[0,T]}\|u(x,t)\|_{B^3_{2,T}}, \tag{35}$$

Now, consider the operator P in the sphere

$K = K_R(\|u\|_{B^3_{2,T}} \le A(T)+1)$ from $B^3_{2,T}$,

where

$$A(T) = a\|\varphi(x)\|_{L_2(0,1)} + aT\|\psi(x)\|_{L_2(0,1)} + T\sqrt{T}a\|f(x,t)\|_{L_2(D_T)} + 4(\sqrt{2}+2aT)\|\varphi'''(x)\|_{L_2(0,1)} \\ + 4(\sqrt{2}+2a(1+T))\|\psi'''(x)\|_{L_2(0,1)} + 4\sqrt{T}(\sqrt{2}+aT)\|f_{xx}(x,t)\|_{L_2(D_T)} \\ + 8\|\varphi'''(x)(1-b-ax)-3a\varphi''(x)\|_{L_2(0,1)} + 8\|\varphi'''(x)(1-b-ax)-3a\varphi''(x)\|_{L_2(0,1)} \\ + 8\sqrt{T}\|f_x(x,t)(1-b-ax)-af(x,t)\|_{L_2(D_T)} \tag{36}$$

It is seen from (33)-(35) that for any $u, u_1, u_2 \in K_R$ the estimates :

$$\|\mathrm{P}u\|_{B^3_{2,T}} \le A(T) + B(T)\|u(x,t)\|_{B^3_{2,T}}, \tag{37}$$

$$\|\mathrm{P}u_1 - \mathrm{P}u_2\|_{B^3_{2,T}} \le B(T)\|u(x,t)\|_{B^3_{2,T}}, \tag{38}$$

where

$$B(T) = T((1+2\sqrt{2}a)T + 2(1+\sqrt{2}))\|q(t)\|_{C[0,T]}. \tag{39}$$

are valid.

Then it follows from estimates (37), (38) that under sufficiently small values of T the operator P acts in the sphere $K = K_R$ from $B^3_{2,T}$ and it is contractive. Therefore, in the sphere $K = K_R$ the operator P has a unique fixed point $\{u\}$ that is a solution of equation (32).

The function $u(x,t)$, as an element of the space $B^3_{2,T}$, is continuous and has continuous derivatives $u_x(x,t)$, $u_{xx}(x,t)$ on D_T. Now, prove that $u_{tt}(x,t)$ and $u_{ttxx}(x,t)$ are continuous in D_T. From (24)-(26) we have:

$$\|u_0''(t)\|_{C[0,T]} \le \left\|\|f(x,t)+q(t)u(x,t)\|_{C[0,T]}\right\|_{L_2(0,1)}, \tag{40}$$

$$\left(\sum_{k=1}^{\infty}(\lambda_k^3\|u_{2k-1}''(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}} \le \sqrt{3}(1+\|q(t)\|_{C[0,T]})\|u(x,t)\|_{B^3_{2,T}} + \frac{\sqrt{6}}{2}\left\|\|f_x(x,t)\|_{C[0,T]}\right\|_{L_2(0,1)}, \tag{41}$$

$$\left(\sum_{k=1}^{\infty}(\lambda_k^3\|u_{2k}''(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}} \le \sqrt{5}(3+\|q(t)\|_{C[0,T]})\|u(x,t)\|_{B^3_{2,T}} \\ + \frac{\sqrt{10}}{2}\left\|\|f_x(x,t)+f(x,t)\|_{C[0,T]}\right\|_{L_2(0,1)} + 2\sqrt{5}a\left(\sum_{k=1}^{\infty}(\lambda_k^3\|u_{2k-1}''(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}}. \tag{42}$$

It follows from estimates (40)-(42) that $u_{tt}(x,t)$ and $u_{ttxx}(x,t)$ are continuous in D_T .Further, it follows from (27) that

$$u(x,0)=\sum_{k=0}^{\infty}u_k(0)X_k(x)=\sum_{k=0}^{\infty}\varphi_k X_k(x)=\varphi(x)\quad (0\le x\le 1),$$

$$u_t(x,0)=\sum_{k=0}^{\infty}u'_k(0)X_k(x)=\sum_{k=0}^{\infty}\psi_k X_k(x)=\psi(x)\quad (0\le x\le 1),$$

since by the given theorem

$$\sum_{k=1}^{\infty}(\lambda_k^3|\varphi_{2k-1}|)^2<+\infty,\quad \sum_{k=1}^{\infty}(\lambda_k^3|\varphi_{2k}|)^2<+\infty,$$

$$\sum_{k=1}^{\infty}(\lambda_k^3|\psi_{2k-1}|)^2<+\infty,\quad \sum_{k=1}^{\infty}(\lambda_k^3|\psi_{2k}|)^2<+\infty,$$

and the more so,

$$\sum_{k=1}^{\infty}|\varphi_k|<+\infty,\ \sum_{k=1}^{\infty}|\psi_k|<+\infty .$$

Thus, conditions (2) are fulfilled.

It is obvious that conditions (3) is fulfilled for the function

$$u(x,t)=\sum_{k=0}^{\infty}u_k(t)X_k(x)=u_0(t)X_0(x)+\sum_{k=1}^{\infty}u_{2k-1}(t)X_{2k-1}(x)+\sum_{k=1}^{\infty}u_{2k}(t)X_{2k}(x).$$

It is easy to see that

$$u_{tt}(x,t)-u_{ttxx}(x,t)-u_{xx}(x,t)=u''_0(t)+\sum_{k=1}^{\infty}[(1+\lambda_k^2)u''_{2k-1}(t)+\lambda_k^2u_{2k-1}(t)]X_{2k-1}(x)+$$
$$+\sum_{k=1}^{\infty}[(1+\lambda_k^2)u''_{2k}(t)+\lambda_k^2u_{2k}(t)+2a\lambda_k(u''_{2k-1}(t)+u_{2k-1}(t))]X_{2k}(x)\quad (k=1,2,...;\ 0\le t\le T). \qquad (43)$$

Now, if we use systems (25)-(27), equality (40) takes the form:

$$u_{tt}(x,t)-u_{ttxx}(x,t)-u_{xx}(x,t)=F_0(t;u)+\sum_{k=1}^{\infty}F_{2k-1}(t;u)\,X_{2k-1}(x)+\sum_{k=1}^{\infty}F_{2k}(t;u)\,X_{2k}(x)=\sum_{k=0}^{\infty}F_k(t;u)\,X_k(x), \qquad (44)$$

Where the functions $X_k(x)$ $(k=0,1,2,...)$ are determined by relation (14), and

$$F_k(t;u)=f_k(t)+q(t)u_k(t)\quad (k=0,1,2,...).$$

Under the conditions of the theorem it is obvious that

$$\sum_{k=0}^{\infty}|F_k(t;u)|<+\infty . \qquad (45)$$

Then it follows from (45) that for any fixed $t\in[0,T]$:

$$\sum_{k=0}^{\infty}F_k(t;u)\,X_k(x)=F(u(x,t))\equiv q(t)u(x,t)+f(x,t)\quad \forall x\in[0,1]. \qquad (46)$$

Thus, relations (44) and (46) yield

$$u_{tt}(x,t)-u_{ttxx}(x,t)-u_{xx}(x,t)=q(t)u(x,t)+f(x,t)$$

Consequently, the function $u(x,t)$ satisfies equation (1) everywhere in D_T.

So, $u(x,t)$ is a solution of problem (1)-(3), (6), and by lemma 2 it is unique. The theorem is proved.

By means of lemma 1 we prove the following

Theorem 3: Let all the conditions of theorem 2 and agreement conditions (5) be fulfilled. Then for sufficiently small values of T, problem (1)-(3) has a unique classic solution.

4. CONCLUSION:

The following results have been obtained:

1. The existence of the solution of a not self-adjoint boundary value problem for a fourth order pseudohyperbolic equation has been proved;

2. The uniqueness of the solution of a not self-adjoint boundary value problem for a fourth order pseudohyperbolic equation has been shown;

3. The existence of the classic solution of a non-classic boundary value problem with integral boundary for a fourth order pseudohyperbolic equation has been proved;

4. The uniqueness of the classic solution of a non-classic boundary value problem with integral boundary for a fourth order pseudohyperbolic equation has been shown.

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