International Journal of Mathematical Archive-3(2), 2012, Page: 512-518
Available online through www.ijma.info ISSN 2229 – 5046

ON SUFFICIENT CONDITIONS FOR CERTAIN SUBCLASSES OF ANALYTIC AND UNIVALENT FUNCTIONS

S. P. Goyal[1*], Rakesh Kumar[2] and Mamoru Nunokawa[3]

[1]Department of Mathematics, University of Rajasthan, Jaipur-302055, India
E-mail: somprg@gmail.com

[2]Department of Mathematics, Amity University Rajasthan, NH-8, Jaipur-302002, India
E-mail: rkyadav11@gmail.com

[3]Department of Mathematics, University of Gunma, Hoshikuki-Cho 798-8, Chuou-Ward, Chiba-2600808, Japan
E-mail: Mamoru_nuno@doctor.nifty.jp

(Received on: 25-01-12; Accepted on: 20-02-12)

ABSTRACT

The aim of the present paper is to obtain sufficient conditions for starlike functions of order β. We establish two theorems. The first theorem provides improvement of the sufficient conditions for starlikeness obtained earlier by several research workers such as Lewandowski et al. [5], Li and Owa [6], Nunokawa et al. [8,9], Ramesha et al. [10] and Ravichandran et al. [11]. The second theorem gives sufficient conditions for a function f to be in the subclass $B_n(\mu,\alpha)$ of analytic and univalent functions on the unit disk.

***AMS Subject Classifications** : 30C45; 30C50.*

***Key words** : Analytic functions, Univalent functions, Starlike functions,*

1. INTRODUCTION

Let A_n be the class of functions of the form

$$f(z) = z + \sum_{m=n+1}^{\infty} a_m z^m \tag{1.1}$$

which are analytic in the open unit disk $U = \{z \in C : |z| < 1\}$. Let $A_1 \equiv A$.

A function $f(z) \in A$ is said to be starlike of order α, if

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha \quad (0 \le \alpha < 1; z \in U). \tag{1.2}$$

The class of all starlike functions of order α is denoted by $S^*(\alpha)$. Let $S^*(0) \equiv S^*$. Also, a function f belonging to S is said to be convex of order α if

$$\operatorname{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \alpha \quad (0 \le \alpha < 1; z \in U). \tag{1.3}$$

We denote by $K(\alpha)$, the subclass of A consisting of functions which are convex of order α in U. Also a function $f \in S$ is said to be in close-to-convex of order α if

***Corresponding author: S. P. Goyal[1]*, *E-mail: somprg@gmail.com**

$$\mathrm{Re} f'(z) > \alpha \quad (0 \le \alpha < 1; z \in U). \tag{1.4}$$

We denote this class by $CC(\alpha)$, the subclass of A consisting of functions which are close-to-convex of order α in U.

Following Frasin and Jahangiri [3] (see also [1]), we introduce the class $B_n(\mu,\alpha)$ defined as follows:

Definition. A function $f(z) \in A_n$ is said to be member of $B_n(\mu,\alpha)$ if and only if

$$\left| f'(z)\left(\frac{z}{f(z)}\right)^{\mu+1} - 1 \right| < 1-\alpha \qquad (z \in U) \tag{1.5}$$

for some $\alpha (0 \le \alpha < 1)$ and $\mu \ge -1$.

Note that the condition (1.5) is equivalent to

$$\mathrm{Re}\left\{ f'(z)\left(\frac{z}{f(z)}\right)^{\mu+1} \right\} > \alpha \quad (0 \le \alpha < 1,\, \mu \ge -1; z \in U). \tag{1.6}$$

Clearly $B_n(0,\alpha) \equiv S_n^*(\alpha)$, $B_n(-1,\alpha) \equiv CC_n(\alpha)$ and $B_1(1,\alpha) \equiv B(\alpha)$, the class which has been introduced and studied by Frasin and Darus [4] (see also [2]).

The following theorems dealing with the sufficient conditions for starlikeness were obtained.

Theorem A: ([5]) *Let* $f(z) \in A$ *satisfy the condition*

$$\mathrm{Re}\left\{ \frac{zf'(z)}{f(z)}\left(1+\frac{zf''(z)}{f'(z)}\right) \right\} > 0 \quad (z \in U),$$

then $f(z) \in S^*$.

Theorem B: ([10]) *Let* $f(z) \in A$ *satisfy the condition*

$$\mathrm{Re}\left\{ \frac{zf'(z)}{f(z)}\left(1+\alpha\frac{zf''(z)}{f'(z)}\right) \right\} > 0 \quad (z \in U),$$

where $\alpha \ge 0$, then $f(z) \in S^*$.

Theorem C: [6] *Let* $f(z) \in A$ *satisfy the condition*

$$\mathrm{Re}\left\{ \frac{zf'(z)}{f(z)}\left(1+\alpha\frac{zf''(z)}{f'(z)}\right) \right\} > -\frac{\alpha}{2} \quad (z \in U),$$

where $\alpha \ge 0$, then $f(z) \in S^*$.

Theorem D: [6] *Let* $f(z) \in A$ *satisfy the condition*

$$\mathrm{Re}\left\{ \frac{zf'(z)}{f(z)}\left(1+\alpha\frac{zf''(z)}{f'(z)}\right) \right\} > -\frac{\alpha^2}{4}(1-\alpha) \quad (z \in U),$$

where $0 \le \alpha < 2$, then $f(z) \in S^*(\alpha/2)$.

Theorem E: [6] *Let* $f(z)\in A$ *satisfy the condition*

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\left(1+\frac{zf''(z)}{f'(z)}\right)\right\}>0 \quad (z\in U),$$

then $f(z)\in S^*(1/2)$.

Theorem F: [9] *Let* $f(z)\in A$ *satisfy the condition*

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\left(1+\alpha\frac{zf''(z)}{f'(z)}\right)\right\}>-\frac{\alpha}{2}\left\{1+3\left(\mathrm{Im}\frac{zf'(z)}{f(z)}\right)^2\right\} \quad (z\in U),$$

where $\alpha\geq 0$ then $f(z)\in S^*$.

The results discussed in Theorem A, B, C, D, E and F were obtained earlier by Lewandowski et al. [5], Ramesha et al. [10], Li and Owa [6] and Nunokawa et al. [9] respectively.

Obviously Theorems C, D and E are improvement of the Theorems A and B while Theorem F is an improvement of Theorem C.

Recently Nunokawa, Goyal and Kumar [8] proved the following theorem:

Theorem G: *Let* $f(z)\in A$ *satisfy the condition*

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\left(1+\alpha\frac{zf''(z)}{f'(z)}\right)\right\}>-\frac{\alpha}{2}\left\{1+3\left|\frac{zf'(z)}{f(z)}\right|^2\right\} \quad (z\in U)$$

where $\alpha\geq 0$ then $f(z)\in S^*$.

The above theorem provides an improvement of all the aforementioned Theorems A to F.

Ravichandran et al. [11] discussed the sufficient conditions for a starlike function of order β and obtained the result:

Theorem H: *Let* $f(z)\in A_n$ *satisfy the condition*

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\left(1+\alpha\frac{zf''(z)}{f'(z)}\right)\right\}>\left\{\beta-\alpha(1-\beta)\left(\beta+\frac{n}{2}\right)\right\} \quad (z\in U)$$

where $\alpha\geq 0$ then $f(z)\in S_n^*(\beta)$.

To prove our main results, we shall require the following lemma:

Lemma 1.1:. [7] *Let* $p(z)=b_0+b_nz^n+b_{n+1}z^{n+1}+...(n\in N)$ *be analytic in* U *with* $p(z)\neq b_0$. *If* $0<|z_0|<1$ *and* $\mathrm{Re}\{p(z_0)\}=\min_{|z|\leq|z_0|}\mathrm{Re}\{p(z)\}$, *then*

$$z_0p'(z_0)\leq-\frac{n}{2}\frac{|b_0-p(z_0)|^2}{\mathrm{Re}\{b_0-p(z_0)\}}.$$

In this paper, we establish two theorems. These theorems provide sufficient conditions for the functions to be starlike of order β. The first theorem is an improvement of the Theorem H and is a generalization of Theorem G which in turn is an improvement of Theorems A to F. Our second theorem generalize and improve the result obtained by Frasin [1].

2. MAIN RESULTS

Theorem 2.1: *If* $f(z)\in A_n$ *satisfies*

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\left(1+\alpha\frac{zf''(z)}{f'(z)}\right)\right\}>\left\{\beta-\alpha(1-\beta)\left(\beta+\frac{n}{2}\right)\right\}-\left\{\alpha\left(1+\frac{n}{2(1-\beta)}\right)\left|\frac{zf'(z)}{f(z)}-\beta\right|^2\right\}. \tag{2.1}$$

then $f(z)\in S_n^*(\beta)$.

Proof: Let us put

$$(1-\beta)p(z)+\beta=\frac{zf'(z)}{f(z)}, \tag{2.2}$$

then $p(z)$ is analytic in U and $p(0)=1$.

If there exists a point $z_0\in U\setminus\{0\}$ such that

$\mathrm{Re}\{p(z)\}>0$ for $|z|<|z_0|$, $\quad p(z_0)=i\gamma$

where γ is a real number. Then applying Lemma 1.1, we have

$$z_0p'(z_0)\le-\frac{n}{2}(1+\gamma^2)$$

or

$$z_0p'(z_0)\le-\frac{n}{2}\left(1+|p(z_0)|^2\right). \tag{2.3}$$

On the other hand

$$\begin{aligned}\mathrm{Re}\left\{\frac{z_0f'(z_0)}{f(z_0)}\left(1+\alpha\frac{z_0f''(z_0)}{f'(z_0)}\right)\right\}&=\mathrm{Re}\left\{[(1-\beta)p(z_0)+\beta]\left[\alpha[(1-\beta)p(z_0)+\beta]+\frac{\alpha(1-\beta)z_0p'(z_0)}{(1-\beta)p(z_0)+\beta}+1-\alpha\right]\right\}\\
&=\mathrm{Re}\left\{\alpha[(1-\beta)p(z_0)+\beta]^2+\alpha(1-\beta)z_0p'(z_0)+(1-\alpha)(1-\beta)p(z_0)+\beta(1-\alpha)\right\}\\
&=\mathrm{Re}\Big\{\alpha(1-\beta)^2\left(p(z_0)\right)^2+\beta^2\alpha+2\beta\alpha(1-\beta)p(z_0)+\alpha(1-\beta)z_0p'(z_0)\\
&\qquad+(1-\alpha)(1-\beta)p(z_0)+\beta(1-\alpha)\Big\}\\
&=\alpha(1-\beta)^2\mathrm{Re}\{p(z_0)\}^2+\beta^2\alpha+2\beta\alpha(1-\beta)\mathrm{Re}\{p(z_0)\}+\alpha(1-\beta)\mathrm{Re}\{z_0p'(z_0)\}\\
&\qquad+(1-\alpha)(1-\beta)\mathrm{Re}\{p(z_0)\}+\beta(1-\alpha)\\
&\le-\alpha(1-\beta)^2|p(z_0)|^2+\beta^2\alpha-\alpha(1-\beta)\frac{n}{2}\left(1+|p(z_0)|^2\right)+\beta(1-\alpha)\\
&\le\left\{\beta-\alpha(1-\beta)\left(\beta+\frac{n}{2}\right)\right\}-\left\{\alpha\left(1+\frac{n}{2(1-\beta)}\right)\left|\frac{z_0f'(z_0)}{f(z_0)}-\beta\right|^2\right\}.\end{aligned}$$

This is a contradiction and therefore the proof of the Theorem 2.1 is complete.

Remark: For $\beta=0$ and for $n=1$, the Theorem 2.1 will reduce to the result given in Theorem G. For $\beta=1/2,\alpha=1$ and $n=1$, we get the following result:

Corollary 2.2: *If If $f(z)\in A$ satisfies*

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\left(1+\frac{zf''(z)}{f'(z)}\right)\right\} > -\left[1+\left|\frac{zf'(z)}{f(z)}-\frac{1}{2}\right|^2\right]$$

then $f(z)\in S^*(1/2)$.

The next theorem gives sufficient conditions for a function $f(z)$ to be in the class $B_n(\mu,\alpha)$.

Theorem 2.3: *If $f(z)\in A_n, \mu\ge -1, \mu\ne 0,$ satisfies*

$$\mathrm{Re}\left\{\beta f'(z)\left(\frac{z}{f(z)}\right)^{\mu+1}+(1-\beta)\left[\frac{\mu-1}{\mu}z\left(\left(\frac{z}{f(z)}\right)^{\mu}\right)-\frac{z^2}{\mu}\left(\left(\frac{z}{f(z)}\right)^{\mu}\right)\right]\right\} > \left[\beta\delta-\frac{n}{2}(1-\beta)(1-\delta)\right]-\frac{n}{2}\frac{(1-\beta)}{(1-\delta)}\left|f'(z)\left(\frac{z}{f(z)}\right)^{\mu+1}-\delta\right|^2 \tag{2.4}$$

then $f(z)\in B_n(\mu,\alpha)(\mu\ne 0)$.

Proof: Let us put

$$(1-\delta)p(z)+\delta = f'(z)\left(\frac{z}{f(z)}\right)^{\mu+1}, \tag{2.5}$$

then $p(z)$ is analytic in U and $p(0)=1$..

Since

$$f'(z)\left(\frac{z}{f(z)}\right)^{\mu+1} = \left(\frac{z}{f(z)}\right)^{\mu}-\frac{z}{\mu}\left(\left(\frac{z}{f(z)}\right)^{\mu}\right). \tag{2.6}$$

It follows from (2.5) and (2.6) that

$$(1-\delta)zp'(z) = \frac{\mu-1}{\mu}z\left(\left(\frac{z}{f(z)}\right)^{\mu}\right)-\frac{z^2}{\mu}\left(\left(\frac{z}{f(z)}\right)^{\mu}\right). \tag{2.7}$$

If there exists a point $z_0\in U\setminus\{0\}$ such that

$$\mathrm{Re}\{p(z)\}>0 \text{ for } |z|<|z_0|,\quad p(z_0)=i\gamma$$

where γ is a real number. Then applying Lemma 1.1, we have

$$z_0p'(z_0)\le -\frac{n}{2}(1+\gamma^2)$$

or

$$z_0p'(z_0)\le -\frac{n}{2}\left(1+|p(z_0)|^2\right). \tag{2.8}$$

From (2.5), (2.7) and (2.8), we have

$$\begin{aligned}
&\mathrm{Re}\left\{\beta f'(z_0)\left(\frac{z_0}{f(z_0)}\right)^{\mu+1}+(1-\beta)\left[\frac{\mu-1}{\mu}z_0^2\left(\left(\frac{z_0}{f(z_0)}\right)^{\mu}\right)-\frac{z_0^2}{\mu}\left(\left(\frac{z_0}{f(z_0)}\right)^{\mu}\right)\right]\right\}\\
&= \mathrm{Re}\left\{\beta\delta+\beta(1-\delta)p(z_0)+(1-\beta)(1-\delta)z_0p'(z_0)\right\}\\
&= \beta\delta+\beta(1-\delta)\mathrm{Re}\{p(z_0)\}+(1-\beta)(1-\delta)\mathrm{Re}\{z_0p'(z_0)\}\\
&\le \beta\delta-\frac{n}{2}(1-\beta)(1-\delta)\left[1+|p(z_0)|^2\right]\\
&\le \left[\beta\delta-\frac{n}{2}(1-\beta)(1-\delta)\right]-\frac{n}{2}\frac{(1-\beta)}{(1-\delta)}\left|f'(z_0)\left(\frac{z_0}{f(z_0)}\right)^{\mu+1}-\delta\right|^2.
\end{aligned}$$

This is a contradiction and therefore the proof of the Theorem 2.3 is complete.

Putting $n = 1, \mu = 1$, we get the following result:

Corollary 2.4: *If* $f(z) \in A$ *satisfies*

$$\mathrm{Re}\left\{\beta\frac{z^2 f'(z)}{f^2(z)} - (1-\beta)z^2\left(\frac{z}{f(z)}\right)\right\} > \frac{(\delta+1)\beta+\delta-1}{2} - \frac{1}{2}\frac{(1-\beta)}{(1-\delta)}\left|\frac{z^2 f'(z)}{f^2(z)} - \delta\right|^2 \qquad (z \in U)$$

then $f(z) \in B(\delta)$.

The above corollary is an improvement to the result in [1, Theorem 2.3].

Putting $\beta = 0$ in Theorem 2.3, we get the following result:

Corollary 2.5: *If* $f(z) \in A_n$ *satisfies*

$$\mathrm{Re}\left\{\frac{\mu-1}{\mu} z\left(\left(\frac{z}{f(z)}\right)^{\mu}\right)' - \frac{z^2}{\mu}\left(\left(\frac{z}{f(z)}\right)^{\mu}\right)''\right\} > -\frac{n}{2}(1-\delta)\left[1+\frac{1}{(1-\delta)^2}\left|f'(z)\left(\frac{z}{f(z)}\right)^{\mu+1} - \delta\right|^2\right]$$

then $f(z) \in B_n(\mu, \delta)$.

If we put $\beta = 0, \mu = 1$ in Theorem 2.3, we get the following result:

Corollary 2.6. *If* $f(z) \in A_n$ *satisfies*

$$\mathrm{Re}\left\{-z^2\left(\frac{z}{f(z)}\right)''\right\} > -\frac{n}{2}(1-\delta)\left[1+\frac{1}{(1-\delta)^2}\left|\frac{z^2 f'(z)}{f^2(z)} - \delta\right|^2\right]$$

then $f(z) \in B_n(\delta)$.

ACKNOWLEDGMENT: The first author (S.P.G.) is thankful to CSIR, New Delhi, India for awarding Emeritus Scientistship, under scheme number 21(084)/10/EMR-II.

REFERENCES:

[1] B.A. Frasin, Some properties of certain analytic and univalent functions, Tamsui Oxford Jour. Math. Sci., **23** (1) (2007), 67-77.

[2] B.A. Frasin, A note on certain analytic and univalent functions, *Southeast Asian J. Math.*, **28** (2004), 829-838.

[3] B.A. Frasin and J. Jahangiri, A new and comprehensive class of analytic functions, *Analele Universitatii din Oradea*, 15 (2008), 61-64.

[4] B.A. Frasin and M. Darus, On certain analytic univalent functions, *Internat. J. Math. and Math. Sci.*, **25** (5) (2001), 305-310.

[5] Z. Lewandowski, S.S. Miller and E. Zlotkiewicz, Generating functions for some classes of univalent functions, *Proc. Amer. Math. Soc.*, **56** (1976), 111-117.

[6] J.-L. Li and S. Owa, Sufficient conditions for starlikeness, *Indian J. Pure Appl. Math.*, **33** (9) (2002), 313-318.

[7] S.S. Miller and P.T. Mocanu, Second-order differential inequalities in the complex plane, *J. Math. Anal. Appl.*, **65** (2) (1978), 289-305.

[8] M. Nunokawa, S.P. Goyal and R. Kumar, Sufficient conditions for starlikeness, Submitted.

[9] M. Nunokawa, S. Owa, S.K. Lee, M. Obradovic, M.K. Aouf, H. Saitoh, H. Ikada and N. Koika, Sufficient conditions for starlikeness, *Chinese J. Math.*, **24** (1996), 265-270.

[10] C. Ramesha, S. Kumar and K.S. Padmanbham, A sufficient condition for starlikeness,*Chinese J. Math.*, 23 (1995), 167-171.

[11] V. Ravichandran, C. Selvaraj and R. Rajalaksmi, Sufficient conditions for starlike functions of order α, *J. Ineq. Pure Appl. Math.*, **3** (5) (2002), 1-6.

****