



ON SUFFICIENT CONDITIONS FOR CERTAIN SUBCLASSES OF ANALYTIC AND UNIVALENT FUNCTIONS

S. P. Goyal^{1*}, Rakesh Kumar² and Mamoru Nunokawa³

¹Department of Mathematics, University of Rajasthan, Jaipur-302055, India

E-mail: somprg@gmail.com

²Department of Mathematics, Amity University Rajasthan, NH-8, Jaipur-302002, India

E-mail: rkyadav11@gmail.com

³Department of Mathematics, University of Gunma, Hoshikuki-Cho 798-8, Chuou-Ward, Chiba-2600808, Japan

E-mail: Mamoru_nuno@doctor.nifty.jp

(Received on: 25-01-12; Accepted on: 20-02-12)

ABSTRACT

The aim of the present paper is to obtain sufficient conditions for starlike functions of order β . We establish two theorems. The first theorem provides improvement of the sufficient conditions for starlikeness obtained earlier by several research workers such as Lewandowski et al. [5], Li and Owa [6], Nunokawa et al. [8,9], Ramesha et al. [10] and Ravichandran et al. [11]. The second theorem gives sufficient conditions for a function f to be in the subclass $B_n(\mu, \alpha)$ of analytic and univalent functions on the unit disk.

AMS Subject Classifications : 30C45; 30C50.

Key words : Analytic functions, Univalent functions, Starlike functions,

1. INTRODUCTION

Let A_n be the class of functions of the form

$$f(z) = z + \sum_{m=n+1}^{\infty} a_m z^m \tag{1.1}$$

which are analytic in the open unit disk $U = \{z \in C : |z| < 1\}$. Let $A_1 \equiv A$.

A function $f(z) \in A$ is said to be starlike of order α , if

$$\operatorname{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} > \alpha \quad (0 \leq \alpha < 1; z \in U). \tag{1.2}$$

The class of all starlike functions of order α is denoted by $S^*(\alpha)$. Let $S^*(0) \equiv S^*$. Also, a function f belonging to S is said to be convex of order α if

$$\operatorname{Re} \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > \alpha \quad (0 \leq \alpha < 1; z \in U). \tag{1.3}$$

We denote by $K(\alpha)$, the subclass of A consisting of functions which are convex of order α in U . Also a function $f \in S$ is said to be in close-to-convex of order α if

Corresponding author: S. P. Goyal^{1}, *E-mail: somprg@gmail.com

$$\operatorname{Re}f'(z) > \alpha \quad (0 \leq \alpha < 1; z \in U). \quad (1.4)$$

We denote this class by $CC(\alpha)$, the subclass of A consisting of functions which are close-to-convex of order α in U .
Following Frasin and Jahangiri [3] (see also [1]), we introduce the class $B_n(\mu, \alpha)$ defined as follows:

Definition. A function $f(z) \in A_n$ is said to be member of $B_n(\mu, \alpha)$ if and only if

$$\left| f'(z) \left(\frac{z}{f(z)} \right)^{\mu+1} - 1 \right| < 1 - \alpha \quad (z \in U) \quad (1.5)$$

for some $\alpha(0 \leq \alpha < 1)$ and $\mu \geq -1$.

Note that the condition (1.5) is equivalent to

$$\operatorname{Re} \left\{ f'(z) \left(\frac{z}{f(z)} \right)^{\mu+1} \right\} > \alpha \quad (0 \leq \alpha < 1, \mu \geq -1; z \in U). \quad (1.6)$$

Clearly $B_n(0, \alpha) \equiv S_n^*(\alpha)$, $B_n(-1, \alpha) \equiv CC_n(\alpha)$ and $B_1(1, \alpha) \equiv B(\alpha)$, the class which has been introduced and studied by Frasin and Darus [4] (see also [2]).

The following theorems dealing with the sufficient conditions for starlikeness were obtained.

Theorem A: ([5]) Let $f(z) \in A$ satisfy the condition

$$\operatorname{Re} \left\{ \frac{zf'(z) \left(1 + \frac{zf''(z)}{f'(z)} \right)}{f(z)} \right\} > 0 \quad (z \in U),$$

then $f(z) \in S^*$.

Theorem B: ([10]) Let $f(z) \in A$ satisfy the condition

$$\operatorname{Re} \left\{ \frac{zf'(z) \left(1 + \alpha \frac{zf''(z)}{f'(z)} \right)}{f(z)} \right\} > 0 \quad (z \in U),$$

where $\alpha \geq 0$, then $f(z) \in S^*$.

Theorem C: [6] Let $f(z) \in A$ satisfy the condition

$$\operatorname{Re} \left\{ \frac{zf'(z) \left(1 + \alpha \frac{zf''(z)}{f'(z)} \right)}{f(z)} \right\} > -\frac{\alpha}{2} \quad (z \in U),$$

where $\alpha \geq 0$, then $f(z) \in S^*$.

Theorem D: [6] Let $f(z) \in A$ satisfy the condition

$$\operatorname{Re} \left\{ \frac{zf'(z) \left(1 + \alpha \frac{zf''(z)}{f'(z)} \right)}{f(z)} \right\} > -\frac{\alpha^2}{4}(1 - \alpha) \quad (z \in U),$$

where $0 \leq \alpha < 2$, then $f(z) \in S^*(\alpha/2)$.

Theorem E: [6] Let $f(z) \in A$ satisfy the condition

$$\operatorname{Re} \left\{ \frac{zf'(z)}{f(z)} \left(1 + \frac{zf''(z)}{f'(z)} \right) \right\} > 0 \quad (z \in U),$$

then $f(z) \in S^*(1/2)$.

Theorem F: [9] Let $f(z) \in A$ satisfy the condition

$$\operatorname{Re} \left\{ \frac{zf'(z)}{f(z)} \left(1 + \alpha \frac{zf''(z)}{f'(z)} \right) \right\} > -\frac{\alpha}{2} \left\{ 1 + 3 \left(\operatorname{Im} \frac{zf'(z)}{f(z)} \right)^2 \right\} \quad (z \in U),$$

where $\alpha \geq 0$ then $f(z) \in S^*$.

The results discussed in Theorem A, B, C, D, E and F were obtained earlier by Lewandowski et al. [5], Ramesha et al. [10], Li and Owa [6] and Nunokawa et al. [9] respectively.

Obviously Theorems C, D and E are improvement of the Theorems A and B while Theorem F is an improvement of Theorem C.

Recently Nunokawa, Goyal and Kumar [8] proved the following theorem:

Theorem G: Let $f(z) \in A$ satisfy the condition

$$\operatorname{Re} \left\{ \frac{zf'(z)}{f(z)} \left(1 + \alpha \frac{zf''(z)}{f'(z)} \right) \right\} > -\frac{\alpha}{2} \left\{ 1 + 3 \left| \frac{zf'(z)}{f(z)} \right|^2 \right\} \quad (z \in U)$$

where $\alpha \geq 0$ then $f(z) \in S^*$.

The above theorem provides an improvement of all the aforementioned Theorems A to F.

Ravichandran et al. [11] discussed the sufficient conditions for a starlike function of order β and obtained the result:

Theorem H: Let $f(z) \in A_n$ satisfy the condition

$$\operatorname{Re} \left\{ \frac{zf'(z)}{f(z)} \left(1 + \alpha \frac{zf''(z)}{f'(z)} \right) \right\} > \left\{ \beta - \alpha(1 - \beta) \left(\beta + \frac{n}{2} \right) \right\} \quad (z \in U)$$

where $\alpha \geq 0$ then $f(z) \in S_n^*(\beta)$.

To prove our main results, we shall require the following lemma:

Lemma 1.1: [7] Let $p(z) = b_0 + b_n z^n + b_{n+1} z^{n+1} + \dots (n \in N)$ be analytic in U with $p(z) \neq b_0$. If $0 < |z_0| < 1$ and $\operatorname{Re} \{ p(z_0) \} = \min_{|z| \leq |z_0|} \operatorname{Re} \{ p(z) \}$, then

$$z_0 p'(z_0) \leq -\frac{n |b_0 - p(z_0)|^2}{2 \operatorname{Re} \{ b_0 - p(z_0) \}}.$$

In this paper, we establish two theorems. These theorems provide sufficient conditions for the functions to be starlike of order β . The first theorem is an improvement of the Theorem H and is a generalization of Theorem G which in turn is an improvement of Theorems A to F. Our second theorem generalize and improve the result obtained by Frasin [1].

2. MAIN RESULTS

Theorem 2.1: If $f(z) \in A_n$ satisfies

$$\operatorname{Re} \left\{ \frac{zf'(z)}{f(z)} \left(1 + \alpha \frac{zf''(z)}{f'(z)} \right) \right\} > \left\{ \beta - \alpha(1-\beta) \left(\beta + \frac{n}{2} \right) \right\} - \left\{ \alpha \left(1 + \frac{n}{2(1-\beta)} \right) \left| \frac{zf'(z)}{f(z)} - \beta \right|^2 \right\}. \quad (2.1)$$

then $f(z) \in S_n^*(\beta)$.

Proof: Let us put

$$(1-\beta)p(z) + \beta = \frac{zf'(z)}{f(z)}, \quad (2.2)$$

then $p(z)$ is analytic in U and $p(0) = 1$.

If there exists a point $z_0 \in U \setminus \{0\}$ such that

$$\operatorname{Re}\{p(z)\} > 0 \text{ for } |z| < |z_0|, \quad p(z_0) = i\gamma$$

where γ is a real number. Then applying Lemma 1.1, we have

$$z_0 p'(z_0) \leq -\frac{n}{2}(1+\gamma^2)$$

or

$$z_0 p'(z_0) \leq -\frac{n}{2}(1+|p(z_0)|^2). \quad (2.3)$$

On the other hand

$$\begin{aligned} \operatorname{Re} \left\{ \frac{z_0 f'(z_0)}{f(z_0)} \left(1 + \alpha \frac{z_0 f''(z_0)}{f'(z_0)} \right) \right\} &= \operatorname{Re} \left\{ [(1-\beta)p(z_0) + \beta] \left[\alpha[(1-\beta)p(z_0) + \beta] + \frac{\alpha(1-\beta)z_0 p'(z_0)}{(1-\beta)p(z_0) + \beta} + 1 - \alpha \right] \right\} \\ &= \operatorname{Re} \left\{ \alpha[(1-\beta)p(z_0) + \beta]^2 + \alpha(1-\beta)z_0 p'(z_0) + (1-\alpha)(1-\beta)p(z_0) + \beta(1-\alpha) \right\} \\ &= \operatorname{Re} \left\{ \alpha(1-\beta)^2 (p(z_0))^2 + \beta^2 \alpha + 2\beta\alpha(1-\beta)p(z_0) + \alpha(1-\beta)z_0 p'(z_0) \right. \\ &\quad \left. + (1-\alpha)(1-\beta)p(z_0) + \beta(1-\alpha) \right\} \\ &= \alpha(1-\beta)^2 \operatorname{Re}\{p(z_0)\}^2 + \beta^2 \alpha + 2\beta\alpha(1-\beta)\operatorname{Re}\{p(z_0)\} + \alpha(1-\beta)\operatorname{Re}\{z_0 p'(z_0)\} \\ &\quad + (1-\alpha)(1-\beta)\operatorname{Re}\{p(z_0)\} + \beta(1-\alpha) \\ &\leq -\alpha(1-\beta)^2 |p(z_0)|^2 + \beta^2 \alpha - \alpha(1-\beta) \frac{n}{2} (1+|p(z_0)|^2) + \beta(1-\alpha) \\ &\leq \left\{ \beta - \alpha(1-\beta) \left(\beta + \frac{n}{2} \right) \right\} - \left\{ \alpha \left(1 + \frac{n}{2(1-\beta)} \right) \left| \frac{z_0 f'(z_0)}{f(z_0)} - \beta \right|^2 \right\}. \end{aligned}$$

This is a contradiction and therefore the proof of the Theorem 2.1 is complete.

Remark: For $\beta = 0$ and for $n = 1$, the Theorem 2.1 will reduce to the result given in Theorem G. For $\beta = 1/2, \alpha = 1$ and $n = 1$, we get the following result:

Corollary 2.2: If $f(z) \in A$ satisfies

$$\operatorname{Re} \left\{ \frac{zf'(z)}{f(z)} \left(1 + \frac{zf''(z)}{f'(z)} \right) \right\} > - \left[1 + \left| \frac{zf'(z)}{f(z)} - \frac{1}{2} \right|^2 \right]$$

then $f(z) \in S^*(1/2)$.

The next theorem gives sufficient conditions for a function $f(z)$ to be in the class $B_n(\mu, \alpha)$.

Theorem 2.3: If $f(z) \in A_n, \mu \geq -1, \mu \neq 0$, satisfies

$$\operatorname{Re} \left\{ \beta f'(z) \left(\frac{z}{f(z)} \right)^{\mu+1} + (1-\beta) \left[\frac{\mu-1}{\mu} z \left(\frac{z}{f(z)} \right)^\mu - \frac{z^2}{\mu} \left(\left(\frac{z}{f(z)} \right)^\mu \right) \right] \right\} > \left[\beta \delta - \frac{n}{2} (1-\beta)(1-\delta) \right] - \frac{n(1-\beta)}{2(1-\delta)} \left| f'(z) \left(\frac{z}{f(z)} \right)^{\mu+1} - \delta \right|^2 \quad (2.4)$$

then $f(z) \in B_n(\mu, \alpha) (\mu \neq 0)$.

Proof: Let us put

$$(1-\delta)p(z) + \delta = f'(z) \left(\frac{z}{f(z)} \right)^{\mu+1}, \quad (2.5)$$

then $p(z)$ is analytic in U and $p(0) = 1$.

Since

$$f'(z) \left(\frac{z}{f(z)} \right)^{\mu+1} = \left(\frac{z}{f(z)} \right)^\mu - \frac{z}{\mu} \left(\left(\frac{z}{f(z)} \right)^\mu \right). \quad (2.6)$$

It follows from (2.5) and (2.6) that

$$(1-\delta)zp'(z) = \frac{\mu-1}{\mu} z \left(\left(\frac{z}{f(z)} \right)^\mu \right) - \frac{z^2}{\mu} \left(\left(\frac{z}{f(z)} \right)^\mu \right). \quad (2.7)$$

If there exists a point $z_0 \in U \setminus \{0\}$ such that

$$\operatorname{Re}\{p(z)\} > 0 \text{ for } |z| < |z_0|, \quad p(z_0) = i\gamma$$

where γ is a real number. Then applying Lemma 1.1, we have

$$z_0 p'(z_0) \leq -\frac{n}{2} (1+\gamma^2)$$

or

$$z_0 p'(z_0) \leq -\frac{n}{2} (1+|p(z_0)|^2). \quad (2.8)$$

From (2.5), (2.7) and (2.8), we have

$$\begin{aligned} \operatorname{Re} \left\{ \beta f'(z_0) \left(\frac{z_0}{f(z_0)} \right)^{\mu+1} + (1-\beta) \left[\frac{\mu-1}{\mu} z_0^2 \left(\left(\frac{z_0}{f(z_0)} \right)^\mu \right) - \frac{z_0^2}{\mu} \left(\left(\frac{z_0}{f(z_0)} \right)^\mu \right) \right] \right\} \\ = \operatorname{Re} \{ \beta \delta + \beta(1-\delta)p(z_0) + (1-\beta)(1-\delta)z_0 p'(z_0) \} \\ = \beta \delta + \beta(1-\delta)\operatorname{Re}\{p(z_0)\} + (1-\beta)(1-\delta)\operatorname{Re}\{z_0 p'(z_0)\} \\ \leq \beta \delta - \frac{n}{2} (1-\beta)(1-\delta) [1+|p(z_0)|^2] \\ \leq \left[\beta \delta - \frac{n}{2} (1-\beta)(1-\delta) \right] - \frac{n(1-\beta)}{2(1-\delta)} \left| f'(z_0) \left(\frac{z_0}{f(z_0)} \right)^{\mu+1} - \delta \right|^2. \end{aligned}$$

This is a contradiction and therefore the proof of the Theorem 2.3 is complete.

Putting $n = 1, \mu = 1$, we get the following result:

Corollary 2.4: If $f(z) \in A$ satisfies

$$\operatorname{Re} \left\{ \beta \frac{z^2 f'(z)}{f^2(z)} - (1-\beta) z^2 \left(\frac{z}{f(z)} \right) \right\} > \frac{(\delta+1)\beta + \delta - 1}{2} - \frac{1}{2} \frac{(1-\beta)}{(1-\delta)} \left| \frac{z^2 f'(z)}{f^2(z)} - \delta \right|^2 \quad (z \in U)$$

then $f(z) \in B(\delta)$.

The above corollary is an improvement to the result in [1, Theorem 2.3].

Putting $\beta = 0$ in Theorem 2.3, we get the following result:

Corollary 2.5: If $f(z) \in A_n$ satisfies

$$\operatorname{Re} \left\{ \frac{\mu-1}{\mu} z \left(\left(\frac{z}{f(z)} \right)^\mu \right) - \frac{z^2}{\mu} \left(\left(\frac{z}{f(z)} \right)^\mu \right) \right\} > -\frac{n}{2} (1-\delta) \left[1 + \frac{1}{(1-\delta)^2} \left| f'(z) \left(\frac{z}{f(z)} \right)^{\mu+1} - \delta \right|^2 \right]$$

then $f(z) \in B_n(\mu, \delta)$.

If we put $\beta = 0, \mu = 1$ in Theorem 2.3, we get the following result:

Corollary 2.6. If $f(z) \in A_n$ satisfies

$$\operatorname{Re} \left\{ -z^2 \left(\frac{z}{f(z)} \right) \right\} > -\frac{n}{2} (1-\delta) \left[1 + \frac{1}{(1-\delta)^2} \left| \frac{z^2 f'(z)}{f^2(z)} - \delta \right|^2 \right]$$

then $f(z) \in B_n(\delta)$.

ACKNOWLEDGMENT: The first author (S.P.G.) is thankful to CSIR, New Delhi, India for awarding Emeritus Scientistship, under scheme number 21(084)/10/EMR-II.

REFERENCES:

- [1] B.A. Frasin, Some properties of certain analytic and univalent functions, *Tamsui Oxford Jour. Math. Sci.*, **23** (1) (2007), 67-77.
- [2] B.A. Frasin, A note on certain analytic and univalent functions, *Southeast Asian J. Math.*, **28** (2004), 829-838.
- [3] B.A. Frasin and J. Jahangiri, A new and comprehensive class of analytic functions, *Analele Universitatii din Oradea*, **15** (2008), 61-64.
- [4] B.A. Frasin and M. Darus, On certain analytic univalent functions, *Internat. J. Math. and Math. Sci.*, **25** (5) (2001), 305-310.
- [5] Z. Lewandowski, S.S. Miller and E. Zlotkiewicz, Generating functions for some classes of univalent functions, *Proc. Amer. Math. Soc.*, **56** (1976), 111-117.
- [6] J.-L. Li and S. Owa, Sufficient conditions for starlikeness, *Indian J. Pure Appl. Math.*, **33** (9) (2002), 313-318.
- [7] S.S. Miller and P.T. Mocanu, Second-order differential inequalities in the complex plane, *J. Math. Anal. Appl.*, **65** (2) (1978), 289-305.
- [8] M. Nunokawa, S.P. Goyal and R. Kumar, Sufficient conditions for starlikeness, Submitted.

- [9] M. Nunokawa, S. Owa, S.K. Lee, M. Obradovic, M.K. Aouf, H. Saitoh, H. Ikada and N. Koika, Sufficient conditions for starlikeness, *Chinese J. Math.*, **24** (1996), 265-270.
- [10] C. Ramesha, S. Kumar and K.S. Padmanbham, A sufficient condition for starlikeness, *Chinese J. Math.*, 23 (1995), 167-171.
- [11] V. Ravichandran, C. Selvaraj and R. Rajalaksmi, Sufficient conditions for starlike functions of order α , *J. Ineq. Pure Appl. Math.*, **3** (5) (2002), 1-6.
