**International Journal of Mathematical Archive-3(2), 2012, Page: 485-490**
**Available online through [www.ijma.info](http://www.ijma.info) ISSN 2229 – 5046**

# HYERS-ULAM-RASSIAS STABILITY
# OF GENERALIZED QUADRATIC FUNCTIONAL EQUATIONS

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***(Received on: 06-02-12; Accepted on: 27-02-12)***

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### *ABSTRACT*

*In this paper, we consider a generalized form of the quadratic functional equations and established the stability in the spirit of D. H. Hyers, S. M. Ulam and Th. M. Rassias, for the function* $f : E_1 \to E_2$, *where* $E_1$ *is a normed space and* $E_2$ *a Banach space.*

***Keywords:*** *- Hyers-Ulam-Rassias stability, Generalized quadratic functional equation.*

***MSC 2010:-*** *39B05, 39B62, 39B82, 30D05.*

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## 1. INTRODUCTION

The history of the stability theory of functional equations started with a problem concerning group homomorphisms posed by S.M. Ulam [5] in 1940:

*Let* $G_1$ *be a group and let* $G_2$ *be a metric group with metric* $d(\cdot, \cdot)$. *Given* $\varepsilon > 0$, *does there exists a* $\delta > 0$, *such that if a function* $h : G_1 \to G_2$ *satisfies the inequality* $d(h(x, y), h(x)\, h(y)) < \delta$ *for all* $x, y \in G_1$, *then there exists a homomorphism* $H : G_1 \to G_2$ *with* $d(h(x), H(x)) < \varepsilon$ *for all* $x \in G_1$?

In other words, we are looking for situations when the homomorphisms are stable. i.e. if a mapping is almost a homomorphism, then there exists a true homomorphism near it. If we turn our attention to the case of functional equations, then we can ask the question, How do the solutions of the inequality differ from those of the given functional equations?

In 1941, D. H. Hyers [2] gave the first affirmative answer to the question of S. M. Ulam [5] under the assumption that $G_1$ and $G_2$ are Banach spaces. Hyers result was generalized by T. Aoki [10] for additive mappings and by Th. M. Rassias [7] for linear mappings by considering an unbounded Cauchy difference. Th. M. Rassias [7] has provided a lot of influence in the development of what we now call Hyers – Ulam – Rassias stability of functional equations.

The quadratic function $f(x) = cx^2$ satisfies the functional equation

$$f(x + y) + f(x - y) = 2f(x) + 2f(y) \tag{1.1}$$

and therefore the equation is called the quadratic functional equation. The first stability result for the quadratic functional equation (1.1) was proved by $f$. Skof [3] for the function $f : X \to Y$ where $X$ is a normed space and $Y$ a Banach space. The result of F. Skof [3] is still true if the relevant domain $X$ is replaced by an abelian group and this was dealt by P. W. Cholewa [4]. This result was further generalized by Th.M. Rassias [8], C. Borelli and G. L Forti [1].

In this paper, we prove the Hyers – Ulam – Rassias stability of the following generalized quadratic functional equation

$$F(x + my) + f(x - my) = 2[f(x) + m^2 f(y)] \tag{1.2}$$

for the mapping $f : E_1 \to E_2$, where $E_1$ is a normed space and $E_2$ a Banach space.

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## 2. HYERS–ULAM–RASSIAS STABILITY OF (1.2)

**Theorem: 2.1** Let $E_1$ be a normed space, $E_2$ be a Banach space and $\zeta : E_1 \times E_1 \to [0,\infty)$ be a function such that

$$\lim_{n\to\infty} \frac{\zeta(m^n x,\, m^n y)}{m^{2n}} = 0 \tag{2.1}$$

for all $x, y \in E_1$. Suppose that a function $f : E_1 \to E_2$ with $f(0) = 0$, satisfies

$$\| f(x + my) + f(x - my) - 2[f(x) + m^2 f(y)]\| \le \zeta(x, y) \tag{2.2}$$

for all $x, y \in E_1$. Then there exists a unique quadratic function $T: E_1 \to E_2$ which satisfies the inequality

$$\|f(y) - T(y)\| \le \frac{1}{2m^2}\sum_{i=0}^{\infty} \frac{1}{m^{2i}}\zeta(0,\, m^i y) \tag{2.3}$$

for all $y \in E_1$. The function $T$ is given by

$$T(y) = \lim_{n\to\infty} \frac{f(m^n y)}{m^{2n}} \tag{2.4}$$

for all $y \in E_1$.

**Proof:** Let $x = 0$ in (2.2), we get

$$\|2f(my) - 2m^2 f(y)\| \le \zeta(0, y) \tag{2.5}$$

for all $y \in E_1$, so

$$\left\| \frac{f(my)}{m^2} - f(y) \right\| \le \frac{1}{2m^2}\, \zeta(0, y) \tag{2.6}$$

for all $y \in E_1$. Replacing $y$ by $my$ in (2.6) and dividing by $m^2$ and summing the resulting inequality with (2.6), we get

$$\left\| \frac{f(m^2 y)}{m^4} - f(y) \right\| \le \frac{1}{2m^4}\zeta(0, m\,y) + \frac{1}{2m^2}\zeta(0, y) \le \frac{1}{2m^2}\left[\frac{1}{m^2}\zeta(0, my) + \zeta(0, y)\right] \tag{2.7}$$

for all $y \in E_1$. Using induction on a positive integer $n$, we obtain that

$$\left\| \frac{f(m^n y)}{m^{2n}} - f(y) \right\| \le \frac{1}{2m^2}\sum_{i=0}^{n-1}\frac{1}{m^{2i}}\, \zeta(0, m^i y) \le \frac{1}{2m^2}\sum_{i=0}^{\infty}\frac{1}{m^{2i}}\zeta(0,\, m^i y) \tag{2.8}$$

for all $y \in E_1$. In order to prove the convergence of the sequence $\left\{ \frac{f(m^n y)}{m^{2n}} \right\}$, we replace $y$ by $m^k y$ and divide inequality (2.8) by $m^{2k}$ to find that for $n, k > 0$,

$$\left\| \frac{f(m^n . m^k y)}{m^{2n+2k}} - \frac{f(m^k y)}{m^{2k}} \right\| = \left\| \frac{f(m^{n+k} y)}{m^{2(n+k)}} - \frac{f(m^k y)}{m^{2k}} \right\|$$

$$\le \frac{1}{m^{2k}} \left\| \frac{f(m^{n+k} y)}{m^{2n}} - f(m^k y) \right\|$$

$$\begin{aligned}
&\le \frac{1}{2m^2}\frac{1}{m^{2k}}\sum_{i=0}^{\infty}\frac{\zeta(0, m^{i+k}y)}{m^{2i}} \\
&\le \frac{1}{2m^2}\sum_{i=0}^{\infty}\frac{\zeta(0, m^{i+k}y)}{m^{2(i+k)}} \qquad (2.9)
\end{aligned}$$

Since the right hand side of the inequality (2.9) tends to zero as $k \to \infty$, the sequence $\left\{\frac{f(m^n y)}{m^{2n}}\right\}$ is a Cauchy sequence for all $y \in E_1$. Since $E_2$ is complete, the sequence $\left\{\frac{f(m^n y)}{m^{2n}}\right\}$ converges to a fixed point $T(y) \in E_1$. So, one can define the function $T: E_1 \to E_2$ by

$$T(y) = \lim_{n\to\infty}\frac{f(m^n y)}{m^{2n}}$$

for all $y \in E_1$. To show that $T$ satisfies the equation (1.2) replacing $x$ and $y$ by $m^n x$, $m^n y$ respectively in (2.2) and dividing by $m^{2n}$, then it follows that

$$\left\|\frac{f(m^n(x+my))}{m^{2n}} + \frac{f(m^n(x-my))}{m^{2n}} - \frac{2[f(m^n x) + m^2 f(m^n y)]}{m^{2n}}\right\| \le \frac{\zeta(m^n x,\, m^n y)}{m^{2n}}$$

Taking limit as $n \to \infty$ and using (2.1), it shows that $T$ satisfies (1.2) for all $y \in E_1$.

Now, let $T' : E_1 \to E_2$ be another quadratic function satisfying (1.2) and (2.3). Then we have

$$\begin{aligned}
\|T(y) - T'(y)\| &= \frac{1}{m^{2n}}\|T(m^n y) - T'(m^n y)\| \\
&\le \frac{1}{m^{2n}}(\|T(m^n y) - f(m^n y)\| + \|T'(m^n y) - f(m^n y)\|) \\
&\le \frac{1}{m^2}\sum_{i=0}^{\infty}\frac{\zeta(0, m^{i+n}y)}{m^{2(i+n)}} \qquad (2.10)
\end{aligned}$$

which tends to zero as $n \to \infty$ for all $y \in E_1$.

So, we can conclude that $T(y) = T'(y)$ for all $y \in E_1$ which shows the uniqueness of the function $T$. This completes the proof of theorem.

**Corollary: 2.1 (Hyers Stability).** Let $E_1$ and $E_2$ be a real normed space and Banach space, respectively, and let $\varepsilon \ge 0$ be a real number. Suppose that a function $f: E_1 \to E_2$ with $f(0) = 0$ satisfies

$$\| f(x + my) + f(x - my) - 2[f(x) + m^2 f(y)]\| \le \varepsilon$$

for all $x, y \in E_1$. Then there exists a unique quadratic function $T: X \to Y$ defined by $T(y) = \lim_{n\to\infty}\frac{f(m^n y)}{m^{2n}}$ which satisfies the equation (1.2) and the inequality $\|f(y) - T(y)\| \le \frac{\varepsilon}{2(m^2 - 1)}$ for all $y \in E_1$. Further, if for each fixed $y \in E_1$ the mapping $t \to f(ty)$ from $R$ to $E_2$ is continuous, then $T(my) = m^2\, T(y)$.

**Corollary: 2.2 (Rassias Stability)** Let $E_1$ and $E_2$ be a real normed space and a Banach space, respectively, and let $\varepsilon \geq 0$, $0 < p < 2$ be real numbers. Suppose that a function $f : E_1 \to E_2$ with $f(0) = 0$ satisfies

$$\| f(x + my) + f(x - my) = 2[f(x) + m^2 f(y)]\| \leq \varepsilon(\|x\|^p + \|y\|^p)$$

for all $x, y \in E_1$. Then there exists a unique quadratic mapping $T$: $E_1 \to E_2$ which satisfies the equation (1.2) and the inequality

$$\|f(y) - T(y)\| \leq \frac{\varepsilon}{2(n^2 - m^p)} \|y\|^p$$

for all $y \in E_1$. The function $T$ is given by

$$T(y) = \lim_{n\to\infty} \frac{f(m^n y)}{m^{2n}}$$

for all $y \in E_1$. Further, if for each fixed $y \in E_1$ the mapping $t \to f(ty)$ from $R$ to $E_2$ is continuous, then $T(my) = m^2\, T(y)$ for all $m \in R$.

**Theorem: 2.2** Let $E_1$ be a normed space, $E_2$ be a Banach space and $\zeta : E_1 \times E_2 \to [0,\infty)$ be a function such that

$$\lim_{n\to\infty} m^n\, \zeta\left(\frac{x}{m^n}, \frac{y}{m^n}\right) = 0 \tag{2.11}$$

for all $x, y \in E_1$. Suppose that a function $f : E_1 \to E_2$ with $f(0) = 0$, satisfies

$$\| f(x + my) + f(x - my) - 2[f(x) + m^2 f(y)]\| \leq \zeta(x, y) \tag{2.12}$$

for all $x, y \in E_1$. Then there exists a unique quadratic function $T$: $E_1 \to E_2$ which satisfies the inequality

$$\|f(y) - T(y)\| \leq \frac{1}{2}\sum_{i=0}^{\infty} m^{2i}\, \zeta\left(0, \frac{y}{m^{i+1}}\right) \tag{2.13}$$

for all $y \in E_1$. The function $T$ is given by

$$T(y) = \lim_{n\to\infty} m^{2n} f\left(\frac{y}{m^n}\right) \tag{2.14}$$

**Proof:** Replacing $y = \dfrac{y}{m}$ and multiplying by $m^2$ in (2.6) we get

$$\left\| f(y) - m^2 f\left(\frac{y}{m}\right)\right\| \leq \frac{1}{2}\,\zeta\left(0, \frac{y}{m}\right) \tag{2.15}$$

Again replacing $y$ with $y/m$ and multiplying by $m^2$ in (2.15).

$$\left\| m^4 f\left(\frac{y}{m^2}\right) - f(y)\right\| \leq \frac{m^2}{2}\,\zeta\left(0, \frac{y}{m^2}\right) + \frac{1}{2}\zeta\left(0, \frac{y}{m}\right)$$

for all $y \in E_1$. Hence

$$\| T(y) - f(y)\| \leq \frac{1}{2}\sum_{i=0}^{\infty} m^{2i}\, \zeta\left(0, \frac{y}{m^{i+1}}\right) \tag{2.16}$$

To prove the convergence of the sequence $\left\{m^{2n} \ f\left(\frac{y}{m^n}\right)\right\}$, we replace $y$ by $\frac{y}{m^k}$ and multiplying inequality by $m^{2k}$,

we have

$$\left\| m^{2n+2k} f\left(\frac{y}{m^{n+k}}\right) - m^{2k} f\left(\frac{y}{m^k}\right) \right\| \le \frac{1}{2}\sum_{i=0}^{\infty} m^{2(i+k)} \ \zeta\left(0, \frac{y}{m^{i+k}}\right)$$

It follows from (2.16) that $\left\{m^{2n} \ f\left(\frac{y}{m^n}\right)\right\}$, is a Cauchy sequence for all $y \in E_1$. Since $E_2$ is complete the sequence $\left\{m^{2n} \ f\left(\frac{y}{m^n}\right)\right\}$ converges to a unique point $T \in E_2$. So, we can define the mapping $f : E_1 \to E_2$ by

$$T(y) = \lim_{n\to\infty} m^{2n} \ f\left(\frac{y}{m^n}\right)$$

for all $y \in E_1$. Also by Theorem 2.1. $T$: $E_1 \to E_2$ is quadratic mapping. The rest of the proof is similar to the proof of Theorem 2.1.

**Corollary: 2.3 (Hyers stability)** If a function $f$: $E_1 \to E_2$ satisfies $f(0) = 0$ and the inequality

$$\| f(x + my) + f(x - my) - 2[f(x) + m^2 f(y)]\| \le \varepsilon$$

for all $x, y \in E_1$, then there exists a unique quadratic mapping $T : E_1 \to E_2$ such that

$$\|f(y) - T(y)\| \le \frac{q}{2(1-m^2)}$$

for all $y \in E_1$. The function $T$ is given by $T(y) = \lim_{n\to\infty} m^{2n} \ f\left(\frac{y}{m^n}\right)$ for all $y \in E_1$.

**Corollary: 2.4 (Rassias stability)** If a function $f$: $E_1 \to E_2$ with $f(0) = 0$ satisfies the inequality

$$\| f(x + my) + f(x - my) - 2[f(x) + m^2 f(y)] \| \le \varepsilon(\|x\|^p + \|y\|^p)$$

for some $p > 2$ and for all $x, y \in E_1$, then there exists a unique quadratic function $T$:$E_1 \to E_2$ such that

$$\| f(y) - T(y) \| \le \frac{1}{2}\frac{\varepsilon}{(m^p - m^2)}\|y\|^p$$

for all $y \in E_1$. The function $T$ is given by $T(y) = \lim_{n\to\infty} m^{2n} f\left(\frac{y}{m^n}\right)$ for all $y \in E_1$.

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