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ON THE USE OF THE *q*-MELLIN TRANSFORM TO SOLVE SOME *q*-HEAT AND *q*-WAVE EQUATIONS

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ABSTRACT

In this paper, we show that the q-integral transforms can be used to solve some q-heat and q-wave equations.

Key words: *q-Mellin transform, Dunkl transform, q-heat and q-wave equations*

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1. INTRODUCTION:

In classical analysis, it is well known that one of the powerful technique of solving differential or partial differential equations, with initial values or boundary conditions, is the method of integral transforms, since these later transform a differential equation to an algebraic or functional one, which can be solved easily (see [5, 8] and references therein).

From the seventies, the interest on the q-deformation theory, that its origin back to the eighteenth century, have witnessed a great development, due to the role of this theory in many areas such as physics and quantum groups. For instance, partial q-difference equations and q-difference-differential equations with more than one variables are one of this interest. But they are generally studied by means of the method of separation of variables or by the techniques of Lie symmetry (see [1, 13, 14] and references therein). However, in literature few papers studied these equations by using integral transforms (see [3, 4, 15]).

In this paper, we use the q-Mellin transform and the properties of some integral transforms to study the Dunkl and q-Dunkl q-wave, and q-heat equations. We note that the majority of the so-mentioned works turn out to be particular cases of ours. For instance, if we take $\alpha = -1/2$, the Dunkl q-wave and q-heat equations reduce to those studied in [4], and the q-Dunkl q-wave and q-heat equations reduce to those studied in [3].

This paper is organized as follows: in Section 2, we present some preliminary results and notations that will be useful in the sequel. In Sections 3, we study the q-heat and q-wave equations for the Dunkl operator. Finally, Section 4 is devoted to study the q-heat and q-wave equations for the q-Dunkl operator.

2. PRELIMINARIES:

Throughout this paper, we fix $q \in]0,1[$ and we write $\mathbb{R}_q = \{\pm q^n,\ n \in \mathbb{Z}\}$ and $\mathbb{R}_{q,+} = \{q^n,\ n \in \mathbb{Z}\}$.

2.1. Basic symbols: We follow the general reference [10] for the definitions, notations and properties of the q-shifted factorials:

For complex number a, the q-shifted factorials are defined by:

$$(a;q)_0 = 1, \quad (a;q)_n = \prod_{k=1}^{n-1}\left(1 - a\,q^k\right),\ n = 1, 2, \ldots, \infty.$$

We also write

$$[x]_q = \frac{1-q^x}{1-q}, \ x \in \mathbb{C} \quad \text{and} \quad n!_q = \frac{(q;q)_n}{(1-q)^n}, \ n \in \mathbb{N}.$$

The forward Jackson's q-derivative D_q is defined by

$$\begin{cases} D_q f(z) = \dfrac{f\left(q^{-1}z\right) - f(z)}{(1-q)z}, \ z \neq 0 \\ \\ D_q f(0) = \lim\limits_{z \to 0} D_q f(z) \end{cases}$$

The Rubin's differential operator is defined by (see [15])

$$\partial_q(f)(z) = \frac{f(q^{-1}z) + f(-q^{-1}z) - f(qz) + f(-qz) - 2f(-z)}{2(1-q)z} \tag{1}$$

Note that if f is differentiable at z, then $\lim\limits_{q \to 1} D_q f(z) = \lim\limits_{q \to 1} \partial_q f(z) = f'(z).$

The q-Jackson's integrals from 0 to $+\infty$ and from $-\infty$ to $+\infty$ are defined by (see [10, 11])

$$\int_0^\infty f(x) d_q x = (1-q) \sum_{n=-\infty}^{\infty} f(q^n) q^n$$

and

$$\int_{-\infty}^\infty f(x) d_q x = (1-q) \sum_{n=-\infty}^{\infty} f(q^n) q^n + (1-q) \sum_{n=-\infty}^{\infty} f(-q^n) q^n ,$$

provided the sums converge absolutely.

For $p > 0$, the notation $L^p_{\alpha,q} = L^p_{\alpha,q}\left(\mathbb{R}_q\right)$ will stand for the Banach space induced by the norm

$$\| f \|_{p,\alpha,q} = \left(\int_{-\infty}^{\infty} |f(x)|^p |x|^{2\alpha+1} d_q x \right)^{\frac{1}{p}}$$

and $L^\infty_q = L^\infty_q\left(\mathbb{R}_q\right)$ will stand for the Banach space induced by the norm

$$\| f \|_{\infty,q} = \sup_{x \in \mathbb{R}_q} |f(x)|.$$

A q-analogue of the exponential function is given by (see [10])

$$E_q^z = \sum_{n=0}^{\infty} \frac{q^{\frac{n(n-1)}{2}} z^n}{n!_q}.$$

This exponential function induces two q-trigonometric functions given by:

$$Cos_q(x) = \frac{E_q^{ix} + E_q^{-ix}}{2} \quad \text{and} \quad Sin_q(x) = \frac{E_q^{ix} - E_q^{-ix}}{2}. \tag{2}$$

It is easy to verify that

$$D_q E_q^x = \frac{1}{q} E_q^x, \quad D_q Cos_q(x) = -\frac{1}{q} Sin_q(x) \quad \text{and} \quad D_q Sin_q(x) = \frac{1}{q} Cos_q(x) \tag{3}$$

Jackson defined a q-analogue of the Gamma function by (see [11])

$$\Gamma_q(x) = \frac{(q;q)_\infty}{(q^x;q)_\infty}(1-q)^{1-x}, \quad x \neq 0, -1, -2, ...$$

It satisfies the following properties

$$\Gamma_q(x+1) = [x]_q \Gamma_q(x), \ \Gamma_q(1) = 1 \quad \text{an d} \ \lim_{q \to 1} \Gamma_q(x) = \Gamma(x)$$

and it has the q-integral representation

$$\Gamma_q(s) = \int_0^{\frac{1}{1-q}} t^{s-1} E_q^{-qt} d_q t.$$

In the particular case $\frac{Log(1-q)}{Log(q)} \in \mathbb{Z}$, we get

$$\Gamma_q(s) = \int_0^\infty t^{s-1} E_q^{-qt} d_q t.$$

In the remainder, we assume that this condition holds.

2.2. The q-Mellin transform: The q-Mellin transform of a suitable function f on $\mathbb{R}_{q,+}$ is given by (see [9])

$$M_q(f)(s) = \int_0^\infty t^{s-1} f(t) d_q t.$$

when the q-integral converges. It is analytic on a strip $\langle \alpha_{q,f}; \beta_{q,f} \rangle$, called the fundamental strip.

The inversion formula for the q-Mellin transform is given by (see [9])

$$\forall x \in \mathbb{R}_q, \quad f(x) = \frac{Log(q)}{2i\pi(1-q)} \int_{c-\frac{i\pi}{Log(q)}}^{c+\frac{i\pi}{Log(q)}} M_q(f)(s) x^{-s} ds,$$

where $c \in \langle \alpha_{q,f}; \beta_{q,f} \rangle$.

The q-Mellin transform satisfies also the two following properties (see [9]):

(1) For $a \in \mathbb{R}_q$ and $s \in \langle \alpha_{q,f}; \beta_{q,f} \rangle$, we have

$$M_q[f(a\ )](s) = a^{-s} M_q[f](s).$$

(2) For $s \in \langle \alpha_{q,f}+1; \beta_{q,f}+1 \rangle$,

we have

$$M_q[D_q f](s) = -[s-1] M_q[f](s-1).$$

By induction, we have for all positive integer n and all $s\in\left\langle \alpha_{q,f}+n;\ \beta_{q,f}+n\right\rangle$,

$$M_q\left[D_q^n f\right](s)=(-1)^n\left[s-1\right]_q\left[s-2\right]_q\ldots\left[s-n\right]_q M_q\left[f\right](s-n). \tag{4}$$

3. q-HEAT AND Q-WAVE EQUATIONS FOR THE DUNKL OPERATOR:

3.1. The Dunkl operator and the Dunkl transform: The Dunkl operator on $\mathbb{R}$ of index $\left(\alpha+\frac{1}{2}\right)$ associated with the reflection group $\mathbb{Z}_2$ is the differential-difference operator introduced by C. F. Dunkl in [6] by

$$\Lambda_\alpha f(x)=\frac{df}{dx}(x)+\left(\alpha+\frac{1}{2}\right)\frac{f(x)-f(-x)}{2},\quad \alpha\geq-\frac{1}{2}. \tag{5}$$

It was shown (see [12]) that for each $\lambda\in\mathbb{C}$, the differential-reflection problem

$$\begin{cases}\Lambda_\alpha f=i\lambda f\\ \\ f(0)=1,\end{cases}$$

has a unique analytic solution ψ_λ^α called Dunkl kernel and given by

$$\psi_\lambda^\alpha(x)=j_\alpha(\lambda x)+\frac{i\lambda x}{2\alpha+2}j_{\alpha+1}(\lambda x),\quad x\in\mathbb{R},$$

where j_α is the normalized spherical Bessel function of order α defined by

$$j_\alpha(z)=\Gamma(\alpha+1)\sum_{n=0}^{\infty}\frac{(-1)^n\left(z/2\right)^{2n}}{n!\Gamma(\alpha+n+1)},\quad z\in\mathbb{C}.$$

Note that $\psi_\lambda^{-\frac{1}{2}}(x)=e^{i\lambda x}$ and for all $\alpha\geq-\frac{1}{2}$ and all $\lambda,x\in\mathbb{R}$

$$\left|\psi_\lambda^\alpha(x)\right|\leq 1 \tag{6}$$

In the remainder, we fix $\alpha\geq-\frac{1}{2}$ and we note μ_α for the weighted Lebesgue measure on $\mathbb{R}$, given by

$$d\mu_\alpha(x)=\frac{|x|^{2\alpha+1}}{2^{\alpha+1}\Gamma(\alpha+1)}dx$$

For $p\in\left[0,+\infty\right]$, the notation $L_\alpha^p=L_\alpha^p\left(\mathbb{R},d\mu_\alpha\right)$ will stand for the Banach space induced by the norm

$$\|f\|_{p,\alpha}=\begin{cases}\left(\int_{-\infty}^{\infty}|f(x)|^p\,d\mu_\alpha(x)\right)^{\frac{1}{p}} & \text{if}\quad p>0\\ \\ \operatorname*{ess\,sup}_{x\in\mathbb{R}}|f(x)| & \text{if}\quad p=\infty.\end{cases}$$

Note that $L^p_{-\frac{1}{2}}$ is the Lebesgue space $L^p(\mathbb{R})$.

The Dunkl transform F_D^α is defined on L^1_α by (see [7])

$$F_D^\alpha(f)(\lambda) = \int_{-\infty}^{\infty} f(x)\psi_{-\lambda}^\alpha(x)d\mu_\alpha(x)$$

According to [12], the Dunkl transform F_D^α satisfies the following properties:

- For all $f \in L^1_\alpha$ such that $F_D^\alpha(f) \in L^1_\alpha$, we have the inversion formula

$$f(x) = \int_{-\infty}^{\infty} F_D^\alpha(f)(\lambda)\psi_\lambda^\alpha(x)d\mu_\alpha(\lambda), \quad \text{ae} \quad x \in \mathbb{R}. \tag{7}$$

- For all $f \in S(\mathbb{R})$ and $\lambda \in \mathbb{R}$
  $F_D^\alpha(\Lambda_\alpha f) = i\lambda F_D^\alpha(f)$ an d $\Lambda_\alpha F_D^\alpha(f) = -i\lambda F_D^\alpha(f),$

where $S(\mathbb{R})$ is the Schwartz space.

- **Plancherel theorem**: F_D^α is an isomorphism from L^2_α (resp. $S(\mathbb{R})$) onto itself and for $f \in L^2_\alpha$, we have

$\left\|F_D^\alpha(f)\right\|_{2,\alpha} = \|f\|_{2,\alpha}$.

3.2. q-Heat equation for the Dunkl operator: Let us consider the following q-heat equation for the Dunkl operator

$$D_{q,t}u(x,t) = \Lambda_{\alpha,x}^2 u(x,t), \qquad x \in \mathbb{R}, \quad t \in \mathbb{R}_{q,+}, \tag{8}$$

with the initial condition: $u(x,0) = f(x).$

We assume that $f \in L^1_\alpha$ and $F_D^\alpha(f) \in L^1_\alpha$. By taking a Dunkl transform in x and a q-Mellin transform in t, the equation (7) becomes

$$\left[s-1\right]_q U(\xi, s-1) = \xi^2 U(\xi, s). \tag{9}$$

A solution of the equation (9) is given by

$$U(\xi,s) = A(\xi)\xi^{-2s}\Gamma_q(s), \tag{10}$$

where $A(\xi)$ is a function only in the variable ξ .

According to the q-integral representation of the q-Gamma function, the inversion q-Mellin transform of $\xi^{-2s}\Gamma_q(s)$ is

$$\frac{Log(q)}{2i\pi(1-q)}\int_{c-\frac{i\pi}{Log(q)}}^{c+\frac{i\pi}{Log(q)}} \xi^{-2s}\Gamma_q(s)x^{-s}ds = \frac{Log(q)}{2i\pi(1-q)}\int_{c-\frac{i\pi}{Log(q)}}^{c+\frac{i\pi}{Log(q)}} \Gamma_q(s)\left(\xi^2 x\right)^{-s} ds = E_q^{-q\xi^2 t}.$$

Moreover, by the inversion formula (7), it follows that

$$u(x,t) = \int_{-\infty}^{\infty} A(\xi)E_q^{-q\xi^2 t}\psi_\xi^\alpha(x)d\mu_\alpha(\xi). \tag{11}$$

In particular, for $t = 0$, we get

$$f(x) = u(x,0) = \int_{-\infty}^{\infty} A(\xi)\psi_{\xi}^{\alpha}(x)d\mu_{\alpha}(\xi). \tag{12}$$

Then, since f and $F_D^{\alpha}(f)$ are in L_{α}^1, we have

$$A(\xi) = \int_{-\infty}^{\infty} f(x)\psi_{-\xi}^{\alpha}(x)d\mu_{\alpha}(x) = F_D^{\alpha}(f)(\xi).$$

Therefore, a solution of (8) is

$$u(x,t) = \int_{-\infty}^{\infty} F_D^{\alpha}\left(f\right)(\xi)E_q^{-q\xi^2 t}\psi_{\xi}^{\alpha}(x)d\mu_{\alpha}(\xi). \tag{13}$$

Conversely, from the relations (3) and (6), and the Lebesgue convergence theorem, it is easy to conclude that (13) satisfies (8).

3.3. q-Wave Equation for the Dunkl operator: We consider the following q-wave equation for the Dunkl operator:

$$D_{q,t}^2 u(x,t) = \Lambda_{\alpha,x}^2 u(x,t), \qquad x \in \mathbb{R}, \quad t \in \mathbb{R}_{q,+}, \tag{14}$$

with the initial conditions: $u(x,0) = f(x)$, $D_{q,t}u(x,0) = g(x)$.

We assume that $f \in S(\mathbb{R})$.

By applying the Dunkl transform in x and the q-Mellin transform in t, the equation (14) turn out to be

$$\left[s-1\right]_q\left[s-2\right]_q U(\xi, s-2) = -\xi^2 U(\xi,s). \tag{15}$$

A simple calculation shows that a solution of the equation (15) is given by

$$U(\xi,s) = \left[A(\xi)\left(-i\xi\right)^{-s} + B(\xi)\left(i\xi\right)^{-s}\right]\Gamma_q(s), \tag{16}$$

where $A(\xi)$ and $B(\xi)$ are functions only in the variable ξ.

From the q-Mellin inversion formula, we get

$$F_D^{\alpha}(u)(\xi,t) = A(\xi)E_q^{iq\xi t} + B(\xi)E_q^{-iq\xi t}, \tag{17}$$

where $F_D^{\alpha}(u)(\xi,t)$ is the Dunkl transform of $u(x,t)$ with respect to the variable x.

It follows from the relations (2) that

$$F_D^{\alpha}(u)(\xi,t) = C(\xi)Cos_q(q\xi t) + D(\xi)Sin_q(q\xi t), \tag{18}$$

where $C(\xi)$ and $D(\xi)$ are functions in ξ.

Now, the inverse-Dunkl transform (7) gives

$$u(x,t) = \int_{-\infty}^{\infty}\left[C(\xi)Cos_q\left(q\xi t\right) + D(\xi)Sin_q\left(q\xi t\right)\right]\psi_{\xi}^{\alpha}(x)d\mu_{\alpha}(\xi). \tag{19}$$

By taking $t=0$ in (19) and using the fact that $f \in S(\mathbb{R})$, we obtain

$$C(\xi) = F_D^{\alpha}(f)(\xi).$$

On the other hand, by using the relation (3), we get

$$g(x) = D_{q,t}u(x,0) = \int_{-\infty}^{\infty} \xi D(\xi)\psi_{\xi}^{\alpha}(x)d\mu_{\alpha}(\xi)$$

So, since $g \in S(\mathbb{R})$, we obtain $\xi D(\xi) \in S(\mathbb{R})$ and

$$F_D^{\alpha}(g)(\xi) = \xi D(\xi). \tag{20}$$

Therefore the final solution of (14) is

$$u(x,t) = \int_{-\infty}^{\infty}\left[F_D^{\alpha}(f)(\xi)Cos_q\left(q\xi t\right) + \frac{F_D^{\alpha}(f)(\xi)}{\xi}Sin_q\left(q\xi t\right)\right]\psi_{\xi}^{\alpha}(x)d\mu_{\alpha}(\xi). \tag{21}$$

4. q-Heat and q-wave equations for the q-Dunkl operator:

4.1. The q-Dunkl operator and the q-Dunkl transform: For $\alpha \geq -\frac{1}{2}$, the q-Dunkl operator is defined in [2] by

$$\Lambda_{\alpha,q}(f) = \partial_q\left[f_e + q^{2\alpha+1}f_o\right](x) + \left[2\alpha+1\right]_q \frac{f(x)-f(-x)}{2x},$$

with f_e and f_O are respectively the even and the odd parts of f.

It was shown in [2] that for each $\lambda \in \mathbb{C}$, the function

$$\psi_{\lambda}^{\alpha,q}(x) = j_{\alpha}(\lambda x;q^2) + \frac{i\lambda x}{\left[2\alpha+2\right]_q} j_{\alpha+1}(\lambda x;q^2)$$

is the unique solution of the q-differential-difference equation:

$$\begin{cases} \Lambda_{\alpha,q} f = i\lambda f \\ \\ f(0) = 1, \end{cases}$$

where $j_{\alpha}(.;q^2)$ is the normalized third Jackson's q-Bessel function given by

$$j_{\alpha}(z;q^2) = \sum_{n=0}^{\infty}(-1)^n \frac{q^{n(n+1)}}{(q^2;q^2)_n(q^{2(\alpha+1)};q^2)_n}\left[\left(1-q\right)z\right]^{2n}.$$

For $\alpha \geq -\frac{1}{2}$, the q-Dunkl transform $F_D^{\alpha,q}$ is defined on $L_{\alpha,q}^1$ (see [2]) by

$$F_D^{\alpha,q}(f)(\lambda) = K_{\alpha}\int_{-\infty}^{\infty} f(x)\psi_{-\lambda}^{\alpha,q}(x)\left|x\right|^{2\alpha+1}d_qx,$$

where $K_{\alpha} = \dfrac{(1+q)^{-\alpha}}{2\Gamma_{q^2}(\alpha+1)}.$

It satisfies the following properties:

- For all $f \in L_{\alpha,q}^1$ such that $xf(x) \in L_{\alpha,q}^1$, we have

  $F_D^{\alpha,q}(\Lambda_{\alpha,q}f) = i\lambda F_D^{\alpha,q}(f)$ and $\Lambda_{\alpha,q}F_D^{\alpha,q}(f) = -iF_D^{\alpha,q}(xf).$

- Plancheral theorem : $F_D^{\alpha,q}$ is an isomorphism from $L^2_{\alpha,q}$ (resp. $S_q(\mathbb{R}_q)$)into itself and for $f \in L^1_{\alpha,q}$, we have

$$\left\|F_D^{\alpha,q}(f)\right\|_{2,\alpha,q} = \|f\|_{2,\alpha,q} \text{ and } \left(F_D^{\alpha,q}\right)^{-1}(f)(x) = K_\alpha \int_{-\infty}^{\infty} f(\lambda)\psi_x^{\alpha,q}(\lambda)|\lambda|^{2\alpha+1}\, d_q\lambda.$$

Here, $S_q(\mathbb{R}_q)$ is is the q-analogue of the Shwartz space, constituted of functions f defined on $\mathbb{R}_q$ satisfying:

$$\forall m,n \in \mathbb{N}, \quad P_{n,m,q}(f) = \sup_{x\in\mathbb{R}_q}\left|x^m \partial_q^n f(x)\right| < \infty \quad \text{and} \quad \lim_{\substack{x\to 0\\ x\in\mathbb{R}_q}} \partial_q^n f(x) \quad \text{exists.}$$

4.2. q-Heat equation for the q-Dunkl operator: Consider the following q-heat equation for the q-Dunkl operator $\Lambda_{\alpha,q}$:

$$D_{q,t}u(x,t) = \Lambda^2_{\alpha,q,x}u(x,t), \qquad x \in \mathbb{R}, \quad t \in \mathbb{R}_{q,+}, \tag{22}$$

with the initial condition: $u(x,0) = f(x), \qquad f \in L^2_{\alpha,q}.$

By applying the q-Dunkl transform respecting to x and the q-Mellin transform respecting to t, equation (22) can be transformed to

$$[s-1]_q U(\xi, s-1) = \xi^2 U(\xi, s). \tag{23}$$

It is easy to verify that a solution of (23) is given by

$$U(\xi,s) = A(\xi)\xi^{-2s}\Gamma_q(s), \tag{24}$$

where $A(\xi)$ is a function only in the variable ξ.

As we proved, the inversion q-Mellin transform of $\xi^{-2s}\Gamma_q(s)$ is $E_q^{-q\xi^2 t}$. Then, by applying the inversion theorems for the q-Mellin and the q-Dunkl transforms, we obtain

$$u(x,t) = K_\alpha \int_{-\infty}^{\infty} A(\xi)E_q^{-q\xi^2 t}\psi_\xi^{\alpha,q}(x)|\xi|^{2\alpha+1}\, d_q\xi. \tag{25}$$

For $t = 0$, we obtain

$$f(x) = u(x,0) = K_\alpha \int_{-\infty}^{\infty} A(\xi)\psi_\xi^{\alpha,q}(x)|\xi|^{2\alpha+1} d_q\xi. \tag{26}$$

Then, since $f \in L^2_\alpha$, we get by the Plancheral theorem that $A \in L^2_{\alpha,q}$ and

$$A(\xi) = \int_{-\infty}^{\infty} f(x)\psi_{-\xi}^{\alpha,q}(x)|x|^{2\alpha+1}\, d_q x = F_D^{\alpha,q}(f)(\xi).$$

Hence, the final solution of (22) is

$$u(x,t) = \int_{-\infty}^{\infty} F_D^{\alpha,q}(f)(\xi)E_q^{-q\xi^2 t}\psi_\xi^{\alpha,q}(x)|\xi|^{2\alpha+1}\, d_q\xi. \tag{27}$$

Conversely, from the relations (3) and the fact that $\Lambda_{\alpha,q}\psi_\lambda^{\alpha,q}(x) = i\lambda\,\psi_\lambda^{\alpha,q}(x)$, we conclude that (27) satisfies (22).

4.3. q-Wave Equation for the q-Dunkl operator: We consider the following q-wave equation for the q-Dunkl operator

$$D^2_{q,t} u(x,t) = \Lambda^2_{\alpha,q,x} u(x,t), \qquad x \in \mathbb{R}, \quad t \in \mathbb{R}_{q,+}, \tag{28}$$

with the initial conditions: $u(x,0) = f(x)$, $D_{q,t}u(x,0) = g(x)$.

We assume that $f, g \in S_q(\mathbb{R}_q)$.

By putting applying $U = M_{q,t}\left[F^{\alpha,q}_{D,x}(u)\right]$, the equation (28) can be transformed to the following functional equation

$$[s-1]_q [s-2]_q U(\xi, s-2) = -\xi^2 U(\xi, s). \tag{29}$$

A simple calculation shows that a solution of the equation (29) is given by

$$U(\xi,s) = \left[A(\xi)(-i\xi)^{-s} + B(\xi)(i\xi)^{-s}\right]\Gamma_q(s), \tag{30}$$

where $A(\xi)$ and $B(\xi)$ are functions only in the variable ξ.

Then, from the q-Mellin inversion formula and the relations (2), we get

$$F^{\alpha,q}_D(u)(\xi,t) = C(\xi)Cos_q(q\xi t) + D(\xi)Sin_q(q\xi t), \tag{31}$$

where $C(\xi)$ and $D(\xi)$ are functions in ξ.

So, by the q-Dunkl inversion formula we obtain

$$u(x,t) = K_\alpha \int_{-\infty}^{\infty} \left[C(\xi)Cos_q(q\xi t) + D(\xi)Sin_q(q\xi t)\right]\psi^{\alpha,q}_\xi(x)|\xi|^{2\alpha+1} d_q\xi. \tag{32}$$

By taking $t = 0$ in (31) and using the Plancheral theorem, since $f \in S_q(\mathbb{R}_q)$, we obtain $C \in S_q(\mathbb{R}_q)$ and $C(\xi) = F^{\alpha,q}_D(f)(\xi)$.

Moreover, using the relation (3), we get

$$g(x) = D_{q,t}u(x,0) = K_\alpha \int_{-\infty}^{\infty} \xi D(\xi)\psi^{\alpha,q}_\xi(x)|\xi|^{2\alpha+1} d_q\xi.$$

So, since $g \in S_q(\mathbb{R}_q)$, we obtain $\xi D(\xi) \in S_q(\mathbb{R}_q)$ and

$$F^{\alpha,q}_D(g)(\xi) = \xi D(\xi). \tag{33}$$

Therefore the final solution of (28) is

$$u(x,t) = K_\alpha \int_{-\infty}^{\infty} \left[F^{\alpha,q}_D(f)(\xi)Cos_q(q\xi t) + \frac{F^{\alpha,q}_D(f)(\xi)}{\xi} Sin_q(q\xi t)\right]\psi^{\alpha,q}_\xi(x)|\xi|^{2\alpha+1} d_q\xi. \tag{34}$$

Finally, as in the previous sections, it is easy to verify that this function is a solution of the equation (28).

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