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FIXED POINTS IN FUZZY METRIC SPACES

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ABSTRACT

In this paper, we prove a common fixed point theorem for two pairs of weakly compatible mappings satisfying expansion type condition in fuzzy metric space.

Key words: *Fuzzy metric space, common fixed point, weakly compatible mappings.*

Mathematical subject classification (2000): *54H25, 47H10.*

INTRODUCTION:

The concept of fuzzy sets was introduced by Zadeh [20] in 1965. Since then, to use this concept in topology and analysis, many authors have extensively developed the theory of fuzzy sets and its applications. Especially, Deng [4], Erceg [5], Kaleva and Seikkala [11], Kramosil and Michalek [12] have introduced the concept of fuzzy metric space in different ways.

In 1988, Grabiec [9] extended the fixed point theorem of Banach [2] to fuzzy metric space. Moreover, it appears that the study of Kramosil and Michalek [12] of fuzzy metric space paved the way for developing this theory to the field of fixed point theorems, in particular, for the study of contractive type maps. There have been several attempts to formulate fixed point theorems in fuzzy mathematics (see for instance; [1], [3], [6], [7], [10], [15], [16], [18], [19]).

George and Veeramani [8] have modified the concept of fuzzy metric space introduced by Kramosil and Michalek [12] and have defined the Hausdorff topology on fuzzy metric spaces. They have also shown that every metric induces a fuzzy metric.

Recently, Kumar and Pant [14] have given a common fixed point theorem for two pairs of compatible mappings satisfying expansion type condition in probabilistic Menger space. In that result one of the mappings has been taken continuous. In this paper we extend the result of Kumar and Pant [14] to fuzzy metric spaces. We improve that result by dropping the condition of continuity of the mapping and using weak compatibility of the mappings in place of compatibility

PRELIMINARIES:

Definition 1 [17]: A binary operation $*: [0,1] * [0,1] \to [0,1]$ is called a continuous t-norm if $([0,1],*)$ is an abelian topological monoid with unit 1 such that $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$ for all $a, b, c, d, \in [0,1]$. Examples of t-norm are $a * b = ab$ and $a * b = \min\{a, b\}$.

Definition 2 [12]: The 3-tuple $(X, M, *)$ is called a fuzzy metric space (shortly, FM-space) if X is an arbitrary set, $*$ is a continuous t-norm and M is a fuzzy set in $X^2 \times [0, \infty)$ satisfying following conditions: for all $x, y, z \in X$ and $s, t > 0$

(FM-1) $M(x, y, 0) = 0$,
(FM-2) $M(x, y, t) = 1$, for all $t > 0$ if and only if $x = y$,
(FM-3) $M(x, y, t) = M(y, x, t)$,
(FM-4) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$,
(FM-5) $M(x, y, .) : [0, \infty) \to [0,1]$ is left continuous.

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Definition 3 [9]: Let $(X, M,*)$ be a fuzzy metric space:

(1) A sequence $\{x_n\}$ in X is said to be convergent to a point $x \in$ X, (denoted by $\lim_{n\to\infty} x_n = x$), if $\lim_{n\to\infty} M(x_n, x, t) = 1$ for all $t > 0$.

(2) A sequence $\{x_n\}$ in X is called a Cauchy sequence if $\lim_{n\to\infty} M(x_{n+p}, x_n, t) = 1$ for all $t > 0$ and $p > 0$.

(3) A fuzzy metric space in which every Cauchy sequence is convergent is said to be complete.

Let (X, M,$*$) be fuzzy metric space with following condition:
(FM-6) $\lim_{t\to\infty} M(x, y, t) = 1$ for all $x, y \in X$.

In [9] Grabiec has given two important lemmas for contraction condition. We have the following lemmas for expansion type condition.

Lemma 1: Let $\{x_n\}$ be a sequence in a fuzzy metric space (X, M, $*$) with (FM-6). If there exists a number h > 1 such that

$M(x_{n+1}, x_n, ht) \le M(x_{n+2}, x_{n+1}, t)$ for all $t > 0$ and $n = 1, 2, 3, \ldots$ then $\{x_n\}$ is Cauchy sequence in X.

Lemma 2: If, for all $x, y \in$ X, $t > 0$ and for a number $h > 1$

$M(x, y, ht) \le M(x, y, t)$, then $x = y$.

Definition 4 [18]: A function M is continuous in fuzzy metric space iff whenever $x_n \to x,\ y_n \to y,$ then

$\lim_{n\to\infty} M(x_n, y_n, t) = M(x, y, t)$ for all $t > 0$.

Definition 5 [13]: Let A and B be mappings from a fuzzy metric space (X, M,$*$) into itself. The mappings A and B are said to be weakly compatible if they commute at coincidence points i.e. Ax = Bx ⇒ ABx = BAx.

RESULT:

Theorem: Let $(X, M,*)$ be a complete fuzzy metric space where $*$ is continuous and satisfies $x * x \ge x$ for all $x \in [0,1]$. Let A, B, S and T be self mapping of $X : X \to X$ satisfying the following conditions.

i. A and B are surjective.
ii. A, S and B, T are weakly compatible.
iii. $\mathrm{M(Au, Bv, hx)} \le \mathrm{M(Su, Tv, x)}$ for all $\mathrm{u, v} \in \mathrm{X}$ and $\mathrm{h} > 1$.

Then A, B, S and T have a unique common fixed point in X.

Proof: Let $u_o \in X$.

Since A and B are surjective, we choose a point $\mathrm{u_1} \in \mathrm{X}$ such that $A\mathrm{u_1} = T\mathrm{u_0} = \mathrm{v_0}$, and for this point $\mathrm{u_1}$, there exist a point $\mathrm{u_2}$ in X such that

$\mathrm{Bu_2} = \mathrm{Su_1} = \mathrm{v_1}$.

Continuing in this manner, we obtain a sequence $\{v_n\}$ in X as follows

$Au_{2n+1} = Tu_{2n} = v_{2n}$ and $Bu_{2n+2} = Su_{2n+1} = v_{2n+1}$

Using (iii) we have

$$\begin{aligned} M(v_{2n}, v_{2n+1}, hx) &= M(Au_{2n+1}, Bu_{2n+2}, hx) \\ &\le M(Su_{2n+1}, Tu_{2n+2}, x) \\ &= M(v_{2n+1}, v_{2n+2}, x) \end{aligned}$$

Therefore by Lemma 1, $\{v_n\}$ is a Cauchy sequence.

Since X is complete, $\{v_n\}$ converges to some point $z \in X$. Consequently, the subsequences $\{Au_{2n+1}\}, \{Bu_{2n}\}, \{Su_{2n+1}\}$ and $\{Tu_{2n}\}$ also converge to z.

Let there exist $q \in X$, such that $Aq = z$, then by (iii)

$M(Aq, Bu_{2n}, hx) \leq M(Sq, Tu_{2n}, x)$.

As $n \to \infty$
$M(Aq, z, hx) \leq M(Sq, z, x)$

$M(z, z, hx) \leq M(Sq, z, x)$

$\Longrightarrow Sq = z$

$\Longrightarrow Aq = Sq = z$

$\therefore$ q is coincidence point of A and S.

Since A and S are weakly compatible and $Aq = Sq = z$

$\therefore$ $ASq = SAq$

$\Longrightarrow Az = Sz$

Again by (iii)

$M(Az, Bu_{2n}, hx) \leq M(Sz, Tu_{2n}, x)$As $n \to \infty$

$M(Az, z, hx) \leq M(Sz, z, x)$, which gives

$M(Az, z, hx) \leq M(Az, z, x)$ implying $Az = z$.

$\therefore Az = Sz = z$

Now let $z = Bp$ for some $p \in X$, then by (iii)

$M(Au_{2n+1}, Bp, hx) \leq M(Su_{2n+1}, Tp, x)$.

Letting $n \to \infty$

$M(z, Bp, hx) \leq M(z, Tp, x)$.

But $Bp = z$, therefore

$M(z, z, hx) \leq M(z, Tp, x)$, giving $z = Tp$

$\therefore Bp = Tp = z$

$\Longrightarrow p$ is coincident point of B and T.

Since B and T are weakly compatible and $Bp = Tp = z$

$\therefore TBp = BTp \Longrightarrow Tz = Bz$.

Now by (iii)

$M(Au_{2n+1}, Bz, hx) \leq M(Su_{2n+1}, Tz, x)$.

Letting $n \to \infty$ this gives

$M(z, Bz, hx) \leq M(z, Tz, x)$

But Bz = Tz, therefore

$M(z, Tz, hx) \leq M(z, Tz, x) \Longrightarrow Tz = z$

Hence $Bz = z$.

$\therefore Az = Bz = Tz = Sz = z$.

Hence z is a common fixed point of A, B, S and T.

Finally for the uniqueness of fixed point, let $w \in X$ be another fixed point.

Thus from (iii),

$M(Aw, Bz, hx) \leq M(Sw, Tz, x)$

$\Rightarrow M(w, z, hx) \leq M(w, z, x)$

$\Rightarrow w = z$.

This shows the uniqueness of the fixed point and hence the proof is complete.

In [15] Kumar and Pant have proved the result for compatible mappings in probabilistic Menger space, by taking one of A, B, S and T to be continuous. But in the present result, we have taken weakly compatible mappings in fuzzy metric space and have omitted the condition of continuity of one of mappings A, B, S and T. Since weak compatibility is weaker condition therefore our result is the improvement of the result of Kumar and Pant [14] and an extension to fuzzy metric space.

REFERENCES:

[1] Badard R., Fixed point theorems for fuzzy numbers, Fuzzy Sets and Systems 13(1984), 291-302.

[2] Banach S., Theorie des Operations Lineaires, Monografie Mathematyczne, Warszawa, Poland 1932.

[3] Butnariu D., Fixed point for fuzzy mappings, Fuzzy Sets and Systems 7(1982), 191-207.

[4] Deng Z. K., Fuzzy pseudo-metric space, J. Math. Anal. Appl. 86(1982), 74-95.

[5] Erceg M. A., Metric spaces in fuzzy set theory, J. Math. Anal. Appl. 69(1979), 205-230.

[6] Fang J. X., On fixed point theorems in fuzzy metric spaces, Fuzzy Sets and Systems 46(1992), 107-113.

[7] Fang J. X., A note on fixed point theorems of Hadzic, Fuzzy Sets Systems 48(1992), 391-395.

[8] George A. and Veeramani P., On some results of analysis for fuzzy metric spaces, Fuzzy Sets and Systems 90(1997), 365-368.

[9] Grabiec M., Fixed points in fuzzy metric space, Fuzzy Sets and Systems 27(1988), 385-389.

[10] Gregori V. and Sapena A., On fixed point theorem in fuzzy metric spaces, Fuzzy Sets Systems 125(2002), 245-252.

[11] Kaleva O., Seikkala S., On fuzzy metric spaces, Fuzzy Sets and Systems 12(1984), 215-229.

[12] Kramosil I., Michalek J., Fuzzy metric and Statistical metric spaces, Kybernetica 11(1975), 326-334.

[13] Kumar S., Common fixed point theorem for minimal commutativity type mappings in fuzzy metric spaces, Thai Journal of Mathematics, 6(2008), 239-270

[14] Kumar S. and Pant B. D., A common fixed point theorem for expansion mappings in probabilistic metric spaces, Ganita, 57(2006), 89-95.

[15] Pap E., Hadzic O. and Mesiar R., A fixed point theorem in probabilistic metric spaces and an application, J. Math. Anal. Appl. 202(1996), 433-449.

[16] Razani A. and Shirdaryadzi M., Some results on fixed points in fuzzy metric space, J. Appl. Math. Comp. 20 (2006), 401-408.

[17] Schweizer B., Sklar A., Statistical metric spaces, Pacific Journal Math 10(1960), 313-334.

[18] Sharma S., On fuzzy metric space, Southeast Asian Bull. of Math. 26(2002), 133-145.

[19] Sharma S., Common fixed point theorems in fuzzy metric spaces, Fuzzy Sets and Systems 127(2002), 345-352.

[20] Zadeh L.A., Fuzzy sets, Inform. And Control 8(1965), 338-353.

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