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# EXPONENTIAL METHODS FOR CONVEX PROGRAMMING UNDER LINEAR EQUATION CONSTRAINTS

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## ABSTRACT

*The barrier and penalty functions methods are one of the Lagrangian multiplier methods for solving both constrained nonlinear equation and optimization problems. In this paper we introduce a new method called exponential method to solve optimization problem on convex function under linear constraints. It presents some variety of the method. The global behavior of this method will be given. Theorems and algorithms for the method are also given in this paper. The main result of this method as follows; that is, every penalty (barrier) parameters value that converging to infinity, the exponential penalty (barrier) function has unique minimizer (maximizer), respectively. And then, the sequence of the minimizers (maximizers), will converge to the minimizer of the original problem.*

***Keywords:*** *Exponential penalty, convex programming, Linear equation constraints.*

## 1. INTRODUCTION:

This paper is concerned with the study of the exponential method for solving a convex function under linear constraints problem. This is one of classical, almost exhausted area of mathematics. Therefore our aim is to investigate the performance of the method rather than to generate new theorems by means of this method.

This paper is laid out as follows. In Section 2 we formulate outline of the problem. In order to do this, in Section 3 we present general properties and main theorems for the exponential penalty method. Finally, in Section 4 we develop a duality method called exponential barrier method for the problem. In this section we summary the general properties and main theorem concerning the method.

## 2 STATEMENT OF THE PROBLEM:

We shall consider the convex function under linear constraints problem as follows.

Let $A$ be an $m \times n$ matrix, and $b$ be column vector with $m$ components. We consider the problem in following form:

Minimize $c(x)$

subject to $Ax = b$ ,

where $c(x)$ be a convex function. In what follows it is supposed that $m < n$ and that the solution $x^*$ to the problem exists. Let $f^*$ denotes the optimum value of the problem. In this paper we solve the problem using exponential method including both penalty and barrier terms.

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## 3 EXPONENTIAL PENALTY METHOD:

Let us introduce the exponential penalty function $E_p(x,\alpha,\beta)$ associated with the problem as follows:

$$E_p(x,\alpha,\beta) = c(x) + \sum_{i=1}^{m} \alpha_i \exp\left\{-\beta_i (A_i x - b_i)\right\} \quad (1)$$

where $\alpha = (\alpha_1, \ldots, \alpha_m) > 0$ be a penalty parameter of the function and $\beta = (\beta_1, \ldots, \beta_m) > 0$ . The penalty is formed from a sum of exponential of constraint violations and the parameter $\alpha_i$ determines the amount of the penalty.

It is easy to show that $E_p(x,\alpha,\beta)$ is a convex function for each $\alpha$ and $\beta$. This property is stated in the following theorem.

**Theorem 1 (Convexity)** The exponential penalty function $E_p(x,\alpha,\beta)$ is convex in its domain.

**Proof:** It is straightforward to prove convexity of $E_p(x,\alpha,\beta)$ using the convexity of $c(x)$ and $\sum_{i=1}^{m} \alpha_i \exp\left\{-\beta_i (A_i x - b_i)\right\}$
Then this theorem is proven.

As a consequence of this theorem we derive the local and global behavior of the exponential penalty function defined by (1) which is stated in the theorem.

**Theorem 2 (Local and global behavior)**

**(a)** $E_p(x,\alpha,\beta)$ has a finite unconstrained minimizer in its domain and the set $M_\sigma$ of unconstrained minimizers of $E_p(x,\alpha,\beta)$ in its domain is convex and compact for every parameters $\alpha$ and $\beta$.

**(b)** Any unconstrained local minimizer of $E_p(x,\alpha,\beta)$ in its domain is also a global unconstrained minimizer of $E_p(x,\alpha,\beta)$.

**Proof:** It follows from Theorem 1 that the smooth function $E_p(x,\alpha,\beta)$ achieves its minimum in its domain. We conclude that $E_p(x,\alpha,\beta)$ has at least one finite unconstrained minimizer. Then the Theorem 2 (a) is established.

By Theorem 1 $E_p(x,\alpha,\beta)$ is convex, so any local minimizer is also a global minimizer. Thus the set $M_\sigma$ of unconstrained minimizers of $E_p(x,\alpha,\beta)$ is bounded and closed, because the minimum value of $E_p(x,\alpha,\beta)$ is unique, and it follows that $M_\sigma$ is compact. Clearly, the convexity of $M_\sigma$ follows from the fact that set of optimal points $E_p(x,\alpha,\beta)$ is convex. Theorem 2 (b) has been verified.

As consequence of Theorem 2 we derive the monotonicity behavior of the minimum value of the exponential penalty function $E_p(x,\alpha,\beta)$. To do this, for any $\alpha^k$, $\beta^k > 0$ we denote $x^k$ and $E_p(x^k,\alpha^k,\beta^k)$ as an unconstrained minimizer and unconstrained minimum value of $E_p(x,\alpha,\beta)$, respectively. We now give the following monotonicity theorem.

**Theorem 3 (Monotonicity)** Let $\{\beta^k\}$ be a bounded sequence of positive real number and $\{\alpha^k\}$ be an increasing sequence of positive penalty parameters such that $\alpha^k \to \infty$ as $k \to \infty$ Then $\{E_p(x^k,\alpha^k,\beta^k)\}$ is non-increasing.

**Proof:** Let $x^k$ and $x^{k+1}$ denote global minimizers of exponential penalty function for the penalty parameters $\alpha^k$ and $\alpha^{k+1}$. By the definition of $x^k$ and $x^{k+1}$ as minimizers and $\alpha^k \le \alpha^{k+1}$, then for a sufficiently large $k$, we have

$$c(x^{k+1}) + \sum_{i=1}^{m} \alpha^{k+1} \exp\left\{-\beta^{k+1}(A_i x^{k+1} - b_i)\right\} \le c(x^k) + \sum_{i=1}^{m} \alpha^{k+1} \exp\left\{-\beta^{k+1}(A_i x^k - b_i)\right\}, \tag{2a}$$

$$c(x^k) + \sum_{i=1}^{m} \alpha^{k+1} \exp\left\{-\beta^{k+1}(A_i x^k - b_i)\right\} \le c(x^k) + \sum_{i=1}^{m} \alpha^{k} \exp\left\{-\beta^{k}(A_i x^k - b_i)\right\}, \tag{2b}$$

$$c(x^k) + \sum_{i=1}^{m} \alpha^{k} \exp\left\{-\beta^{k}(A_i x^k - b_i)\right\} \le c(x^{k+1}) + \sum_{i=1}^{m} \alpha^{k} \exp\left\{-\beta^{k}(A_i x^{k+1} - b_i)\right\}. \tag{2c}$$

Using inequalities (2a) and (2b), we obtain

$$c(x^{k+1}) + \sum_{i=1}^{m} \alpha^{k+1} \exp\left\{-\beta^{k+1}(A_i x^{k+1} - b_i)\right\} \le c(x^k) + \sum_{i=1}^{m} \alpha^{k} \exp\left\{-\beta^{k}(A_i x^k - b_i)\right\}.$$

It means that

$$E_p(x^{k+1},\alpha^{k+1},\beta^{k+1}) \le E_p(x^k,\alpha^k,\beta^k),$$

as required in theorem. Hence, the theorem is established.

Using the definition of $E_p(x^k,\alpha^k,\beta^k)$ and Theorem 3, we have

$$c(x^{k+1}) \le E_p(x^{k+1},\alpha^{k+1},\beta^{k+1}) \le E_p(x^k,\alpha^k,\beta^k). \tag{3}$$

Assume that $\{x^k\}$ is sequence such that $Ax^k = b$ and that

$$c(x^k) \to f^* \text{ as } k \to \infty. \tag{4}$$

By using (3),

$$f^* \le \ldots \le E_p(x^{k+1},\alpha^{k+1},\beta^{k+1}) \le E_p(x^k,\alpha^k,\beta^k). \tag{5}$$

Therefore, the sequence $\{E_p(x^k,\alpha^k,\beta^k)\}$ of exponential penalty function values is non increasing and bounded from below, and must converge monotonically from above to a limit, say $g^*$, where $g^* \ge f^*$.

Suppose that $g^* > f^*$. In this case, we define a positive number

$$\gamma := \tfrac{1}{2}\left(g^* - f^*\right).$$

It follows from (4) that there exist a positive real number $k_0$ such that for all $k \ge k_0$,

$$c(x^k) \le g^* - \gamma. \tag{6}$$

Since $\alpha^k > 0$, $\{\beta^k\}$ be a bounded sequence and $\alpha^k \to \infty$,

$$\sum_{i=1}^{m} \alpha^k \exp\left\{-\beta^k(A_i x^k - b_i)\right\} \to 0 \text{ as } k \to \infty.$$

So, there exist $k_1$ such that for all $k \ge k_1$,

$$\sum_{i=1}^{m} \alpha^k \exp\left\{-\beta^k(A_i x^k - b_i)\right\} < \tfrac{1}{2}\gamma. \tag{7}$$

If we apply (6)–(7) and take $k \ge \max\{k_0, k_1\}$, the result is

$$\begin{aligned} E_p(x^k,\alpha^k,\beta^k) &= c(x^k) + \sum_{i=1}^{m} \alpha^k \exp\left\{-\beta^k(A_i x^k - b_i)\right\} \\ &\le g^* - \gamma + \tfrac{1}{2}\gamma \\ &= g^* - \tfrac{1}{2}\gamma. \end{aligned} \tag{8}$$

Taking $k \to \infty$ and using (8), we have

$$g^* \le g^* - \tfrac{1}{2}\gamma,$$

that is,

$$\gamma \le 0,$$

which contradicts with the assumption that $\gamma > 0$. We conclude that $g^* = f^*$ and $E_p(x^k, \alpha^k, \beta^k) \to f^*$ as $k \to \infty$, which gives result of the following theorem.

**Theorem 4 (Convergence of exponential penalty function)** Let $\{\beta^k\}$ be a bounded sequence and $\{\alpha^k\}$ be an increasing sequence of positive penalty parameters such that $\alpha^k \to \infty$ as. Assume that $\{x^k\}$ is sequence such that $Ax^k = b$ and that $c(x^k) \to f^*$ as $k \to \infty$. Then $E_p(x^k, \alpha^k, \beta^k) \to f^*$ as $k \to \infty$.

The implication of this theorem is remarkable strong. For the problem that is studied in this paper, the exponential penalty function has a finite unconstrained minimizer for every value of the penalty parameter, and every limit point of a minimizing sequence for the barrier function is a constrained minimizer of the problem.

## 4 EXPONENTIAL BARRIER METHOD:

In this section we present an exponential barrier function method to solve the problem. This idea is derived from the exponential penalty function presented in Section 2. The exponential barrier function is given by

$$E_b(x, \mu, \sigma) = c(x) - \sum_{i=1}^{m} \mu_i \exp\left\{\sigma_i (A_i x - b_i)\right\} \quad (9)$$

where $\mu = (\mu_1, \ldots \mu_m) > 0$ is a barrier parameter of the function and $\beta = (\beta_1, \ldots, \beta_m) > 0$. The barrier is formed from a sum of exponential of constraint violations and the parameter $\mu$ determines the amount of the barrier.

We summary the concavity, local and global optimality, monotonicity and convergence behaviors of the function defined in (9) for solving the problem. All proofs of these behaviors are omitted because similar with the proofs in Section 3.

**Theorem 5 (Concavity)** The exponential barrier function $E_b(x, \mu, \sigma)$ is concave in its domain.

**Theorem 6 (Local and global behavior)** For every $\mu > 0$ and $\sigma > 0$ we have

**(a)** $E_b(x, \mu, \sigma)$ has a finite unconstrained maximizer in its domain for and the set $M_\mu$ of unconstrained maximizers of $E_b(x, \mu, \sigma)$ in its domain is convex and compact.

**(b)** Any unconstrained local maximizer of $E_b(x, \mu, \sigma)$ in its domain is also a global unconstrained maximizer of $E_b(x, \mu, \sigma)$.

**Theorem 7 (Monotonicity)** Let $\{\sigma^k\}$ be a bounded sequence of positive real number and $\{\mu^k\}$ be an increasing sequence of positive barrier parameters such that $\mu^k \to \infty$ as $k \to \infty$. Then $\{E_b(x^k, \mu^k, \sigma^k)\}$ is non-decreasing.

**Theorem 8 (Convergence of exponential barrier function)** Let $\{\sigma^k\}$ be a bounded sequence of positive real number and $\{\mu^k\}$ be an increasing sequence of positive barrier parameters such that $\mu^k \to \infty$ as $k \to \infty$. Assume that $\{x^k\}$ is sequence such that $Ax^k = b$ and that $c(x^k) \to f^*$ as $k \to \infty$. Then $E_b(x^k, \mu^k, \sigma^k) \to f^*$ as $k \to \infty$.

The implication of this theorem is remarkable strong. For the problem that is studied in this paper, the exponential barrier function has a finite unconstrained maximizer for every value of the barrrier parameter, and every limit point of a maximizing sequence for the barrier function is a constrained minimizer of the problem.

Theorems 4 and 8 have a weakness. That is, it assumes that the sequence $\{x^k\}$ is taken such that $\{c(x^k)\}$ has a limit point $f^*$. However, this situation can be remedied by taking sequence $\{x^k\}$ of minimizers (maximizers) of $E_p(x^k, \alpha^k, \beta^k)$ ($E_b(x^k, \mu^k, \sigma^k)$). The proofs of these results will appear elsewhere.

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