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A SINGLE SERVER M/G/1 QUEUE WITH SERVICE INTERRUPTION AND BERNOULLI SCHEDULE SERVER VACATION HAVING GENERAL VACATION TIME DISTRIBUTION

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ABSTRACT

This paper studies an M/G/1 queue with service interruption and Bernoulli schedule server vacation. The server provides essential service to all arriving customers with service time following general distribution. However after a service completion, the server may take a vacation with probability p, or to continue to stay in the system with probability $1-p$. We consider a general distribution for vacation time. The server is interrupted at random and the duration of attending interruption follows exponential distribution. The steady state solutions have been found by using Supplementary variables technique. Also the mean queue length and the mean waiting time are computed.

Keywords: *M/G/1 queue, Bernoulli Schedule Server Vacation, Steady state, the Mean number in the system, Mean waiting time.*

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1. INTRODUCTION:

Vacation queues have been studied by numerous researchers since Levy and Yechiali [5] and Doshi [3, 4] due to their wide applications in manufacturing and telecommunication systems. Vacation queues with c servers have been studied by Tian etal [7]. Choudhury and Borthakur [2] have studied vacation queues with batch arrivals. Multiple vacations have been studied by Tian and Zhang [8].

Madan and Maraghi[6] have studied batch arrival queueing system with random breakdowns and Bernoulli schedule server vacation having general vacation time. They have obtained steady state results in terms of the probability generating functions for the number of customers in the queue. Queues with interruptions were considered by many others. To mention White and Christie [10] have studied queue with interruption. Times of interruptions and services generally distributed are considered by Avi – Itzhak and Naor [1] and Thiruvengadan [9].

In this paper we consider queue with interruption and interrupt it through a different angle. The vacations follow a Bernoulli distribution, that is, after a service completion, the server may go for a vacation with probability p ($0 \leq p \leq 1$) or may continue to serve the next customer, if any, with probability $1-p$. The vacation time and service time are assumed to be general, while the time of attending interruption follows exponential distribution. Also we assume the customer whose service is interrupted goes back to the head of the queue where the arrivals are Poisson. The customer arrives to the system one by one and are served on a first come first served basis.

2. MATHEMATICAL MODEL:

- Customers arrive at the system one by one according to a Poisson stream with arrival rate λ (>0).

- There is a single server and the service time follows a general (arbitrary) distribution with distribution function $G(s)$ and density function $g(s)$. While serving the customers we assume interruptions arrive at random and assumed to occur according to a Poisson process with mean rate $\alpha > 0$ 0 .Let β be the server rate of attending interruption. Further we assume that once the interruption arrives the customer whose service is interrupted comes back to the head of the queue.

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- Let $\mu(x)dx$ be the conditional probability of service completion during the interval $(x, x + dx]$, given that the elapsed service time is x, so that

$$\mu(x) = \frac{g(x)}{1-G(x)} \tag{1}$$

and therefore,

$$g(s) = \mu(s)\, e^{-\int_0^s \mu(x)dx} \tag{2}$$

- As soon as the essential service is completed, then with probability p the server may take vacation of random length, or with probability $1 - p$ he may stay in the system providing service, where $0 \le p \le 1$.

- The server's vacation time follows a general (arbitrary) distribution with distribution function $H(r)$ and density function $h(r)$. Let $\gamma(x)dx$ be the conditional probability of a completion of a vacation during the interval $(x, x + dx]$ given that the elapsed vacation time is x, so that

$$\gamma(x) = \frac{h(x)}{1-H(x)} \tag{3}$$

and, therefore

$$h(r) = \gamma(r)\, e^{-\int_0^r \gamma(x)dx} \tag{4}$$

- On returning from vacation the server instantly starts serving the customer at the head of the queue if any.
- The customers are provided service one by one on a first come first served rule.
- Various stochastic processes involved in the system are assumed to be independent of each other.

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3. DEFINITIONS AND NOTATIONS:

We define $P_n(x,t)$ = Probability that at time t, the server is active providing essential service and there are n (≥ 0) customers in the queue excluding one customer in service and the elapsed service time for this customer is x.

Consequently $P_n(t) = \int_0^\infty P_n(x,t)\,dx$ denotes the probability that at time t there are n customers in the queue excluding the one customer in essential service irrespective of the value of x.

$V_n(x,t)$ = Probability that at time t, there are n (≥ 0) customers in the queue and the server is on vacation with elapsed vacation time is x. Consequently $V_n(t) = \int_0^\infty V_n(x,t)\,dx$ denotes the probability that at time t there are n (≥ 0) customers in the queue and the server is on vacation irrespective of the value of x.

$R_n(t)$ = Probability that at time t, the server is inactive due to the arrival of interruption

$Q(t)$ = Probability that at time t, there are no customers in the queue or in service and the server is idle but available in the system.

4. STEADY STATE CONDITION:

Let

$$\lim_{t\to\infty} A_n(x,t) = An(x) \qquad \lim_{t\to\infty} A_n(t) = \lim_{\lim t\to\infty} \int_0^\infty An(x,t)dx = An$$

$$\lim_{t\to\infty} \frac{dA_n(t)}{dt} = 0 \qquad n \geq 0$$

Where A= P, V.

$$\lim_{t\to\infty} Q(t)=Q \qquad \lim_{t\to\infty} R(t)=R$$

denote the corresponding steady state probabilities.

5. EQUATIONS GOVERNING THE SYSTEM:

According to the mathematical model mentioned above, the system has the following set of differential-difference equations

$$\frac{\partial}{\partial x} P_n(x) + (\lambda + \mu(x) + \alpha)\, P_n(x) = \lambda\, P_{n-1}(x), \qquad n = 1, 2, 3 \ldots \tag{5}$$

$$\frac{\partial}{\partial x} P_0(x) + (\lambda + \mu(x) + \alpha)\, P_0(x) = 0 \tag{6}$$

$$\frac{\partial}{\partial x} V_n(x) + (\lambda + \gamma(x))\, V_n(x) = \lambda\, V_{n-1}(x), \qquad n = 1, 2, 3\ldots \tag{7}$$

$$\frac{\partial}{\partial x} V_0(x) + (\lambda + \gamma(x))\, V_0(x) = 0 \tag{8}$$

$$(\lambda + \beta)\, R_n = \lambda\, R_{n-1} + \alpha \int_0^\infty P_{n-1}(x)\, dx \qquad n=1, 2, 3\ldots. \tag{9}$$

$$(\lambda + \beta)\, R_0 = 0 \tag{10}$$

$$\lambda Q = R_0\, \beta + \int_0^\infty V_0(x)\, \gamma(x)\, dx + (1-p) \int_0^\infty P_0(x)\, \mu(x)\, dx \tag{11}$$

The above equations are to be solved subject to the following boundary conditions

$$P_n(0) = (1-p) \int_0^\infty P_{n+1}(x)\, \mu(x)\, dx + \int_0^\infty V_{n+1}(x)\, \gamma(x)\, dx + \beta\, R_{n+1} \qquad n=1,2,3\ldots\ldots \tag{12}$$

$$P_0(0) = (1-p) \int_0^\infty P_1(x)\, \mu(x) dx + \int_0^\infty V_1(x)\, \gamma(x) dx + \lambda Q + \beta\, R_1 \tag{13}$$

$$V_n(0) = p \int_0^\infty P_n(x)\, \mu(x) dx \qquad n=0,1,2,\ldots. \tag{14}$$

6. QUEUE SIZE DISTRIBUTION AT A RANDOM EPOCH:

Defining the following probability generating functions

$$A_q(x, z) = \sum_{n=0}^{\infty} z^n A_n(x) \qquad A_q(z) = \sum_{n=0}^{\infty} z^n A_n$$

Where A=P,V

$$R_q(z) = \sum_{n=0}^{\infty} z^n R_n \tag{15}$$

Now multiplying the equation (5) by z^n and summing over n from 1 to ∞, adding to equation (6) and using the generating functions defined in (15) we obtain

$$\frac{\partial}{\partial x} P_q(x, z) + (\lambda - \lambda z + \mu(x) + \alpha) P_q(x, z) = 0 \tag{16}$$

Performing similar operations to equations (7) and (8) and (9) and (10) we get

$$\frac{\partial}{\partial x} V_q(x, z) + (\lambda - \lambda z + \gamma(x)) V_q(x, z) = 0 \tag{17}$$

$$(\lambda - \lambda z + \beta) R_q(z) = \alpha z \int_0^\infty P_q(x,z)\, dx \tag{18}$$

For the boundary conditions we multiply both sides of equation (12) by z^{n+1}, sum over n from 1 to ∞, multiply both sides of (13) by z, add the two results and using the generating function (15) we obtain

$$z P_q(0, z) = (1-p) \int_0^\infty P_q(x,z)\mu(x)\, dx + \int_0^\infty V_q(x,z)\, \gamma(x)\, dx + \beta R_q(z) + \lambda Q z$$
$$- \left((1-p) \int_0^\infty P_0(x)\mu(x)\, dx + \int_0^\infty V_0(x)\, \beta(x)\, dx + \beta R_0 \right) \tag{19}$$

using equation (11), equation (19) becomes

$$z P_q(0, z) = (1-p) \int_0^\infty P_q(x,z)\mu(x)\, dx + \int_0^\infty V_q(x,z)\, \gamma(x)\, dx + \beta R_q(z) + \lambda Q (z-1) \tag{20}$$

Now multiply equation (14) by z^n and sum over *n* from 0 to ∞, we get

$$V_q(0, z) = p \int_0^\infty P_q(x,z)\mu(x)\, dx \tag{21}$$

Integrating equation (16) from 0 to x we get

$$P_q(x, z) = P_q(0, z)\, e^{-(\lambda - \lambda z + \alpha)x - \int_0^x \mu(t)\, dt} \tag{22}$$

Where $P_q(0, z)$ is given by equation (20). Integrating equation (22) by parts with respect to *x* yields

$$P_q(z) = P_q(0, z)\left(\frac{1 - G^*(\lambda - \lambda z + \alpha)}{\lambda - \lambda z + \alpha}\right) \tag{23}$$

where $G^*(\lambda - \lambda z + \alpha) = \int_0^\infty e^{-(\lambda - \lambda z + \alpha)x}\, dG(x)$ is the Laplace-stieltjes transform of service time G(x).

Multiplying both sides of equation (22) by μ(x) and integrating with respect to x we obtain

$$\int_0^\infty P_q(x, z)\mu(x)\, dx = P_q(0, z)\, G^*(\lambda - \lambda z + \alpha) \tag{24}$$

using equation (24), equation (21) becomes

$$V_q(0, z) = p\, P_q(0, z)\, G^*(\lambda - \lambda z + \alpha) \tag{25}$$

Integrating equation (17) from 0 to x we get

$$V_q(x, z) = V_q(0, z)\, e^{-(\lambda-\lambda z)x - \int_0^x \gamma(t)\,dt} \tag{26}$$

Substituting for $V_q(0, z)$ from equation (25) in (26) we obtain

$$V_q(x, z) = p\,P_q(0, z)\, G^*(\lambda-\lambda z+\alpha)\, e^{-(\lambda-\lambda z)x - \int_0^x \gamma(t)\,dt} \tag{27}$$

Integrating equation (27) by parts with respect to *x* yields

$$V_q(z) = = p\,P_q(0, z)\, G^*(\lambda-\lambda z+\alpha)\left(\frac{1-H^*(\lambda-\lambda z)}{\lambda-\lambda z}\right) \tag{28}$$

where $H^*(\lambda-\lambda z) = \int_0^\infty e^{-(\lambda-\lambda z)x}\, dH(x)$ is the Laplace-stieltjes transform of the vaction time H(x)

Multiplying both sides of equation (27) by $\gamma(x)$ and integrating with respect to x we obtain

$$\int_0^\infty V_q(x,z)\,\gamma(x)\,dx = p\,P_q(0, z)\, G^*(\lambda-\lambda z+\alpha)\, H^*(\lambda-\lambda z) \tag{29}$$

Using equation (22) in equation (18) we get

$$R_q(z) = \alpha z\, P_q(0, z)\left(\frac{1-G^*(\lambda-\lambda z+\alpha)}{\lambda-\lambda z+\alpha}\right)\left(\frac{1}{\lambda+\beta-\lambda z}\right) \tag{30}$$

Assuming $a = (\lambda-\lambda z+\alpha)$, $b = (\lambda-\lambda z+\beta)$ and $c = (\lambda-\lambda z)$

Now using equations (24), (29) and (30) in equation (20) yields

$$P_q(0,z) = \frac{-abcQ}{ab\,\{z-(1-p)G^*(a) - pG^*(a)H^*(c)\} - \beta\alpha z[1-G^*(a)]} \tag{31}$$

Substituting $P_q(0, z)$ in equations (23), (28)and (30) we obtain

$$P_q(z) = \frac{-bcQ(1-G^*(a))}{ab\,\{z-(1-p)G^*(a) - pG^*(a)H^*(c)\} - \beta\alpha z[1-G^*(a)]} \tag{32}$$

$$V_q(z) = \frac{abpQG^*(a)[H^*(c)-1]}{ab\,\{z-(1-p)G^*(a) - pG^*(a)H^*(c)\} - \beta\alpha z[1-G^*(a)]} \tag{33}$$

$$R_q(z) = \frac{-\alpha z c Q(1-G^*(a))}{ab\,\{z-(1-p)G^*(a) - pG^*(a)H^*(c)\} - \beta\alpha z[1-G^*(a)]} \tag{34}$$

Let Sq (z) denote the probability generating function of the queue size irrespective of the state of the system. Then adding equations (32), (33) and (34), we obtain

$$Sq(z) = Pq(z) + Vq(z) + Rq(z)$$

$$= \frac{-Qc[1-G^*(a)][b+\alpha z] + abpQG^*(a)[H^*(c)-1]}{ab\{z-(1-p)G^*(a) - pG^*(a)H^*(c)\} - \beta\alpha z[1-G^*(a)]} \qquad (35)$$

It is easy to verify that for z = 1, *Sq* (z) is indeterminate of the form 0/0 Hence we apply L'Hopitals rule on equation (35), we obtain

$$Sq(1) = \frac{\lambda Q[(\alpha+\beta)(1-G^*(\alpha)) + \alpha p\beta G^*(\alpha)E(V)]}{\alpha\beta G^*(\alpha) - \lambda[1-G^*(\alpha)](\alpha+\beta) - p\alpha\beta\lambda G^*(\alpha)E(V)} \qquad (36)$$

Therefore adding Q to equation (36) and equating to 1 and on simplifying, we get

$$Q = 1 - \lambda\left[\frac{1}{\alpha G^*(\alpha)} + \frac{1}{G^*(\alpha)\beta} - \frac{1}{\alpha} - \frac{1}{\beta} + pE(V)\right] \qquad (37)$$

And hence the utilization factor ρ of the system is given by

$$\rho = \lambda\left[\frac{1}{\alpha G^*(\alpha)} + \frac{1}{G^*(\alpha)\beta} - \frac{1}{\alpha} - \frac{1}{\beta} + pE(V)\right] \qquad (38)$$

where $\rho<1$ is the stability condition under which the steady state exists. Equation (37) gives the probability that the server is idle. Substituting for Q from (37) into (35), we have completely and explicitly determined Sq (z), the probability generating function of the queue size.

7. THE MEAN NUMBER IN THE SYSTEM:

Let L_q denote the mean number of customers in the queue under the steady state. Then

$$L_q = \frac{d}{dz} Sq(z)\Big|_{z=1.} \qquad (39)$$

Since this formula gives 0/0 form, then we write Sq(z) given in (35) as Sq(z) = *N*(*z*) / *D*(*z*) where *N*(*z*) and *D*(*z*) are the numerator and denominator of the right hand side of (35) respectively. Then using the L'Hopital's rule twice we obtain

$$L_q = \lim_{z\to 1} \frac{D'(z)N''(z) - N'(z)D''(z)}{2(D'(z))^2}$$

$$= \lim_{z\to 1} \frac{D'(1)N''(1) - N'(1)D''(1)}{2(D'(1))^2} \qquad (40)$$

where, primes and double primes in (40) denote first and second derivatives at z = 1. Carrying out the derivatives at z = 1,using the fact that $H^*(0)=1$ $H^{*'}(0) = -E(V)$ and $H^{*''}(0) = E(V^2)$, the second moment of the vacation time. After simplification we get

$$N'(1) = \lambda Q\{(\alpha+\beta) + G^*(\alpha)\ [E(V)p\alpha\beta - \alpha - \beta]\} \qquad (41)$$

$$N''(1) = 2\lambda^2 Q[(-1+\alpha/\lambda) + G^*(\alpha)[1 - \alpha/\lambda - p\alpha E(V) - p\beta E(V) + 1/2\, p\alpha\beta E(V^2)] + G^{*'}(\alpha)(\alpha+\beta-\alpha\beta E(V))] \qquad (42)$$

$$D'(1) = -\lambda(\alpha+\beta) + G^*(\alpha)\ [\alpha\beta + \lambda(\alpha+\beta-p\alpha\beta E(V))] \tag{43}$$

$$D''(1) = 2\lambda^2 \left\{ \left(1-\frac{\alpha+\beta}{\lambda}\right) + G^*(\alpha)\ [-1+p\alpha E(V) + p\,\beta E(V) - 1/2\, p\alpha\beta E(V^2)] + G^{*'}(\alpha)\left[-\frac{\alpha\beta}{\lambda} - \alpha - \beta + \alpha\beta\, pE(V)\right] \right\} \tag{44}$$

Using equations (41) to (44) into (40), we obtain Lq in closed form where Qhas been found in equation (37). Further we find the average system size L using Little's formula. Thus we have

$$L = L_q + \rho \tag{45}$$

where L_q has been found in equation (40) and ρ is obtained from equation (38) as ρ

$$\rho = 1 - Q \tag{46}$$

8. THE MEAN WAITING TIME:

Let W_q and W denote the mean waiting time in the queue and in the system respectively. Then using Little's formula, we obtain

$$W_q = \frac{L_q}{\lambda} \tag{47}$$

$$W = \frac{L}{\lambda} \tag{48}$$

Where L_q and L have been found in equations (40) and (45)

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