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A FIXED POINT THEOREM FOR OWC MAPPINGS SATISFYING A CONTRACTIVE CONDITION OF INTEGRAL TYPE

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ABSTRACT

In a recent paper have extension of Banach fixed point theorem for mappings satisfying a contractive condition of integral type. We generalize G. Jungck and B. E. Rhodes [9] results.

Mathematics subject classification: *47H10, 54H25.*

Keywords: *Common fixed point, occasionally weakly compatible mappings, Contractive condition of integral type.*

1. INTRODUCTION:

For an integral type of the Banach contraction principle, that could be extended to more general contractive conditions. We generalize G. Jungck and B. E. Rhodes [9] results. Branciari [1] established the following theorem.

Theorem: 1.1 Let (X, d) be a complete metric space, $c \in (0,1)$ and let $f : X \to X$ be a mapping such that for each $x, y \in X$,

$$\int_0^{d(fx,fy)} \phi(s)ds < c \int_0^{d(x,y)} \phi(s)ds\,,$$

where $\phi : [0,+\infty) \to [0,+\infty)$ is a Lebesgue integrable mapping which is summable on each compact subset of $[0,+\infty)$ and such that for all $\varepsilon > 0$,

$$\int_0^{\varepsilon} \phi(s)ds > 0\,.$$

Then, f admits a unique fixed point $a \in X$ such that for each $x \in X$, $f^n x \to a$ as $n \to +\infty$.

Theorem:1.2 Rhoades [2] proved that Theorem 1.1 holds also if we replace d(x, y) by

$$\max\left\{d(x, y), d(x, fx), d(y, fy), \frac{d(x, fy) + d(y, fx)}{2}\right\}.$$

Fixed point theorems involving more general contractive conditions proved by I. Altun, P. Vijayaraju, A. Djoudi, [see,[3, 4, 5]]. Sessa [6], with the notion of weakly commuting mappings, weakened the concept of commutativity of two mappings. Then, Jungck [7, 8] and Rhoades [9] enlarged the concept of weakly commuting mappings by adding the notion of compatible mappings as well as for occasionally weakly compatible mappings. Our main result is a generalization of Theorem 1 given in [9] by integral type.

Definition: 1.3 Let X be a set f and g selfmaps of X. A point x in X is called a coincidence point of f and g iff fx = gx.

We shall call w = fx = gx a point of coincidence of f and g.

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Definition: 1.4 Two self mappings $f, g : X \to X$ are said to be weakly compatible if they commute at their coincidence points.

Definition: 1.5 Two self mappings $f, g : X \to X$ are said to be occasionally weakly compatible (owc) iff there is a point in X they commute at their coincidence points.

MAIN RESULTS:

Lemma: 1.6 Let X be a set f, g are owc selfmaps of X. If f and g have a unique fixed point of coincidence, w = fx = gx, then w is the unique fixed point of f and g [9].

Theorem: 1.7 Let (X, d) be a metric space and let f, g, S and T be selfmaps of X and the pairs {f, S} and {g, T} are each owc. If

$$\int_0^{d(fx,fy)} \phi(s)ds < c \int_0^{M(x,y)} \phi(s)ds \tag{1}$$

for each $x, y \in X$ such that $fx \neq gy$, where

$$M(x, y) = \max\{\text{d(Sx,Ty), d(Sx, fx), d(Ty, gy),d(Sx, gy),d(Ty, gx)}\}.$$

where $\phi : [0,+\infty) \to [0,+\infty)$ is a Lebesgue integrable mapping which is summable on each compact subset of $[0,+\infty)$ and such that for all $\varepsilon > 0$,

$$\int_0^{\varepsilon} \phi(s)ds > 0 .$$

Then there is a unique fixed point $w \in X$ such that fw = gw = w and a unique point $z \in X$ such that gz = Tz = z. Moreover z = w, so that there is a unique common fixed point of f, g, S and T.

Proof: Since the pairs f, S and g, T are each owc, there exist a points $x, y \in X$ such that fx = Sx and gy = Ty. We claim fx = gy. Suppose that $fx \neq gy$,so we get

M(x, y) = max{d(fx, gy), d(fx, fx), d(gy, gy), d(fx, gy), d(gy, fx)}

= d(fx, gy).

Form, (1) we get

$$\int_0^{d(fx,gy)} \phi(s)ds < \int_0^{M(x,y)} \phi(s)ds$$

$$= \int_0^{d(fx,gy)} \phi(s)ds$$

is a contradiction. Therefore fx = gy, i.e. fx = Sx = gy = Ty. Moreover, if there is another point z such that fz = Sz, then by (1) it follows that fz = Sz = gy = Ty, or fx = fz and w = fx = Sx is the unique point of coincidence of f and S. By lemma 1.6, w is the only common fixed point of f and S. Also there is a unique point $z \in X$ such that z = gz = Tz. Suppose that $w \neq z$. Using (1), we get

$$\int_0^{d(w,z)} \phi(s)ds = \int_0^{d(fw,gz)} \phi(s)ds$$
$$< \int_0^{M(w,z)} \phi(s)ds$$
$$= \int_0^{d(w,z)} \phi(s)ds,$$

which is a contradiction. Therefore w = z and w is a common fixed point. Hence w is unique fixed point.

Corollary: 1.8 Let (X, d) be a metric space and let f, g, S and T be selfmaps of X and the pairs {f, S} and {g, T} are each owc. If

$$\int_0^{d(fx,gy)} \phi(s)ds < h \int_0^{M(x,y)} \phi(s)ds \tag{2}$$

for each $x, y \in X$, where $0 \le h < 1$

$$M(x, y) = \max\left\{d(Sx, Ty), d(Sx, fx), d(Ty, gy), \frac{d(Sx, gy) + d(Ty, fx)}{2}\right\}.$$

where $\phi : [0,+\infty) \to [0,+\infty)$ is a Lebesgue integrable mapping which is summable on each compact subset of $[0,+\infty)$ and such that for all $\varepsilon > 0$,

$$\int_0^{\varepsilon} \phi(s)ds > 0 .$$

Then f, g, S and T have unique common fixed point.

Proof: From theorem 1.10 result follows, since (2) is special case of (1).

Now we are proving our result for symmetric spaces, which is more general than metric spaces.

Definition: 1.9 Let X be a set. A symmetric on X is a mapping $r : X \times X \to [0,+\infty)$ such that

$$r(x, y) = 0 \text{ iff } x = y \text{ and } r(x, y) = r(y, x) \text{ for } x, y \in X. \tag{3}$$

Theorem: 1.10 Let (X, d) be a set with symmetric r and let f, g, S and T be selfmaps of X and the pairs {f, S} and {g,T} are each owc. If

$$\int_0^{r(fx,fy)} \phi(s)ds < \int_0^{M(x,y)} \phi(s)ds \tag{4}$$

for each $x, y \in X$ such that $fx \neq gy$, where

$$M(x, y) = \max\{r(Sx, Ty), r(Sx, fx), r(Ty, gy), r(Sx, gy), r(Ty, fx)\}.$$

where $\phi : [0,+\infty) \to [0,+\infty)$ is a Lebesgue integrable mapping which is summable on each compact subset of $[0,+\infty)$ and such that for all $\varepsilon > 0$,

$$\int_0^{\varepsilon} \phi(s)ds > 0.$$

Then there is a unique fixed point $w \in X$ such that fw = gw = w and a unique point $z \in X$ such that gz = Tz = z. Moreover z = w, so that there is a unique common fixed point of f, g, S and T.

Proof. Since the pairs f, S and g, T are each owc, there exist a points $x, y \in X$ such that fx = Sx and gy = Ty. we claim fx = gy. Suppose that s $fx \neq gy$,so we get

$$M(x, y) = \max\{r(fx, gy), r(fx, fx), r(gy, gy), r(fx, gy), r(gy, fx)\}$$
$$= r(fx, gy).$$

Form, (4)

$$\int_0^{r(fx,gy)} \phi(s)ds < \int_0^{M(x,y)} \phi(s)ds$$

$$= \int_0^{r(fx,gy)} \phi(s)ds$$

is a contradiction. Therefore fx = gy, i.e. fx = Sx = gy = Ty. Moreover, if there is another point z such that fz = Sz, then by (4) it follows that fz = Sz = gy = Ty, or fx = fz and w = fx = Sx is the unique point of coincidence of f and S. By lemma 1.6, w is the only common fixed point of f and S. Also by symmetry there is a unique point $z \in X$ such that z = gz = Tz. Suppose that $w \neq z$. Using (4), we get

$$\int_0^{r(w,z)} \phi(s)ds = \int_0^{r(fw,gz)} \phi(s)ds$$

$$< \int_0^{M(w,z)} \phi(s)ds$$

$$= \int_0^{r(w,z)} \phi(s)ds,$$

which is a contradiction. Therefore w = z and w is a common fixed point. Hence w is unique fixed point.

Corollary: 1.11 Let X be a set and let f, g, S and T be selfmaps of X and the pairs {f, S} and {g, T} are each owc. If

$$\int_0^{r(fx,fy)} \phi(s)ds < h \int_0^{M(x,y)} \phi(s)ds \qquad (5)$$

for each $x, y \in X$, where $0 \le h < 1$,

$$M(x, y) = \max\left\{r(Sx, Ty), r(Sx, fx), r(Ty, gy), \frac{r(Sx, gy) + r(Ty, fx)}{2}\right\}$$

where $\phi : [0,+\infty) \to [0,+\infty)$ is a Lebesgue integrable mapping which is summable on each compact subset of $[0,+\infty)$ and such that for all $\varepsilon > 0$,

$$\int_0^{\varepsilon} \phi(s)ds > 0.$$

Then f, g, S and T have unique common fixed point.

Proof: From theorem 1.10 result follows, since (5) is special case of (4).

Example 1.12 Let (X, d) be a metric space with X = [2, 20] and d(x, y) =|x − y|. Define f, g, S, T by

f2 = 2, fx = 3 if x > 2,

S2 = 2, Sx = 6 if x > 2,

g2 = 2 or x > 5, gx = 6 if $2 < x \le 5$,

T2 = 2, Tx = 12 if $2 < x \le 5$, Tx = x − 3, if x > 5.

and $\phi(t) = t$, for t > 0 and $\phi(0) = 0$.

Then f, g, S, T satisfy (1) and (4). If we choose $x_n = 5 + 1/n$, then $Tx_n \to 2$, $gx_n = 2$, $Tgx_n = 2$, and $gTx_n = 2$. Clearly, g and T are not compatible. The maps are owc at x = 2.

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