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Some Fixed Point Theorems for A_w^*-maps

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ABSTRACT

In this paper, we introduce the idea of A_w^*-mapping for w-distance and proved some results for fixed point and common fixed point for A_w^*-mappings. These results, generalized and improved the corresponding results of A.Latif, Suzuki and others.

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***Key Words**: A_w^*-maps, Fixed points, Multi-valued maps, w-distance.*

1 INTRODUCTION:

In [6], Kada et al., have introduced a notion of w-distance on a metric space and improved several results replacing the involved metric by a generalized distance. Using this generalized distance, Suzuki and Takahashi in [12] introduced notions of single-valued and multi-valued weakly contractive maps and proved some fixed point results for such maps. Consequently, they generalized the Banach contraction principle and Nadler's (see [10]) fixed point results . While, in [13] Suzuki generalized Kannan's [7] fixed point result in w-distance. Further fixed point results concerning w-distance can be found in [6,7,11]. In [2], M. Akram, et al., introduced the general class of contraction called A-contraction and proved that the class of A-contraction is a proper super class of Kannan's contractions including several other famous contractions like Bianchini's [4] and Reich's [11] contractions. In [1] M. Akram et al., using the idea of A-contractions, proved several fixed point theorems for single valued maps on generalized complete metric spaces. In [3], M. Akram, et al., introduced general multi-valued A^*-maps and proved some fixed point theorems for these maps. In this paper, using the concept of w-distance, we define A_w^*-maps and proved some fixed point and common fixed point results for these maps. These results generalizes the results of [9] and others, for general class of maps called A_w^*-maps.

2 PRELIMINARIES:

Throughout this paper, X is a metric space with metric d. Let 2^X denote a set of all nonempty subsets of X, $Cl(X)$ a family of all nonempty closed subsets of X. Consider a multi-valued map $T: X \to 2^X$. An element $x \in X$ is said to be a fixed point of T if $x \in T(x)$. Let R_+ denote the set of all non-negative real numbers and A^* stands for the set of all functions $\alpha: R_+^3 \to R_+$ satisfying

1. α is continuous on the R_+^3 of all triplets of non-negative reals (with respect to the Euclidean metric on R^3);
2. α is non-decreasing in each coordinate variables;
3. $a \le kb$ for some $k \in [0,1)$, whenever $a \le \alpha(a,b,b)$ or $a \le \alpha(b,a,b)$ or $a \le \alpha(b,b,a)$ for all a, b.

Definition: 2.1 *A map $\phi: X \to R$ is called Lower Semi-Continuous if for any sequence $\{x_n\} \subset X$ with $x_n \to x \in X$ imply that $\phi(x) \le liminf_{n\to\infty}\phi(x_n)$.*

Kada et all. [6], introduced a concept of w-distance as follows:

Definition: 2.2 *A function $w: X \times X \to [0,\infty)$ is called w-distance on X if it satisfies the following conditions, for any $x, y, z \in X$:*

1. $w(x,z) \le w(x,y) + w(y,z)$
2. the map $w(x,.): X \to [0,\infty)$ is lower semi-continuous.
3. for any $\varepsilon > 0$, there exist a $\delta > 0$ such that $w(z,x) \le \delta$ and $w(z,y) \le \delta$ imply that $w(x,y) \le \varepsilon$.

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The metric d on X is a simple example of w-distance on X. Many other examples of w-distance can be found in [8, 10, 13].

Definition: 2.3 *A multi-valued map $T: X \to 2^X$ is A_w^*-map if there exists a w-distance function w such that for any $x \in M, u \in T(x)$ there exist $v \in T(y)$ for all $y \in M$, and*

$$w(u,v) \le \alpha(w(x,y),\ w(x,u),\ w(y,v)),$$

for some $\alpha \in A^*$.

In particular, if we take $w = d$, then A_w^*-map becomes an A^*-map as defined in [3].

Definition: 2.4 *A multi-valued map $T: X \to 2^X$ is said to be K_w-map if there exists a nonnegative number $r \in [0,1)$ and a w-distance function w such that for any $x \in M, u \in T(x)$ there exist $v \in T(y)$ for all $y \in M$, w satisfies the inequality $w(u,v) \le r(w(x,u) + w(y,v))$.*

The following Lemma concerning w-distance is very important.

Lemma: 2.5 *[8] Let X be metric space with metric d and let w be the w-distance on X. Let $\{x_n\}$ and $\{y_n\}$ be sequences in X. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be a sequences in $[0,\infty)$ converging to 0, and let $x, y, z \in X$, then the following hold:*

(a) If $w(x_n, y) \le \alpha_n$ and $w(x_n, z) \le \beta_n$ for any $n \in N$, then $y = z$; in particular, if $w(x,y) = 0$ and $w(x,z) = 0$. Then $y = z$;
(b) If $(x_n, y_n) \le \alpha_n$ and $w(x_n, z) \le \beta_n$ for any $n \in N$, then $\{y_n\}$ converges to z;
(c) If $w(x_n, x_m) \le \alpha_n$ for any $n, m \in N$ with $m > n$, then $\{x_n\}$ is Cauchy sequence;
(d) If $w(y, x_n) \le \alpha_n$ for any $n \in N$, then $\{x_n\}$ be Cauchy sequence.

3. MAIN RESULTS:

In the sequel, we take X complete and M a non-empty closed subset of X.

Theorem 3.1 *Let $T: M \to Cl(M)$ be a multi-valued A_w^*-map such that for any $x \in M$, there exists $u \in Tx$, there exist $v \in Ty$ for all $y \in M$ and*

$$\inf\{w(x,u) + \ w(x,Tx): x \in X\} > 0$$

for every $u \in X$ with $u \in T(u)$. Then T has a fixed point.

Proof. Let u_0 be an element of M, and $u_1 \in T(u_0)$ be fixed. Since T is A_w^*-map, there exist $u_2 \in T(u_1)$ s.t

$$\begin{aligned} w(u_1,u_2) &\le \alpha(w(u_0,u_1),\ w(u_0,u_1),\ w(u_1,u_2)) \\ &\le k\ w(u_0,u_1), \end{aligned}$$

where $k \in [0,1)$. By repeating this process, we have

$$w(u_n,u_{n+1}) \le k^n\ w(u_0,u_1).$$

Now, for $m > n$, we have

$$\begin{aligned} w(u_m,u_n) &\le w(u_n,u_{n+1}) + w(u_{n+1},u_{n+2}) + \ldots + w(u_{m-1},u_m) \\ &\le k^n\ w(u_0,u_1) + k^{n+1}\ w(u_0,u_1) + \ldots + k^{m-1}\ w(u_0,u_1) \\ &\le (k^n + k^{n+1} + \ldots + k^{m-1})\ w(u_0,u_1) \\ &\le \{k^n(1-k^m)/(1-k)\}\ w(u_0,u_1) \to 0 \ \ as\ \ n \to \infty. \end{aligned}$$

Hence, by lemma(1.4), $\{u_n\}$ is a Cauchy sequence. From the completeness of X, we get that $\{u_n\}$ converges to some $v_0 \in X$. Since M being closed, we have $v_0 \in M$. Let $n \in N$ be fixed. Since $\{u_m\}$ converges to some v_0 and $w(u_n,\cdot)$ is lower semi-continuous, we get

$$w(u_n,v_0) \le \liminf_{m\to\infty} w(u_n,u_m) \le \{k^n(1-k^m)/(1-k)\}\ w(u_0,u_1)$$

So, as $n \to \infty$, we have $w(u_n,v_0) \to 0$. Assume that $v_0 \in T(v_0)$. Then by hypothesis, we have

$$\begin{aligned} 0 &< inf\{w(u,v_0) + \ w(u,Tu): u \in X\} \\ &\le \inf\{w(u_n,v_0) + \ w(u_n,Tu_n): n \in N\} \\ &\le \inf\{w(u_n,v_0) + \ w(u_n,u_{n+1}): n \in N\} \end{aligned}$$

$$\leq \inf\{k^n(1-k^m)/(1-k)\ w(u_0,u_1) + \ k^n w(u_0,u1): n \in N\} = 0.$$

Which is impossible and hence $v_0 \in T(v_0)$.

Note that Theorem 3.1 of [9] becomes the corollary of our Theorem 3.1 as below.

Corollary: 3.2 *Let $T: M \to Cl(M)$ be a multi-valued K_w-map such that*

$$\inf\{w(x,u) + w(x,T(x)): x \in X\} > 0,$$

for every $u \in X$ with $u \in T(u)$. Then T has a fixed point.

Proof: Define $\alpha: R_+^3 \to R^+$ by $\alpha(u,v,w) = r(u+v)$, where $0 < r < 1/2$. One can easily see that $\alpha \in A^*$. For any $x \in M, u \in T(x)$ there exist a $v \in T(y)$ for all $y \in M$, If we take $u = w(x,u)$, $v = w(y,v)$ and $w = w(x,y)$, in the definition of K_w-map, we have

$$w(u,v) \leq h\big(w(x,u) + w(y,v)\big) = h(u+v) = \alpha(u,v,w).$$

This gives $w(u,v) \leq \alpha(w(x,y), w(x,u),\ w(y,v))$. That is T is a A_w^*-map. Hence by Theorem 3.1, we have the conclusion.

Theorem 3.3 *Each A_w^*-map $T: M \to Cl(M)$ has a fixed point, provided that for any iterative sequence $\{u_n\}$ in M with $u_n \to v_0 \in M$, the sequence of real numbers $\{w(v_0,u_n)\}$ converges to zero.*

Proof. In a similar way, as in Theorem 3.1, there exists a convergent iterative sequence $\{u_n\}$ such that $u_n \to v_0 \in M$ with

$$w(u_n,v_0) \leq \liminf_{m\to\infty} w(u_n,u_m) \leq \{k^n(1-k^m)/(1-k)\}\ w(u_0,u_1)$$

and $$w(u_n,u_{n+1}) \leq k^n\ w(u_0,u_1).$$

Note that $w(u_n,v_0) \to 0$ as $n \to \infty$. Further since $u_n \in T(u_{n-1})$ and T is a A_w^*-map therefore there is $v_n \in T(v_0)$ such that

$$w(u_n,v_n) \leq \alpha(w(u_{n-1},v_0),\ w(u_{n-1},u_n),\ w(v_0,v_n)).$$

For this we use definition of lower semi-continuous

$$w(v_0, \lim_{n\to\infty} v_n) \leq \liminf_{n\to\infty} w(u_n,v_n) \leq \lim_{n\to\infty} w(u_n,v_n).$$

By using this expression given above, we get

$$\begin{aligned}
\lim_{n\to\infty} w(u_n,v_n) &\leq \lim_{n\to\infty}(\alpha(w(u_{n-1},v_0),\ w(u_{n-1},u_n),\ w(v_0,v_n))) \\
&\leq \alpha(\lim_{n\to\infty} w(u_{n-1},v_0),\ \lim_{n\to\infty} w(u_{n-1},u_n),\ \lim_{n\to\infty} w(v_0,v_n)) \\
&\leq \alpha(0,0,\ w(v_0,\ \lim_{n\to\infty} v_n)) \\
&\leq \alpha(0,0,\ \lim_{n\to\infty} w(u_n,v_n)) \\
&\leq k(0) = 0.
\end{aligned}$$

Thus $w(u_n,v_n) \to 0$ as $n \to \infty$. Then by Lemma 2.5, we get that $v_n \to v_0$. As $v_n \in T(v_0)$, and $T(v_0)$ is closed, $v_0 \in T(v_0)$.

Theorem 3.2 of [9] becomes the special case of our Theorem 3.3 as bellow.

Corollary: 3.4 *Each K_w-map $T: M \to Cl(M)$ has a fixed point, provided that for any iterative sequence $\{u_n\}$ in M with $u_n \to v_0 \in M$, the sequence of real numbers $\{w(v_0,u_n)\}$ converges to zero.*

Theorem: 3.5 *Let $\{T_n\}$ be a sequence of multi-valued maps of M into $Cl(M)$. For any two maps $T_i, T_j \in \{T_n\}$ and for any $x, y \in M$ there exists $u \in T_i(x)$ and $v \in T_j(y)$ with*

$$w(u,v) \leq \alpha(w(x,y),\ w(x,u),\ w(y,v)),$$

and for each $n \geq 1$, $\inf\{w(x,u) + w(x,T_n(x)): x \in X\} > 0, \quad for\ any\ \ u \in T_n(u).$

Then the sequence $\{T_n\}$ has a common fixed point.

Proof. Let u_0 be an arbitrary element of M and let $u_1 \in T_1(u_0)$, then there is $u_2 \in T_2(u_1)$ such that $w(u_1, u_2) \le k\ w(u_0, u_1)$. So, there exist a sequence $\{u_n\}$ such that $u_{n+1} \in T_{n+1}(u_n)$ for all $n \ge 1$. Then as $n \to \infty$, we get that $\{u_n\}$ is a Cauchy sequence in X. Let $p = \lim_{n\to\infty} u_n$ in M. Now we show that $p \in \bigcap\ T_n(p)$. Let T_m be any arbitrary member of $\{T_n\}$. Since $u_n \in T_n(u_{n-1})$, by hypothesis there is $s_n \in T_m(p)$ such that

$$w(u_n, s_n) \le \alpha(w(u_{n-1}, p),\ w(u_{n-1}, u_n), w(p, s_n))$$

when $n \to \infty$, we have $w(u_{n-1}, p) \to 0$ and $w(u_{n-1}, u_n) \to o$

$$\begin{aligned}\lim_{n\to\infty} w(u_n, s_n) &\le \alpha(0,0, \lim_{n\to\infty} w(u_n, s_n))\\ &\le k(0) = 0.\end{aligned}$$

This implies that $w(u_n, s_n) \to 0$ as $n \to \infty$ and we get the following

$$w(u_n, p) \le \liminf w(u_n, u_m) \le \{k^n(1-k^m)/(1-k)\}\ w(u_0, u_1) \to 0, n \to \infty.$$

This implies that $w(u_n, p) \to 0$ as $n \to \infty$. Now, assume that $p \in T_m(p)$. Then by given hypothesis and for $n > m$ and $m \ge 1$, we have

$$\begin{aligned}0 &< inf\{w(u, p) + w(u, T_m(u)) : u \in X\}\\ &\le \inf\{w(u_{m-1}, p) + w(u_{m-1}, T_m(u_{m-1})) : m \in N\}\\ &\le \inf\{w(u_{m-1}, p) + w(u_{m-1}, u_m) : m \in N\}\\ &\le \inf\{k^{m-1}(1 - k^n/1 - k)\ w(u_0, u_1) + k^{m-1}\ w(u_0, u_1) : m \in N\} = 0.\end{aligned}$$

which is impossible and hence $p \in T_m(p)$. But T_m is an arbitrary, hence p is a common fixed point.

Next corollary, is the Theorem 3.3 of [9].

Corollary: 3.6 *Let $\{T_n\}$ be a sequence of multi-valued maps of M into $Cl(M)$. Suppose that there exist a constant $0 \le r < 1/2$ such that for any two maps $T_i, T_j \in \{T_n\}$ and for any $x \in M, u \in T_i(x)$, there exists $v \in T_j(y)$ for all $y \in M$ with*

$$w(u, v) \le r\ \{w(x, u) + w(y, v)\},$$

and for each $n \ge 1$, $\inf\{w(x, u) + w(x, T_n(x)) : x \in X\} > 0$,

for any $u \in T_n(u)$. Then the sequence $\{T_n\}$ has a common fixed point.

Theorem: 3.7 *Let $\{T_n\}$ be a sequence of multi-valued maps of M into $Cl(M)$. For any two maps $T_i, T_j \in \{T_n\}$ and for any $x, y \in M$ there exists $u \in T_i(x)$ and $v \in T_j(y)$ with*

$$w(u, v) \le \alpha(w(x, y),\ w(x, u),\ w(y, v)).$$

Then $\{T_n\}$ has a common fixed point, provided that for any iterative sequence $\{u_n\}$ in M with $u_n \to v_0 \in M$. The sequence of real numbers $\{w(v_0, u_n)\}$ converges to 0.

Proof. This result can be proved in a similar way, as Theorem 3.5 is proved.

Theorem 3.4 of [9] becomes the corollary of our theorem given above as follows.

Corollary 3.8 *Let $\{T_n\}$ be a sequence of multi-valued maps of M into $Cl(M)$. Suppose that there exit a constant $0 \le r < 1/2$ such that for any two maps $T_i,\ T_j$ and for any $x \in M,\ u \in T_i(x)$. There exists $v \in T_j(y)$ for all $y \in M$ with*

$$w(u.v) \le r\{w(x, u) + w(y, v)\}.$$

Then $\{T_n\}$ has a common fixed point, provided that for any iterative sequence $\{u_n\}$ in M with $u_n \to v_0 \in M$. The sequence of real numbers $\{w(v_0, u_n)\}$ converges to zero.

REFERENCES:

[1] M. Akram, A. A. Siddique, A. A. Zafar, Common fixed point theorems for self maps of a generalized metric space satisfying A-contraction type condition, Int. Journal of Math. Analysis, Vol. 5, (2011),757 - 763.

[2] M. Akram, A.A. Siddique, A. A. Zafar, A general class of contractions: *A*-contractions, Novi Sad J. Math. Vol. 38, (2008), 25-33.

[3] M. Akram, A. A. Siddique, A. A. Zafar, Some fixed point theorems for multi-valued A^*- mappings, Korean J. Math. Science Vol. 10, (2003),7-12.

[4] R. Bianchini, Su un problema di S.Reich riguardante la teori dei punti fissi, Boll. Un. Math, Ital., Vol. 5 (1972), 103-108.

[5] Y. Feng and S. Liu, Fixed point theorems for multi-valued contractive mappings and multi-valued Caristi Type mappings, J. Math. Anal. Appl., 317 (2006), 103-112.

[6] O. Kada, T. Susuki and W. Takahashi, Nonconvex minimization theorems and fixed point theorems in complete metric spaces, Math. Japon., 44 (1996), 381-391.

[7] R. Kannan, Some results on fixed points, Bull. math. Calcutta, 6 (1968), 405-408.

[8] T.H. Kim, K. Kim and J.S. Ume, Fixed point theorems on complete metric spaces, Panamer. Math. J., 7 (1997), 41-51.

[9] A. Latif, Fixed point results for multivalued maps, Int. J. Contemp. Math. Science, Vol. 2, (2007), no. 23, 1129-1136.

[10] S. B. Nadler, Multivalued contraction mappings, Pacific J. Math., 30 (1969), 475-488.

[11] Reich S., Kannan's fixed point theorem, Boll.Un.Math.Ital., Vol.4 (1971), 1-11.

[12] T. Suzuki and W. Takahashi, Fixed point theorems and characterizations of metric completeness, Topol. Methods Nonlinear Anal., 8 (1996), 371-382.

[13] T. Suzuki, Several fixed point theorems in complete metric spaces,Yokohama Math. J., 44 (1997), 61-72.

[14] W. Takahashi, Nonlinear functional analysis: Fixed point theory and its applications, Yokohama Publishers, 2000.

[15] J. S. Ume, B. S. Lee and S. J. Cho, Some results on fixed point theorems for multivalued mappings in complete metric spaces, IJMMS, 30 (2002),319-325.

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