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SOME FIXED POINT THEOREMS FOR MULTIVALUED MAPPINGS IN D*- METRIC SPACES

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ABSTRACT

In this paper we establish some Fixed Point Theorems for multivalued contraction and some generalized multivaled contraction mappings in D - metric space. In natural setting it has very important application in control theory which is initated by Nadler. Jr[18] and subsequently developed by Assad and Kirk[17]Browder[14] Goebel [16] Iami-Doze [17] in metric spaces. In what follows (X, D*) will denote D* - metric space, (CB(X),H) is a Hausdorff metric space introduced by D*-- metric ,N, the set of all natural number and R^+, the set of all positive real number.*

Mathematics Subject Classification: *54E40; 54E35; 54H25*

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1. INTRODUCTION:

There have been an number of generalization in generalized metric space (or D-Metric space) initiated by Dhage [2] in 1992. He proved the existence of unique fixed point theorems of a self map satisfying a contractive condition in complete and bounded D- Metric space. Dealing with D- Metric space, B. Ahmad M. Ashraf, and B.E. Rhoades [1] , Dhage [2,3,4] Rhoades [8], Singh and Sharma [9] and others made a significant contribution in fixed point theory of D-Metric space. Unfortunately almost all theorems in D-Metric space are not valid (See S.V.R Naidu and others [5-7]). Here our aim is to prove some common fixed point theorems using some generalized contractive conditions in D*-Metric space as a probable

Definition: 1.1 Let X be a non empty set. A generalized metric (or D* - metric) on X is a function $D^*: X^3 \to [0,\infty)$ that satisfies the following conditions for each $x, y, z, a \in X$.
1. $D^*(x, y, z) \geq 0$
2. $D^*(x, y, z) = 0$ if and only if $x = y = z$
3. $D^*(x, y, z) = D^*(\rho\{x, y, z\})$ where ρ is permutation function.
4. $D^*(x, y, z) \leq D^*(x, y, a) + D^*(a, z, z)$.

The pair (X, D*) is called generalized metric (or D*- metric) space.

Examples: 1.2
(a) $D^*(x, y, z) = \max\{d(x, y), d(y, z), d(z, x)\}$,
(b) $D^*(x, y, z) = d(x, y) + d(y, z) + d(z, x)$. Here, d is the ordinary metric on X.
(c) If $X = R^n$ then we define

$$D^*(x, y, z) = (\|x - y\|^p + \|y - z\|^p + \|z - x\|^p)^{1/p} \text{ for every } p \in R^+$$

ce $D^*(x, y, y) = D^*(x, x, y)$

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Definition: 1.4 A open ball in a D* - metric space X with centre x and radius r is denoted by B_{D^*} (x, r) and is defined by $B_{D^*}(x, r) = \{y \in X: D^*(x, y, y) < r\}$.

Example: 1.5 Let X = R Denote $D^*(x, y, z) = |x - y| + |y - z| + |z - x|$ for all $x, y, z \in R$.

Thus $B_{D^*}(0, 2) = \{y \in R / D^*(0, y, y) < 2\}$

$= \{y \in R / |0 - y| + |y - y| + |y - 0| < 2\}$

$= \{y \in R / |y| + |y| < 2\}$

$= \{y \in R / |y| < 1\}$

$= \{y \in R / -1 < y < 1\}$

$= (-1, 1)$

Definition: 1.6 Let (X, D*) be a D* - metric space and $A \subseteq X$

1. If for every $x \in A$, there exist $r > 0$ such that $B_{D^*}(x, r) \subseteq A$, then subset A is called open subset of X.
2. Subset A of X is said to be D* - bounded if there exist $r > 0$ such that

$D^*(x, y, y) < r$ for all $x, y \in A$.

3. A subset A of a D*-Metric Space X is closed in X if A^C is open in X
4. A sequence $\{x_n\}$ in X converges to x if and only if $D^*(x_n, x_n, x) = D^*(x, x, x_n) \to 0$ as $n \to \infty$. That is, for each $\varepsilon > 0$ there exist $n_0 \in N$ such that for all $n \geq n_0$ implies $D^*(x, x, x_n) < \varepsilon$. This is equivalent for each $\varepsilon > 0$, there exist $n_0 \in N$ such that for all $n, m \geq n_0$ implies $D^*(x, x_n, x_m) < \varepsilon$ .It is also noted that $D^*(x_n, x_n, x) = D^*(x, x, x_n) < \varepsilon$ for all $n \geq n_0$, for some $n_0 \in N$,
5. A square $\{x_n\}$ in X is called a Cauchy sequence if for each $\varepsilon > 0$, there exist $n_0 \in N$ such that $D^*(x_n, x_n, x_m) < \varepsilon$ for each $n, m \geq n_0$ The D* - metric space (X, D*) is said to complete if every Cauchy sequence is convergent.

Remark: 1.7

(1) D* is continuous function on X^3
(2) If sequence $\{x_n\}$ in X converges to x, then x is unique.
(3) Any convergent sequence in (X, D*) is a Cauchy sequence.

Definition: 1.8 Let CB(X) be set of all non empty bounded closed subsets of D*-Metric space X Define H: CB(X)x CB(X)x CB(X) $\to[0, \infty)$ by $H(A,B,C) = \inf\{\varepsilon / AUB \subseteq B_{D^*}(C, \varepsilon), AUC \subseteq B_{D^*}(B, \varepsilon)$ and $BUC \subseteq B_{D^*}(A, \varepsilon)\}$for A, B, $C \in CB(X)$ where $B_{D^*}(C, \varepsilon) = \{x \in X: D^*(x, c, c) < \varepsilon$ for some $c \in C$, where $\varepsilon>0\}$ Clearly H is metric on CB(X) and it is called Hausdorff metric on CB(X).

Definition: 1.9 Let (X,D*) and (Y,D*) be two D*-Metric spaces. Let $T : X \to CB(Y)$ .Then T is said to be multivalued contraction mapping if and only if $H(Tx, Ty, Tz) \leq k D^*(x, y. z)$ for all $x, y, z \in X$ **and 0** $\leq k < 1$.

Definition: 1.10 A multivalued map T: X → CB(X) is orbitaly continuous if $x_n, \to x$ implies $T x_n, \to Tx$

Definition: 1.11 A point x in X is a fixed point of a multivalued map T: X →CB(X) if $x \in T(x)$

Example: 1.12 Let X = [0, 1] and f: X → X be a map such that

$$f(x) = \begin{cases} ½(x) + ½ & \text{for } 0 \leq x \leq 1/2 \\ -½(x) + 1 & \text{for } ½ \leq x \leq 1 \end{cases}$$

Define T: X → CB(X) by T(x) = {0} U {f(x)} for all $x \in X$. Then $2/3 \in T(2/3)$

Clearly T is a multivalued map from X to CB(X) and 2/3 is fixed point of T.

2. MAIN RESULTS:

Theorem: 2.1 Let (X, D*) be a D*-complete metric space s) and (CB(X), H) be a Haussdorf metric space induced by D*- metric .Let T : X → CB(X) be a orbitaly continuous map such that $H(Tx, Ty, Tz) \leq q D^*(x, y, z)$ for all $x, y, z \in X$ and $0 \leq q < 1$. Then T has a unique fired point

Proof: Let $x_0 \in X$ a fixed arbitrary element.

Define the sequence $\{x_n\}$ in X as $x_{n+1} \in Tx_n$ for n = 0, 1, 2. . .

For $n \geq 0$ we have,

$$
\begin{aligned}
D^*(x_n, x_n, x_{n+1}) &\leq H(Tx_{n-1}, Tx_{n-1}, Tx_n) \\
&\leq q\, D^*(x_{n-1}, x_{n-1}, x_n) \\
&\;\;. \\
&\leq q^n D^*(x_0, x_0, x_1) \to 0, \text{ as } n \to \infty.
\end{aligned}
$$

For $m > n$, we have

$$
\begin{aligned}
D^*(x_n, x_n, x_m) &\leq D^*(x_n, x_n, x_{n+1}) + D^*(x_{n+1}, x_{n+1}, x_m) \\
&\leq D^*(x_n, x_n, x_{n+1}) + D^*(x_{n+1}, x_{n+1}, x_{n+2}) + . \;. + D^*(x_{m-1}, x_{m-1}, x_m) \to 0 \text{ as } m, n \to \propto.
\end{aligned}
$$

Thus $\{x_n\}$ is D*- Cauchy sequence in X. Since X is D*- complete $x_n \to x$ in X and T is orbitaly continuous $T x_n \to Tx$

Now we prove that $x \in Tx$.

$$
\begin{aligned}
H(Tx, x, x) &= \lim_{n\to\infty} H(Tx_n, x_{n+1}, x_{n+1}) \\
&\leq \lim_{n\to\infty} H(Tx_n, Tx_n, Tx_n) \\
&= 0
\end{aligned}
$$

Thus $x \in Tx$

Now we prove the uniqueness. Suppose $x \neq y$ such that $y \in Ty$

$$
\begin{aligned}
D^*(x, y, y) &\leq H(Tx, Ty, Ty) \\
&\leq q\, \{D^*(x, y, y) \\
&< D^*(x, y, y), \text{ which is contradiction.}
\end{aligned}
$$

Hence T has a unique fixed point.

Theorem: 2.2 Let (X, D*) be a D*-complete metric space and (CB(X), H) be a Haussdorf metric space induced by D*- metric .Let T: X→ CB(X) be a orbitaly continuous map such that $H(Tx, Ty, Tz) \leq q\{D^*(x, y, z) + D^*(a, b, c)\}$ for all $x, y, z \in X$ and for all $a \in Tx$, $b \in Ty$, $c \in Tz$ and $0 \leq q < 1./2$ Then T has a unique fired point

Proof: Let $x_0 \in X$ a fixed arbitrary element.

Define the sequence $\{x_n\}$ in X as $x_{n+1} \in Tx_n$ for n = 0, 1, 2 . . .

For $n \geq 0$ we have,

$$
\begin{aligned}
D^*(x_n, x_{n+1}, x_{n+1}) &\leq H(Tx_{n-1}, Tx_n, Tx_n) \\
&\leq q\, \{D^*(x_{n-1}, x_n, x_n) + D^*(x_n, x_{n-+1}, x_{n+1})\}
\end{aligned}
$$

$$
\begin{aligned}
(1-q)\, D^*(x_n, x_{n+1}, x_{n+1}) &\leq q\, D^*(x_{n-1}, x_n, x_n) \\
D^*(x_n, x_{n+1}, x_{n+1}) &\leq (q/1-q)\; D^*(x_{n-1}, x_n, x_n) \\
&\leq b\, D^*(x_{n-1}, x_n, x_n). \quad \text{where } b = (q/1-q) < 1. \\
&\;\;. \\
&\;\;. \\
&\;\;. \\
&\leq b^n D^*(x_0, x_1, x_1) \to 0, \text{ as } n \to \infty.
\end{aligned}
$$

For $m > n$, we have

$$
\begin{aligned}
D^*(x_n, x_n, x_m) &\leq D^*(x_n, x_n, x_{n+1}) + D^*(x_{n+1}, x_{n+1}, x_m) \\
&\leq D^*(x_n, x_n, x_{n+1}) + D^*(x_{n+1}, x_{n+1}, x_{n+2}) + . \;. \;. + D^*(x_{m-1}, x_{m-1}, x_m) \to 0 \text{ as } m, n \to \propto.
\end{aligned}
$$

Thus $\{x_n\}$ is D*- Cauchy sequence in X.

Since X is D*- complete $x_n \to x$ in X and T is orbitaly continuous $Tx_n \to Tx$

Now we prove that $x \in Tx$

$$H(Tx, x, x) = \lim_{n\to\infty} H(Tx_n, x_{n+1}, x_{n+1})$$
$$\leq \lim_{n\to\infty} H(Tx_n, Tx_n, Tx_n)$$
$$=0.$$

Thus $x \in Tx$

.

Now we prove the uniqueness. Suppose $x \neq y$ such that that $y \in Ty$

$$D^*(x, y, y) \leq q\, H(Tx, Ty, Ty)$$
$$\leq q\,\{D(x, y, y) + D^*(x, y, y)\}$$
$$\leq 2qD^*(x, y, y)$$
$$< D^*(x, y, y), \text{ which is contradiction.}$$

Thus T has a uniqe fixed point.

Theorem: 2.3 Let (X, D*) be a D*-complete metric space and (CB(X), H) be a Haussdorf metric space induced by D*-metric .Let T: X→CB(X) be a orbitaly continuous map such that
$H(Tx, Ty, Tz) \leq q\{D^*(x, y, z) + D^*(x, b, c) + D^*(a, b, c)\}$ for all $x, y, z \in X$ and for all $a \in Tx$, $b \in Ty$, $c \in Tz$
and $0 \leq q < 1./4$ Then T has a unique fired point

Proof: Let $x_0 \in X$ a fixed arbitrary element.

Define the sequence $\{x_n\}$ in X as $x_{n+1} \in Tx_n$ for n = 0, 1, 2. . .

For $n \geq 0$ we have,

$$D^*(x_n, x_{n+1}, x_{n+1}) \leq H(Tx_{n-1}, Tx_n, Tx_n)$$
$$\leq q\,\{D^*(x_{n-1}, x_n, x_n) + D^*(x_{n-1}, x_{n-+1}, x_{n+1}) + D^*(x_n, x_{n+1}, x_{n+1})\}$$
$$= q\,\{2\,D^*(x_n, x_{n+1}, x_{n+1}) + 2\,D^*(x_{n-1}, x_n, x_n)\}$$
$$\leq 2q\,D^*(x_{n-1}, x_n, x_n) + 2q\,D^*(x_n, x_{n+1}, x_{n+1})\}$$

$$(1-2q)\,D^*(x_n, x_{n+1}, x_{n+1}) \leq 2q\,D^*(x_{n-1}, x_n, x_n)$$
$$D^*(x_n, x_{n+1}, x_{n+1}) \leq (2q/1-2q)\,D^*(x_{n-1}, x_n, x_n)$$
$$\leq b\,D^*(x_{n-1}, x_n, x_n). \quad \text{where } b = (2q/1-2q) < 1.$$
$$\vdots$$
$$\leq b^n D^*(x_0, x_1, x_1) \to 0, \text{ as } n \to \infty.$$

For $m > n \geq n_0$, we have

$$D^*(x_n, x_n, x_m) \leq D^*(x_n, x_n, x_{n+1}) + D^*(x_{n+1}, x_{n+1}, x_m)$$
$$\leq D^*(x_n, x_n, x_{n+1}) + D^*(x_{n+1}, x_{n+1}, x_{n+2}) + \ldots + D^*(x_{m-1}, x_{m-1}, x_m) \to 0 \text{ as } m, n \to \propto.$$

Thus $\{x_n\}$ is D*- Cauchy sequence in X.

Since X is D*- complete $x_n \to x$ in X and T is orbitaly continuous $Tx_n \to Tx$

Now we prove that $x \in Tx$

$$H(Tx, x, x) = \lim_{n\to\infty} H(Tx_n, x_{n+1}, x_{n+1},)$$
$$\leq \lim_{n\to\infty} H(Tx_n\ Tx_n, Tx_n)$$
$$=0.$$

Thus $x \in Tx$

.

Now we prove the uniqueness. Suppose $x \neq y$ such that that $y \in Ty$

$$\begin{aligned} D^*(x, y, y) &\leq q \{H(Tx, Ty, Ty) \\ &\leq q \{D(x, y, y) + D^*(x, y, y) + D^*(x, y, y)\} \\ &\leq 3qD^*(x, y, y) \\ &< D^*(x, y, y), \text{ which is contradiction.} \end{aligned}$$

Thus T has a uniqe fixed point.

Theorem: 2.4 Let (X, D^*) be a D^*- complete metric space and $(CB(X), H)$ be a Haussdorf metric space induced by D^*- metric. Let $T: X \to CB(X)$ be a orbitaly continuous map such that $H(Tx, Ty, Tz) \leq q\{D^*(x, y, z) + D^*(x, b, c) + D^*(a, y, c) + D^*(a, b. c)\}$ for all $x, y, z \in X$ and for all $a \in Tx$, $b \in Ty$, $c \in Tz$ and $0 \leq q < 1./5$ Then T has a unique fired point.

Proof: Let $x_0 \in X$ a fixed arbitrary element.

Define the sequence $\{x_n\}$ in X as $x_{n+1} \in Tx_n$ for $n = 0, 1, 2 \ldots$

For $n \geq 0$ we have,

$$\begin{aligned} D^*(x_n, x_{n+1}, x_{n+1}) &\leq H(Tx_{n-1}, Tx_n, Tx_n) \\ &\leq q\{D^*(x_{n-1}, x_n, x_n) + D^*(x_{n-1}, x_{n-+1}, x_{n+1}) + D^*(x_n, x_n, x_{n+1}) + D^*(x_n, x_{n-+1}, x_{n+1})\} \\ &\leq q\{2D^*(x_{n-1}, x_n, x_n) + 3D^*(x_n, x_n, x_{n+1})\} \end{aligned}$$

$$\begin{aligned} (1-3q)\, D^*(x_n, x_{n+1}, x_{n+1}) &\leq 2q\, D^*(x_{n-1}, x_n, x_n) \\ D^*(x_n, x_{n+1}, x_{n+1}) &\leq (2q/1-3q)\, D^*(x_{n-1}, x_n, x_n) \\ &\leq b\, D^*(x_{n-1}, x_n, x_n). \quad \text{where } b = (2q/1-3q) < 1. \\ &. \\ &. \\ &. \\ &\leq b^n D^*(x_0, x_1, x_1) \to 0, \text{ as } n \to \infty. \end{aligned}$$

For $m > n$, we have

$$\begin{aligned} D^*(x_n, x_n, x_m) &\leq D^*(x_n, x_n, x_{n+1}) + D^*(x_{n+1}, x_{n+1}, x_m) \\ &\leq D^*(x_n, x_n, x_{n+1}) + D^*(x_{n+1}, x_{n+1}, x_{n+2}) + \ldots + D^*(x_{m-1}, x_{m-1}, x_m) \to 0 \text{ as } m, n \to \propto. \end{aligned}$$

Thus $\{x_n\}$ is D^*- Cauchy sequence in X.

Since X is D^*- complete $x_n \to x$ in X and T is orbitaly continuous $Tx_n \to Tx$

Now we prove that $x \in Tx$

$$\begin{aligned} H(Tx, x, x) &= \lim_{n\to\infty} H(Tx_n, x_{n+1}, x_{n+1}) \\ &\leq \lim_{n\to\infty} H(Tx_n\ Tx_n, Tx_n) \\ &= 0. \end{aligned}$$

Thus $x \in Tx$

.

Now we prove the uniqueness. Suppose $x \neq y$ such that that $y \in Ty$

$$\begin{aligned} D^*(x, y, y) &\leq \{H(Tx, Ty, Ty) \\ &\leq q\{D^*(x, y, y) + D^*(x, y, y) + D^*(x, y, y) + D^*(x, y, y)\} \\ &\leq 4q\, D^*(x, y, y) \\ &< D^*(x, y, y) \end{aligned}$$

Thus T has a uniqe fixed point.

Theorem: 2.5 Let (X, D*) be a D*- complete metric space and (CB(X), H) be a Haussdorf metric space induced by D*- metric .Let $T : X \to CB(X)$ be a orbitaly continuous map such that
$H (Tx , Ty , Tz) \leq q \quad D^*(x, y, z) + D^*(x, b , c) + D^* (a , y , c) + D^*(a, b., z) + D^*(a , b. c) \}$ for all $x, y, z \in X$ and for all $a \in Tx$, $b \in Ty$, $c \in Tz$ and $0 \leq q < 1./6$ Then T has a unique fired point

Proof: Let $x_0 \in X$ a fixed arbitrary element.

Define the sequence $\{x_n\}$ in X as $x_{n+1} \in Tx_n$ for n = 0, 1, 2. . .

For $n \geq 0$ we have,

$$
\begin{aligned}
D^*(x_n, x_{n+1}, x_{n+1}) &\leq H(Tx_{n-1}, Tx_n, Tx_n) \\
&\leq q \{D^*(x_{n-1}, x_n, x_n) + D^*(x_{n-1}, x_{n-+1}, x_{n+1}) + D^*(x_n , x_n, x_{n+1}) + D^*(x_n , x_{n-+1}, x_n) + D^*(x_n , x_{n-+1}, x_{n+1})\} \\
&\leq q \{2 D^*(x_{n-1}, x_n, x_n) + 4 D^*(x_n , x_n, x_{n+1})\}
\end{aligned}
$$

$$
\begin{aligned}
(1- 4q) D^*(x_n, x_{n+1}, x_{n+1}) &\leq 2q D^*(x_{n-1}, x_n , x_n) \\
D^*(x_n, x_{n+1}, x_{n+1}) &\leq (2q/1- 4q) D^*(x_{n-1}, x_n , x_n) \\
&\leq b D^*(x_{n-1}, x_n , x_n). \quad \text{where } b = (2q/1-4q) < 1. \\
&\;\;\vdots \\
&\leq b^n D^* (x_0, x_1, x_1) \to 0, \text{ as } n \to \infty.
\end{aligned}
$$

For $m > n$, we have

$$
\begin{aligned}
D^*(x_n, x_n, x_m) &\leq D^*(x_n, x_n, x_{n+1}) + D^*(x_{n+1}, x_{n+1}, x_m) \\
&\leq D^*(x_n, x_n, x_{n+1}) + D^*(x_{n+1}, x_{n+1}, x_{n+2}) + \ldots + D^*(x_{m-1}, x_{m-1}, x_m) \to 0 \text{ as } m, n \to \propto.
\end{aligned}
$$

Thus $\{x_n\}$ is D*- Cauchy sequence in X.

Since X is D*- complete $x_n \to x$ in X and T is orbitaly continuous $Tx_n \to Tx$

Now we prove that $x \in Tx$

$$
\begin{aligned}
H (Tx, x, x) &= \lim_{n\to\infty} H(Tx_n, x_{n+1}, x_{n+1,}) \\
&\leq \lim_{n\to\infty} H(Tx_n \; Tx_n, Tx_n) \\
&=0.
\end{aligned}
$$

Thus $x \in Tx$
.
Now we prove the uniqueness. Suppose $x \neq y$ such that that $y \in Ty$

$$
\begin{aligned}
D^*(x, y, y) &\leq \{H (Tx, Ty, Ty) \\
&\leq q \{D^*(x, y, y) + D^*(x, y, y) + D^*(x, y, y) + D^*(x, y, y) \} \\
&\leq 4q D^*(x, y, y) \\
&< D^*(x, y, y)
\end{aligned}
$$

Thus T has a uniqe fixed point.

Theorem 2.6: Let (X, D*) be a D*-complete metric space and (CB(X), H) be a Haussdorf metric space induced by D*- metric .Let $T : X \to CB(X)$ be a orbitaly continuous map such that $H(Tx, Ty, Tz) \leq a_1 D^*(x, y, z) + (a_2 /2) \{D^*(x, b , c) + D^*(a, y, c)\} + a_3 \{D^*(a, b , z) + D^*(a , b , c)\}$ for all $a \in Tx$, $b \in Ty$, $c \in Tz$ and for all $x, y, z \in X$, and $0 \leq a_1 + 2a_2 + 2a_3 < 1$. Then T has a unique fixed point.

Proof: Let $x_0 \in X$ be a fixed arbitrary element.

Define the sequence $\{x_n\}$ in X as $x_{n+1} \in Tx_n$, for n = 0, 1, 2 . . .

For $n \geq 0$ we have

$$\begin{aligned}D^*(x_n, x_n, x_{n+1}) &\le H(Tx_{n-1}, Tx_{n-1}, Tx_n)\\ &\le a_1 D^*(x_{n-1}, x_{n-1}, x_n) + a_2\{D^*(x_{n-1}, x_n, x_{n+1}) + D^*(x_n, x_{n-1}, x_{n+1})\} + a_3\{D^*(x_n, x_n, x_n)\\ &\qquad + D^*(x_n, x_n, x_{n+1})\}\\ &= a_1 D^*(x_{n-1}, x_{n-1}, x_n) + a_2 D^*(x_{n-1}, x_n, x_{n+!}) + a_3 D^*(x_n, x_n, x_{n+1})\\ &\le (a_1 + a_2) D^*(x_{n-1}, x_{n-1}, x_n) + (a_2 + a_3) D^*(x_n, x_n, x_{n+1})\end{aligned}$$

$$\begin{aligned}(1 - a_2 - a_3) D^*(x_n, x_n, x_{n+1}) &\le (a_1 + a_2) D^*(x_{n-1}, x_{n-1}, x_n)\\ D^*(x_n, x_n, x_{n+1}) &\le ((a_1 + a_2)/(1 - a_2 - a_3)) D^*(x_{n-1}, x_{n-1}, x_n)\\ D^*(x_n, x_n, x_{n+1}) &\le a D^*(x_{n-1}, x_{n-1}, x_n), \text{ where } a = ((a_1 + a_2)/(1 - a_2 - a_3)) < 1.\\ &\ \vdots\\ &\le a^n D^*(x_0, x_0, x_1) \to 0 \text{ as } n \to \propto\end{aligned}$$

Now we prove that $\{x_n\}$ is D^* - Cauchy sequence in X .

For $m > n$, we have

$$\begin{aligned}D^*(x_n, x_n, x_m) &\le D^*(x_n, x_n, x_{n+1}) + D^*(x_{n+1}, x_{n+1}, x_m)\\ &\le D^*(x_n, x_n, x_{n+1}) + D^*(x_{n+1}, x_{n+1}, x_{n+2}) + \ldots + D^*(x_{m-1}, x_{m-1}, x_m) \to 0 \text{ as } n, m \to \infty.\end{aligned}$$

Thus $\{x_n\}$ is D^* - Cauchy sequence in X.

Since X is D^*- complete $x_n \to x$ in X and T is orbitaly continuous $Tx_n \to Tx$

Now we prove that $x \in Tx$

$$\begin{aligned}H(Tx, x, x) &= \lim_{n\to\infty} H(Tx_n, x_{n+1}, x_{n+1})\\ &\le \lim_{n\to\infty} H(Tx_n\ Tx_n, Tx_n)\\ &= 0.\end{aligned}$$

Thus $x \in Tx$

Uniqueness:

Suppose $x \ne y$ such that $y \in Ty$.

$$\begin{aligned}D^*(x, y, y) &\le H(Tx, Ty, Ty)\\ &\le a_1 D^*(x, y, y) + a_2\{D^*(x, y, y) + D^*(x, y, y)\} + a_3\{D^*(x, y, y) + D^*(x, y, y)\}\\ &= (a_1 + 2a_2 + 2a_3) D^*(x, y, y)\\ &< D^*(x, y, y), \text{ which is contradiction.}\end{aligned}$$

Thus T has a unique fixed point.

Theorem: 2.7 Let (X, D^*) be a D^*-complete metric space and $(CB(X), H)$ be a Haussdorf metric space induced by D^*- metric .Let T $T: X \to CB(X)$ be a orbitaly continuous map and for all $a \in Tx$, $b \in Ty$, $c \in Tz$ such that $H(Tx, Ty, Tz) \le a_1 D^*(x, y, z) + a_2 D^*(x, b, c) + a_3 D^*(a, y, c) + a_4 D^*(a, b, z) + a_5 D^*(a, b, c)$ for all $x, y, z \in X$, and $0 \le a_1 + 2a_2 + a_3 + a_4 + a_5 < 1$. Then T has a unique fixed point.

Proof: Let $x_0 \in X$ be a fixed arbitrary element.

Define the sequence $\{x_n\}$ in X as $x_{n+1} \in Tx_n$, for $n = 0, 1, 2 \ldots$

For $n \ge 0$ we have

$$\begin{aligned}D^*(x_n, x_{n+1}, x_{n+1}) &\le H(Tx_{n-1}, Tx_n, Tx_n)\\ &\le a_1 D^*(x_{n-1}, x_n, x_n) + a_2 D^*(x_{n-1}, x_{n+1}, x_{n+1}) + a_3 D^*(x_n, x_n, x_{n+1}) + a_4\{D^*(x_n, x_{n+1}, x_n)\\ &\qquad + a_5 D^*(x_n, x_{n+1}, x_{n+1})\}\\ &\le (a_1 + a_2) D^*(x_{n-1}, x_n, x_n) + (a_2 + a_3 + a_4 + a_5) D^*(x_n, x_{n+1}, x_{n+1})\end{aligned}$$

$(1 - a_2 - a_3 - a_4 - a_5)\, D^*(x_n, x_{n+1}, x_{n+1}) \le (a_1 + a2)\, D^*(x_{n-1}, x_n, x_n)$

$$D^*(x_n, x_n, x_{n+1}) \le ((a_1 + a_2)/(1 - a_2 - a_3 - a_4 - a_5))\, D^*(x_{n-1}, x_{n-1}, x_n)$$

$$D^*(x_n, x_n, x_{n+1}) \le a\, D^*(x_{n-1}, x_{n-1}, x_n), \text{ where } a = ((a_1 + a_2)/(1 - a_2 - a_3 - a_4 - a_5)) < 1.$$

$$\vdots$$

$$\le a^n D^*(x_0, x_0, x_1) \to 0 \text{ as } n \to \propto$$

Now we prove that $\{x_n\}$ is D* - Cauchy sequence in X .

For m > n, we have

$$D^*(x_n, x_n, x_m) \le D^*(x_n, x_n, x_{n+1}) + D^*(x_{n+1}, x_{n+1}, x_m)$$
$$\le D^*(x_n, x_n, x_{n+1}) + D^*(x_{n+1}, x_{n+1}, x_{n+2}) + \ldots + D^*(x_{m-1}, x_{m-1}, x_m) \to 0 \text{ as } n, m \to \infty.$$

Thus $\{x_n\}$ is D* - Cauchy sequence in X.

Since X is D*- complete $x_n \to x$ in X and T is orbitaly continuous $Tx_n \to Tx$

Now we prove that $x \in Tx$

$$H(Tx, x, x) = \lim_{n\to\infty} H(Tx_n, x_{n+1}, x_{n+1})$$
$$\le \lim_{n\to\infty} H(Tx_n\ Tx_n, Tx_n)$$
$$= 0.$$

Thus $x \in Tx$

Uniqueness:

Suppose $x \ne y$ such that $y \in Ty$.

$$D^*(x, y, y) \le H(Tx, Ty, Ty)$$
$$\le a_1 D^*(x, y, y) + a_2 D^*(x, y, y) + a_3 D^*(x, y, y)\} + a_4 \{D^*(x, y, y) + a_5 D^*(x, y, y)\}$$

$$= (a_1 + a_2 + a_3 + a_4 + a_5)\, D^*(x, y, y)$$
$$< D^*(x, y, y), \text{ which is contradiction.}$$

Thus. T has a unique fixed point.

Theorem: 2.8 Let (X, D*) be a complete D* - metric space and (CB(X), H) be a Haussdorf metric space induced by D*- metric .Let T and T: X→X be a map such that

$H(Tx, T^2x, T^3x) \le a\, D^*(x, Tx, T^2x)$ for all $x \in X$ and $0 \le a < 1$. Then T has a unique fixed point.

Proof: Let $x_0 \in X$ be a fixed arbitrary element.

Define the sequence $\{x_n\}$ in X as $x_{n+1} = Tx_n$ for n = 0, 1, 2. . .

For $n \ge 0$, we have

$D^*(x_n, x_n, x_{n+1}) = D^*(Tx_{n-1}, Tx_{n-1}, Tx_n)$

Theorem: 2.7 Let (X, D*) be a complete D* - metric space and (CB(X), H) be a Haussdorf metric space induced by D*- metric .Let T : X→CB(X)be a map and orbital continuous such that $H(Tx, T^2x, T^3x) \le a\, D^*(x, Tx, T^2x)$ for all $x \in X$ and $0 \le a < 1$. Then T has a unique fixed point.

Proof: Let $x_0 \in X$ be a fixed arbitrary element.

Define the sequence $\{x_n\}$ in X as $x_{n+1} \in Tx_n$ for n = 0, 1, 2. . .

For $n \geq 0$, we have

$$\begin{aligned} D^*(x_n, x_n, x_{n+1}) &\leq H(Tx_{n-1}, T^2 x_{n-2}, T^3 x_{n-2}) \\ &\leq a\, D^*(x_{n-1}, Tx_{n-2}, T^2 x_{n-2}) \\ &\;\;\vdots \\ &\leq a^n D^*(x_0, x_0, x_1) \to 0 \text{ as } n \to \infty. \end{aligned}$$

For $m > n$ we have

$$D^*(x_n, x_n, x_m) \leq D^*(x_n, x_n, x_{n+1}) + D^*(x_{n+1}, x_{n+1}, x_{n+2}) + \ldots + D^*(x_{m-1}, x_{m-1}, x_m) \to 0 \text{ as } m, n \to \propto$$

Hence $\{x_n\}$ is a Cauchy sequence in D* - complete metric space.

Thus $x_n \to x$ in X.

Now we prove $Tx = x$. Supper $x \neq Tx$

$$\begin{aligned} D^*(x, x, Tx) &= \lim_{n\to\infty} D^*(x_{n+3}, x_{n+2}, Tx) \\ &= \lim_{n\to\infty} D^*(T^3x_n, T^2x_n, Tx) \\ &\leq a \lim_{n\to\infty} D^*(T^2x_n, Tx_n, x) \\ &= a \lim_{n\to\infty} D^*(x_{n+2}, x_{n+1}, x) = 0 \end{aligned}$$

Thus $x = Tx$

Uniqueness:

Supper $x \neq y$ such that $Ty = y$.

Then
$$\begin{aligned} D^*(x, y, y) &= D^*(T^3x, T^2y, Ty) \\ &\leq a\, D^*(T^2x, Ty, y) \\ &= a\, D^*(x, y, y) \end{aligned}$$

This implies

$(1- a)\, D^*(x, y, y) \leq 0$

Hence $1 - a < 0$ (since $D^*(x, y, y) > 0$)

Therefore $a > 1$. This is contradiction to $a < 1$.

Thus T has a unique fixed point .

Theorem: 2.5 Let (X, D*) be a complete D* - metric space and (CB(X), H) be a Haussdorf metric space induced by D*- metric .Let $T : X \to X$ be a map such that $H(Tx, Ty, Tz) \leq a \max\{D^*(x, y, z), D^*(x, Tx, Ty), D^*(y, Ty, Tz), D(x, y, Ty), D(y, z, Tz)\}$ for all $x, y, z \in X$ and $0 \leq a < \frac{1}{2}$. Then T has a unique fixed point.

Proof: Let $x_0 \in X$ be a fixed arbitrary element.

Define the sequence $\{x_n\}$ in X as $x_{n+1} = Tx_n$, for n = 0, 1, 2 . . .

For $n \geq 0$, we have

$$\begin{aligned} D^*(x_n, x_n, x_{n+1}) &= D^*(Tx_{n-1}, Tx_{n-1}, Tx_n) \\ &\leq a \max\{D^*(x_{n-1}, x_{n-1}, x_n), D^*(x_{n-1}, Tx_{n-1}, Tx_{n-1}), D^*(x_{n-1}, Tx_{n-1}, Tx_n), D^*(x_{n-1}, x_{n-1}, Tx_{n-1}), \\ &\qquad D^*(x_{n-1}, x_n, Tx_n)\} \\ &= a \max\{D^*(x_{n-1}, x_{n-1}, x_n), D^*(x_{n-1}, x_n, x_n), D^*(x_{n-1}, x_n, x_{n+1}), D^*(x_{n-1}, x_{n-1}, x_n)\, D^*(x_{n-1}, x_n, x_{n+1})\} \end{aligned}$$

$= a \max\{D^*(x_{n-1}, x_{n-1}, x_n), D^*(x_{n-1}, x_n, x_{n+1})\}$
$\leq a \max\{D^*(x_{n-1}, x_{n-1}, x_n), D^*(x_{n-1}, x_{n-1}, x_n) + D^*(x_n, x_n, x_{n+1})\}$
$\leq a D^*(x_{n-1}, x_{n-1}, x_n) + a D^*(x_n, x_n, x_{n+1})$

$(1 - a) D^*(x_n, x_n, x_{n+1}) \leq a D^*(x_{n-1}, x_{n-1}, x_n)$

$$D^*(x_n, x_n, x_{n+1}) \leq \frac{a}{1-a} D^*(x_{n-1}, x_{n-1}, x_n)$$

$$\leq b D^*(x_{n-1}, x_{n-1}, x_n) \text{ where } b = \frac{a}{1-a} < 1 \text{ for all n}$$

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$$\leq b^n D^*(x_0, x_0, x_1) \to 0 \text{ as } n \to \propto$$

Now we prove that $\{x_n\}$ is D^* - Cauchy sequence in X.

For m > n we have,

$D^*(x_n, x_n, x_m) \leq D^*(x_n, x_n, x_{n+1}) + D^*(x_{n+1}, x_{n+1}, x_{n+2}) +. \; . \; . + D^*(x_{m-1}, x_{m-1}, x_m) \to 0$ as m. $n \to \propto$

Thus $\{x_n\}$ is a D^* Cauchy sequence in X and X is D^* - complete $x_n \to x$ in X .

Now we shall prove that Tx = x

$$D^*(Tx, x, x) = \lim_{n\to\infty} D^*(Tx, x_{n+1}, x_{n+1})$$
$$= \lim_{n\to\infty} D^*(Tx, Tx_n, Tx_n)$$
$$\leq a \lim_{n\to\infty} \max\{D^*(x, x_n, x_n), D^*(x, Tx, Tx_n), D^*(x_n, Tx_n, Tx_n) D^*(x, x_n, Tx_n), D^*(x_n, x_n, Tx_n)\}$$
$$\leq a \lim_{n\to\infty} \max\{D^*(x, x_n, x_n), D^*(x, Tx, x_{n+1}), D^*(x_n, x_{n+1}, x_{n+1}), *(x, x_n, x_{n+1}), D^*(x_n, x_n, x_{n+1})\}$$
$$\to a\{D^*(x, Tx, x)\}$$
$< D^*(Tx, x, x)$, Which is a contradiction. Thus x = Tx

Uniqueness:

Suppose $x \neq y$ such that Ty = y

$D^*(x, y, y) = D^*(Tx, Ty, Ty)$
$\leq a \max\{D^*(x, y, y), D^*(x, Tx, Ty), D^*(y, Ty, Ty), D^*(x, y, Ty), D^*(y, y, Ty)\}$
$= a \max\{D^*(x, y, y), D^*(x, x, y), D^*(y, y, y), D^*(x, y, y), D^*(y, y, y)\}$
$= a D^*(x, y, y)$
$< D^*(x, y, y)$, which in contradiction. Thus x = y

Therefore T has a unique fixed point.

Theorem: 2.6 Let (X, D^*) be a complete D^* - metric space and (CB(X), H) be a Haussdorf metric space induced by D^*- metric .Let T: $X \to CB(X)$ be a map such that

$H(Tx, Ty, Tz) \leq a_1 D^*(x, y, z) + a_2 \max\{D^*(x, Tx, Ty), D^*(y, Ty, Tz)\}$ for all x, y, z, $\in$ X and $0 \leq a_1 + 2a_2 < 1$. Then T has a unique fixed point.

Proof: Let $x_0 \in X$ be any arbitrary fixed element and define a sequence element and define a sequence $\{x_n\}$ in X as $x_{n+1} = x_n$ for n = 0, 1, 2 . . .

For $n \geq 0$, we have.

$D^*(x_n, x_n, x_{n+1}) = D^*(Tx_{n-1}, Tx_{n-1}, Tx_n)$
$\leq a_1 D^*(x_{n-1}, x_{n-1}, x_n) + a_2 \max\{D^*(x_{n-1}, Tx_{n-1}, Tx_{n-1}), D^*(x_{n-1}, Tx_{n-1}, Tx_n)\}$

$$= a_1 D^*(x_{n-1}, x_{n-1}, x_n) + a_2 \max \{D^*(x_{n-1}, x_n, x_n), D^*(x_{n-1}, x_n, x_{n+1})\}$$
$$\leq a_1 D^*(x_{n-1}, x_{n-1}, x_n) + a_2 \{D^*(x_{n-1}, x_{n-1}, x_n) + D^*(x_n, x_n, x_{n+1})\}$$

$$(1- a_2) D^*(x_n, x_n, x_{n+1}) \leq (a_1 + a_2) D^*(x_{n-1}, x_{n-1}, x_{n+1})$$
$$D^*(x_n, x_n, x_{n+1}) \leq \frac{a_1 + a_2}{1-a_2} D^*(x_{n-1}, x_{n-1}, x_n)$$

Thus $D^*(x_n, x_n, x_{n+1}) \leq a\, D^*(x_{n-1}, x_{n-1}, x_n)$, where $a = \dfrac{a_1 + a_2}{1-a_2} < 1$.

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$$\leq a^n D^*(x_0, x_0, x_1) \quad \to 0 \text{ as } n \to \infty.$$

Now for $m > n \geq 0$ we have

$$D^*(x_n, x_n, x_m) \leq D^*(x_n, x_n, x_{n+1}) + D^*(x_{n+1}, x_{n+1}, x_m)$$
$$\leq \sum_{k=n}^{m-1} D^*(x_k, x_k, x_{k+1}) \to 0 \text{ as } m, n \to \infty$$

Thus $\{x_n\}$ is a Cauchy sequence in complete D* - metric space.

Hence there exist a point $x \in X$ such that $x_n \to x$ in X.

Now we shall prove that x is a fixed point of T.

$$D^*(x, x, Tx) = \lim_{n\to\infty} D^*(x_{n+1}, x_{n+1}, Tx)$$
$$= \lim_{n\to\infty} D^*(Tx_n, Tx_n, Tx)$$
$$\leq \lim_{n\to\infty} \{a_1 D^*(x_n, x_n, x) + a_2 \max \{D^*(x_n, Tx_n, Tx_n), D^*(x_n, Tx_n, Tx)\}$$
$$\leq \lim_{n\to\infty} \{a_1 D^*(x_n, x_n, x) + a_2 \max \{D^*(x_n, x_{n+1}, x_{n+1}), D^*(x_n, x_{n+1}, Tx)\}\} \to a_1 (0) + a_2 D^*(x, x, Tx)$$
as $n \to \infty$.

Thus $D^*(x, x, Tx) < D^*(x, x, Tx)$. Which is contradiction.

Thus implies $x = Tx$.

Now we shall prove uniqueness. Suppose $x \neq y$ such that $Ty = y$

Then $D^*(x, y, y) = D^*(Tx, Ty, Ty)$
$$\leq a_1 D^*(x, y, y) + a_2 \max \{D^*(x, Tx, Ty), D^*(y, Ty, Ty)\}$$
$$= a_1 D^*(x, y, y) + a_2 \max \{D^*(x, x, y), D^*(y, y, y)\}$$
$$= (a_1 + a_2) D^*(x, y, y)$$

$< D^*(x, y, y)$, which is contradiction. There fore T has a unique fixed point.

Remark: 2.7 If we put $a_2 = 0$ and $a_1 = a$ in the above theorem we get the following Theorem as corollary.

Corollary: 2.8 Let (X, D*) be a complete D* - metric space and (CB(X), H) be a Haussdorf metric space induced by D*- metric .Let T: X → X be a map such that

$H(Tx, Ty, Tz) \leq a\, D^*(x, y, z)$ for all $x, y, z \in X$ and $0 \leq a < 1$. Then T has a unique fixed point. The above Theorem is know as Banach contraction Type Theorem in D* - metric space.

Remark: 2.9 If we put $a_1 = 0$ and $a_2 = a$ in the above theorem 1. We get the following theorem as corollary 2.10.

Corollary: 2.10 Let (X, D*) be a complete D* - metric space and (CB(X), H) be a Haussdorf metric space induced by D*- metric. Let T: X →X be a map such that H (Tx, Ty, Tz) ≤ $\frac{a}{2}$ max {D*(x, Tx, Ty), D*(y, Ty, Tz)} for all x, y, z $\in$ X and $0 \le a < \frac{1}{2}$. Then T has a unique fixed point.

Theorem: 2.11 Let (X, D*) be a complete D* - metric space and (CB(X), H) be a Haussdorf metric space induced by D*- metric .Let T: X → X be a map such that

H (Tx, Ty, Tz) ≤ a_1 D*(x, y, z) + a_2 max {D*(x, Tx, Ty), D*(y, Ty, Tz)}+ a_3 max {D*(x, y, Ty), D*(y, z, Tz)}
for all x, y, z $\in$ X and $0 \le a_1 + 2a_2 + 2a_3 < 1$. Then T has a unique fixed point.

Proof: Let $x_0 \in X$ be a fixed arbitrary element.

Define a sequence $\{x_n\}$ in X as $x_{n+1} = Tx_n$ for n = 0, 1, 2. . .

Now for n ≥ 0 we have.

$$\begin{aligned} D^*(x_n, x_n, x_{n+1}) &= D^*(Tx_{n-1}, Tx_{n-1}, Tx_n) \\ &= \{a_1 D^*(x_{n-1}, x_{n-1}, x_n) + a_2 \max \{D^*(x_{n-1}, Tx_{n-1}, Tx_{n-1}) , D^*(x_{n-1}, Tx_{n-1}, Tx_n)\} \\ &\quad + a_3 \max \{D^*\{x_{n-1}, x_{n-1}, Tx_{n-1}), D^*(x_{n-1}, x_n, Tx_n)\} \\ &= \{a_1 D^*(x_{n-1}, x_{n-1}, x_n) + a_2 \max \{D^*(x_{n-1}, x_n, x_n) , D^*(x_{n-1}, x_n, x_{n+1}) + a_3 \max \{D^*(x_{n-1}, x_{n-1}, x_n), \\ &\qquad D^*(x_{n-1}, x_n, x_{n+1})\} \\ &\le \{a_1 D^*(x_{n-1}, x_{n-1}, x_n) + a_2\{ D^*(x_{n-1}, x_{n-1}, x_n) + D^*(x_n, x_n, x_{n+1})\}+ a_3\{D^*(x_{n-1}, x_{n-1}, x_n) +D^*(x_n, x_n, x_{n+1})\} \end{aligned}$$

$(1-a_2- a_3) D^*(x_n, x_n, x_{n+1}) \le (a_1+ a_2 + a_3) D^*(x_{n-1}, x_{n-1}, x_n)$.

$$D^*(x_n, x_n, x_{n+1}) \le \frac{a_1 + a_2 + a_3}{1 - a_2 - a_3} D^*(x_{n-1}, x_{n+1}, x_n)$$

$$\begin{aligned} D^*(x_n, x_n, x_{n+1}) &\le a D^* (x_{n-1}, x_{n-1}, x_n) \text{ for all } n \ge 0, \quad \text{where } a = \frac{a_1 + a_2 + a_3}{1 - a_2 - a_3} < 1 \\ &\le a^n D^*(x_0, x_0, x_1) \to 0 \text{ as } n \to \propto \end{aligned}$$

Now we shall prove that $\{x_n\}$ is a D* - Cauchy sequence in X.

For m > n ≥ 0, we have

$$\begin{aligned} D^*(x_n, x_n, x_m) &\le D^*(x_n, x_n, x_{n+1}) + D^*(x_{n+1}, x_{n+1}, x_m) \\ &\le \sum_{k=n}^{m-1} D^*(x_k, x_k, x_{k+1}) \to 0 \text{ as } m, n \to \propto \end{aligned}$$

Thus $\{x_n\}$ is a Cauchy sequence in D* - complete metric space X.

Hence there is a point x in X such that $x_n \to x$ in X.

Now we shall prove that x is fixed point of T.

$$\begin{aligned} \text{Now } D^*(x, x, Tx) &= \lim_{n\to\infty} D^*(x_{n+1}, x_{n+1}, Tx) \\ &= \lim_{n\to\infty} D^*(Tx_n, Tx_n, Tx) \\ &\le \lim_{n\to\infty} \{a_1 D^*(x_n, x_n, x) + a_2 \max \{D^*(x_n, Tx_n, Tx_n), D^*(x_n, Tx_n, Tx)\} \\ &\quad + a_3 \max \{D^*(x_n, x_n, Tx_n), D^*(x_n, x, Tx)\} \\ &= \lim_{n\to\infty} \{a_1 D^*(x_n, x_n, x) + a_2 \max \{D^*(x_n, x_{n+1}, x_{n+1}), D^*(x_n, x_{n+1}, Tx)\} \\ &\qquad + a_3 \max \{D^*(x_n, x_n, x_{n+1}), D^*(x_n, x, Tx)\} \\ &= a_1 (0) + a_2 D^*(x, x, Tx) + a_3 D^*(x, x, Tx) \end{aligned}$$

$< D^*(x, x, Tx)$,

which is contradiction. Thus $Tx = x$.

Uniqueness: Suppose $y \neq x$ such that $Ty = y$.

Now $D^*(x, y, y) = D^*(Tx, Ty, Ty)$

$\leq \{a_1 D^*(x, y, y) + a_2 \max \{D^*(x, Tx, Ty), D^*(y, Ty, Ty)\} + a_3 \max \{D^*(x, y, Ty), D^*(y, y, Ty)\}$

$= \{a_1 D^*(x, y, y) + a_2 \max \{D^*(x, x, y), D^*(y, y, y) + a_3 \max \{D^*(x, y, y), D^*(y, y, y)\}$

$= a_1 D^*(x, y, y) + a_2 D^*(x, y, y) + a_3 D^*(x, y, y)$

$= (a_1+a_2+a_3) D^*(x, y, y)$

$< D^*(x, y, y)$, which is contradiction.

Hence T has a unique fixed point.

Theorem: 2.10 Let (X, D^*) be a complete D^* - metric space and $(CB(X), H)$ be a Haussdorf metric space induced by D^*- metric .Let $T X \rightarrow CB(X)$ be a orbital continuous map such that

$$H(Tx, Ty, Tz) \leq a_1 D^*(x, y, z) + a_2 \max\left\{ \left[\frac{D^*(x,Tx,Ty) + D^*(y,Ty,Tz)}{2}\right], \left[\frac{D^*(x,y,Ty) + D^*(y,z,Tz)}{2}\right]\right\}$$

for all $x, y, z \in X$ and $0 \leq a_1 + 3\frac{a_2}{2} < 1$. Then T has a unique fixed point.

Proof: Let $x_0 \in X$ be any fixed arbitrary element.

Define a sequence $\{x_n\}$ in X as $x_{n+1} \in Tx_n$ for $n = 0, 1, 2. . .$

For $n \geq 0$, we have

$$D^*(x_n, x_n, x_{n+1}) \leq H(Tx_{n-1}, Tx_{n-1}, Tx_n)$$

$$\leq a_1 D^*(x_{n-1}, x_{n-1}, x_n) + a_2 \max\left\{ \left\{\frac{D^*(x_{n-1},Tx_{n-1},Tx_{n-1}) + D^*(x_{n-1},Tx_{n-1},Tx_n)}{2}\right\}, \left\{\frac{D^*(x_{n-1},x_{n-1},Tx_{n-1}) + D^*(x_{n-1},x_n,Tx_n)}{2}\right\} \right\}$$

$$\leq a_1 D^*(x_{n-1}, x_{n-1}, x_n) + a_2 \max\left\{ \left\{\frac{D^*(x_{n-1},x_n,x_n) + D^*(x_{n-1},x_n,x_{n+1})}{2}\right\}, \left\{\frac{D^*(x_{n-1},x_{n-1},x_n) + D^*(x_{n-1},x_n,x_{n+1})}{2}\right\} \right\}$$

$$\leq a_1 D^*(x_{n-1}, x_{n-1}, x_n) + a_2 \left\{\frac{D^*(x_{n-1},x_n,x_n) + D^*(x_n,x_n,x_{n+1}) + D^*(x_{n-1},x_{n-1},x_n)}{2}\right\}$$

$$\left(1-\frac{a_2}{2}\right) D^*(x_n, x_n, x_{n+1}) \leq (a_1+a_2) D^*(x_{n-1}, x_{n-1}, x_n)$$

$$D^*(x_n, x_n, x_{n+1}) \leq \frac{a_1+a_2}{1-\frac{a_2}{2}} D^*(x_{n-1}, x_{n-1}, x_n)$$

$\leq a\, D^*(x_{n-1}, x_{n-1}, x_n)$ for all $n \geq 0$ and $a = \dfrac{a_1 + a_2}{1 - \frac{a_2}{2}} < 1$

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$\leq a^n D^*(x_0, x_0, x_1) \to 0$ as $n \to \infty$.

Now we shall prove that $\{x_n\}$ is a Cauchy sequence in X.

For $m > n \geq 0$, we have

$D^*(x_n, x_n, x_m) \leq \sum_{k=n}^{m-1} D^*(x_k, x_k, x_{k+1}) \to 0$ as $m, n \to \infty$.

Thus $\{x_n\}$ is a D^* - Cauchy sequence in X.

Since X is complete D^*- metric space and $x_n \to x$ in X.

Since T is orbital continuous $Tx_n \to Tx$ in X.

Now we prove that $x \in Tx$ suppose $x \neq Tx$.

$$
\begin{aligned}
H(x, x, Tx) &= \lim_{n\to\infty} H(x_{n+1}, x_{n+1}, Tx) \\
&\leq \lim_{n\to\infty} \{a_1 D^*(x_n, x_n, x) + \frac{a_2}{2} \max \{D^*(x_n, Tx_n, Tx_n) + D^*(x_n, Tx_n, Tx), \\
&\qquad \{D^*(x_n, x_n, Tx_n) + D^*(x_n, x, Tx)\}\} \\
&= \lim_{n\to\infty} \{a_1 D^*(x_n, x_n, x) + \frac{a_2}{2} \max \{D^*(x_n, x_{n+1}, x_{n+1}) + D^*(x_n, x_{n+1}, Tx)\}, \\
&\qquad \{D^*(x_n, x_n, x_{n+1}) + D^*(x_n, x, Tx)\}\} \\
&= a_1 (0) + \frac{a_2}{2} \max \{D^*(x, x, Tx) , \{D^*(x, x, Tx)\} \\
&< D^*(x, x, Tx), \text{ which is contradiction. Thus } Tx = x.
\end{aligned}
$$

Uniqueness:

Suppose $y \neq x$ such that $Ty = y$.

$$
\begin{aligned}
\text{Now } D^*(x, y, y) &= D^*(Tx, Ty, Ty) \\
&\leq \left\{ a_1 D^*(x, y, y) + \frac{a_2}{2} \max \{D^*(x, Tx, Ty) + D^*(y, Ty, Ty)\}, \{D^*(x, y, Ty) + D^*(y, y, Ty)\} \right\} \\
&= \left\{ a_1 D^*(x, y, y) + \frac{a_2}{2} \max \{\{D^*(x, x, y) + D^*(y, y, y)\} , \{D^*(x, y, y) + D^*(y, y, y)\} \right\} \\
&\leq a_1 D^*(x, y, y) + \frac{a_2}{2} D^*(x, y, y) \\
&= (a_1 + \frac{a_2}{2}) D^*(x, y, y) \\
&< D^*(x, y, y), \text{ which is contradiction.}
\end{aligned}
$$

Hence T has a unique fixed point.

REFERENCES:

[1] B. Ahmad, M. Ashraf, and B.E. Rhoades, "Fixed point theorems for expansive mappings in D-metric spaces," Indian Journal of Pure and Applied Mathematics, vol. 32, no. 10, pp. 1513-1518, 2001.

[2] B.C. Dhage, "Generalized metric spaces and mappings with fixed point," Bulletin of the Calcutta Mathematical Society, vol. 84, no. 4, pp. 329-336, 1992.

[3] B.C. Dhage, "A common fixed point principle in D-metric spaces," Bulletin of the Calcutta Mathematical Society, vol. 91, no. 6, pp. 475-480, 1999.

[4] B.C. Dhage, A.M. Pathan, and B.E. Rhoades, "A general existence principle for fixed point theorems in D-metric spaces," International Journal of Mathematical Sciences, vol. 23, no. 7, pp. 441-448, 2000.

[5] S.V.R. Naidu, K.P.R. Rao, and N. Srinivasa Rao, "On the topology of D-metric spaces and generation of D-metric spaces from metric spaces," International Journal of Mathematics and Mathematical Sciences vol. 2004, no. 51, pp. 2719-2740, 2004.

[6] S.V.R. Naidu, K.P.R. Rao, and N. Srinivasa Rao, "On the concepts of balls in a D-metric space," International Journal of Mathematics and Mathematical Sciences, vol. 2005, no. 1, pp. 133-141, 2005.

[7] S.V.R. Naidu, K.P.R. Rao, and N. Srinivasa Rao, "On convergent sequence and fixed point theorems in D-metric spaces," International Journal of Mathematical Sciences, vol. 2005, no. 12, pp. 1969-1988, 2005.

[8] Assad, N.A. and Kirk ,W.A.Fixed point theorems for set valued mappings of contactive type, Pac .J. Math , 43 (1972) ,553-562 .

[9] B.E. Rhoades, "A fixed point theorem for generalized metric spaces," International Journal of Mathematics and mathematical Sciences, vol. 19, no. 3, pp. 457-460, 1996.

[10]B. Singh and R.K. Sharma, "Common fixed points via compatible maps in D-metric spaces," Radovi Mathematical, vol. 11, no.1, pp. 145-153, 2002.

[11] Shaban Sedghi, NabiShobe and Haiyun Zhou, "A common Fixed Point Theorem in D*- Metric Spaces," Hindawi Publishing corporation Fixed Point Theory and Applications, vol.2007,Article ID 27906,13 pages doi:10.1155.

[12] Bose, R.K.and Mukherjee, R.N .Common fixed point theorems of some multivalued mappings, Tamukang jour. Math, 8(1977), 245-249

[13] Bose, R. K. and Mukherjee, R.N .On fixed points of nonexpansive set valued mappings, Proc. AMer. Math. Soc., 82 (1978), 97-98.

[14] Browder, F.E. The fixed point theory of multivaled mappings in topological vector spaces, Math. Ann.177 (1968, 283-301.

[15] Downing, D.and Kirk.W.A, Fixed point theorems for set-valed mappingsin metric andBanach spaces, Pacific J. Math 69(1977),1367-1377.

[16] Goebal K.On affixed point theorem for multivalued non expansive mappings, Annales Univ.Maria Curie Sklodowska.

[17] Lami Dozo, E. Multivalued non expansivemappings and Opial's condition, Proc.Amer.Soc.38 (1973), 286-292.

[18] NADLER., s.b., Jr. Multiovalued contaction mappings for fixed point theorems Pac.L.Math27 (1968), 579-585.

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