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ON GLOBAL EXISTENCE OF NONLINEAR FUNCTIONAL VOLTERRA INTEGRODIFFERENTIAL EQUATIONS IN BANACH SPACES

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ABSTRACT

The purpose of the present paper is to establish the global existence of mild solutions of nonlinear functional Volterra integrodifferential equations of more general type in Banach spaces. An application of the topological transversality theorem known as Leray-Schauder alternative, rely on a priori bounds of solutions, Pachpatte's inequality and semigroup theory are used to investigate our result.

Mathematics subject classification (2000): *45N05, 45J05, 45G10, 45D05.*

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1. INTRODUCTION:

Let X denotes a Banach space with norm $\|.\|$. Let $C = C([-r, 0], X)$, $0 < r < \infty$, be the Banach space of all continuous functions $\psi : [-r, 0] \to X$ with supremum norm

$$\|\psi\|_C = sup\,\{\|\psi(\theta)\|: \; -r \le \theta \le 0\}.$$

If x is a continuous function from $[-r, T]$, $T > 0$, to X and $t \in [0, T]$ then x_t stands for the element of C given by $x_t(\theta) = x(t+\theta)$ for $\theta \in [-r, 0]$. Let $B = C([-r, T], X)$ denotes the Banach space of all continuous functions $x : [-r, T] \to X$ endowed with supremum norm $\|x\|_B = sup\,\{\|x(t)\|: \; -r \le t \le T\}$. The purpose of this paper is to prove the global existence of mild solutions of the nonlinear functional integrodifferential equations of the form

$$\frac{d}{dt}\left[x(t) - h(t, x(t))\right] + Ax(t) = f(t, x_t, \int_0^t k(t,s)\, g(s, x_s)\, ds), \qquad t \in [0, T] \tag{1.1}$$

$$x_0(t) = \phi(t), \qquad -r \le t \le 0 \tag{1.2}$$

where $x : [-r, T] \to X$, $-A$ is the infinitesimal generator of a strongly continuous semigroup of bounded linear operators $T(t)$, $t \ge 0$, in X, the functions $k : [0, T] \times [0, T] \to R$, $g : [0, T] \times C \to X$, $f : [0, T] \times C \times X \to X$, $h : [0, T] \times X \to X$ are continuous functions and ϕ is a given element of C.

Many authors have investigated the problems such as existence, uniqueness, boundedness and other properties of solutions of the special forms of the equations (1.1)-(1.2), for example, see [2,4,11,12,13,15] and references cited therein. Also see [6,16]. The equations of the type (1.1)-(1.2) or their special forms serve as an abstract formulation of many partial differential equations or partial integrodifferential equations arising in heat flow in material with memory, viscoelasticity and reaction diffusion problems, for example, see [1,3,5,8,18,19] and some of the references given therein.

The main purpose of the present paper is to study the global existence of mild solutions of (1.1)-(1.2) by using an application of the topological transversality theorem known as Leray-Schauder alternative, semigroup theory and Pachpatte's inequality. The main feature of the method employed here is that it yields simultaneously the existence of mild solutions and the maximal interval of existence.

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The paper is organized as follows: Section 2 presents the preliminaries and statement of our main result. In Section 3, we deal with the proof of the theorem. Finally, in Section 4, we exhibit an example to illustrate the application of our theorem.

2. PRELIMINARIES AND STATEMENT OF RESULT:

Before proceeding to the statement of our main result, we shall setforth some preliminaries from [9,14,18] and the hypotheses on the functions involved in (1.1)-(1.2) that will be used throughout the paper.

By a precompact set S in a Banach space X, we mean $\overline{S}$ is a compact. An operator $T : X \to X$ is said to be compact if it maps bounded sets into precompact sets. An operator $T : X \to X$ is called completely continuous if it is continuous and compact.

A one parameter family $T(t),\ 0 \le t < \infty,$ of bounded linear operators from X into X is said to be a semigroup of bounded linear operators on X if

(i) $T(t+s) = T(t)T(s)$ for every $t, s \ge 0$.

(ii) $T(0) = I$, I is the identify operator on X.

The infinitesimal generator A of a semigroup of bounded linear operators $\{T(t):\ t \ge 0\}$ on X is the linear operator defined by

$$Ax = \lim_{t\to 0_+} \frac{T(t)x - x}{t}, \qquad \text{for } x \in D(A),$$

where $D(A) = \left\{ x \in X : \lim_{t\to o_+} \frac{T(t)x - x}{t} \text{ exists} \right\}$ is the domain of A.

Definition: A semigroup $\{T(t):\ t \ge 0\}$ of bounded linear operators on X is called a strongly continuous semigroup of bounded linear operators if for each $x \in X$, $T(t)x$ is (strongly) continuous in t on $[0, \infty)$, that is

$$\| T(t+\Delta t)x - T(t)x \| \to 0 \ \text{as}\ \Delta t \to 0,\ \ t,\ t+\Delta t \ge 0.$$

A strongly continuous semigroup $\{T(t): 0 \le t < \infty\}$ is also called a C_0-smigroup. For a C_0 semigroup $\{T(t):\ t \ge 0\}$ of bounded linear operators, we have the growth condition

$$\| T(t) \| \le Me^{\beta t}, \qquad \text{for} \qquad 0 \le t < \infty,$$

with some constant $M > 0$ and $\beta \in R$.

Definition: A C_0 semigroup $\{T(t):\ t \ge 0\}$ is called compact for $t > t_0$ if for every $t > t_0$, $T(t)$ is a compact operator on X.

A C_0 semigroup $\{T(t):\ t \ge 0\}$ is called compact if $T(t)$ is compact for $t > 0$. We remark that if $T(t)$ is compact for $t \ge 0$, then, in particular, the identity operator $T(0) = I$ is compact and therefore the Banach space X, in this case, is finite dimensional. We also note that if $T(t_0)$ is compact for some $t_0 > 0$ then $T(t)$ is compact for every $t \ge t_0$ since $T(t) = T(t - t_0)T(t_0)$ and $T(t - t_0)$ is bonded.

Definition: A C_0-semigroup $\{T(t):\ t \ge 0\}$ is called a uniformly continuous semigroup of operators if the map $t \to T(t)$ is continuous in the uniform operator topology, that is,

$$\| T(t+\Delta t) - T(t) \| \to 0 \ \text{as}\ \Delta t \to 0,\ \ t,\ t+\Delta t \ge 0.$$

For convenience, we list the following hypotheses

(H_1) $-A$ is the infinitesimal generator of a compact semigroup of bounded linear operators $T(t), t \ge 0,$ on X such that $\| T(t) \| \le M$ for some $M > 1$.

(H_2) There exists a continuous function $p:[0,T]\to R_+=[0,\infty)$ such that

$$\| f(t,\psi,x)\| \le p(t)\,(\|\psi\|_C + \|x\|) \quad \text{for every } t\in[0,T],\ \psi\in C \text{ and } x\in X.$$

(H_3) There exists a continuous function $q:[0,T]\to R_+$ such that

$$\| g(t,\psi)\| \le q(t)\,\|\psi\|_C \quad \text{for every } t\in[0,T] \text{ and } \psi\in C.$$

(H_4) For each $t\in[0,T]$, the function $f(t,.,.):[0,T]\times C\times X\to X$ is continuous and for each $\psi\in C$ and for each $x\in X$, the function $f(.,\psi,x):[0,T]\times C\times X\to X$ is strongly measurable.

(H_5) For each $t\in[0,T]$, the function $g(t,.):[0,T]\times C\to X$ is continuous and for each $\psi\in C$, the function $g(.,\psi):[0,T]\times C\to X$ is strongly measurable.

(H_6) There exists a constants $0<c_1<1$ and $c_2>0$ such that

$$\| h(t,x(t))\| \le c_1\,\|x(t)\| + c_2 \quad \text{for every } t\in[0,T] \text{ and } x(t)\in X.$$

Let $x_t\in C([-r,0],X)$ and $t\in[0,T]$. The hypotheses (H_3) and (H_5) yield that the function $\tau\to g(\tau,x_\tau)\in X$ is strongly measurable and bounded and therefore the integral $\int_0^s k(s,\tau)\ g(\tau,x_\tau)\,d\tau$ is meaningful and gives a continuous function with values in X. This together with the hypotheses (H_2) and (H_4) imply that the function

$$s\to f\Big(s,x_s,\int_0^s k(s,\tau)\ g(\tau,x_\tau)\,d\tau\Big)\in X$$

is strongly measurable and bounded. This leads us to the idea of mild solution of the initial value problem (1.1)-(1.2). For $\phi\in C$, the function $x\in C([-r,T],X)$ given by

$$x(t)=T(t)\big[\phi(0)-h(0,\phi(0))\big]+T(t)h(t,x(t))+\int_0^t T(t-s)f\Big(s,x_s,\int_0^s k(s,\tau)\,g(\tau,x_\tau)\,d\tau\Big)ds,\quad t\in[0,T] \tag{2.1}$$

$$x_0=\phi(t),\qquad -r\le t\le 0 \tag{2.2}$$

is called mild solution of initial value problem (1.1)-(1.2) on $[-r,T]$. For more details, see [8, 14, 18].

We use the following version of the topological transversality theorem given by in J. Dugundji and A. Granas ([7], p61).

Theorem: 2.1 Let Y be a convex subset of a normed linear space E and assume $0\in Y$. Let $F:Y\to Y$ be a completely continuous operator and let

$$\varepsilon(F)=\{x\in Y: x=\lambda Fx \quad for \quad some \qquad 0<\lambda<1\}$$

Then either $\varepsilon(F)$ is unbounded in Y or F has a fixed point in Y.

The following Pachpatte's inequality is crucial in our subsequent discussion.

Theorem: 2.2 ([17], p758) Let $u(t)$, $p(t)$ and $q(t)$ be real valued non-negative continuous functions defined on R_+, for which the inequality

$$u(t)\le u_0+\int_0^t p(s)u(s)ds+\int_0^t p(s)\Big[\int_0^s q(\tau)u(\tau)d\tau\Big]ds$$

holds for all $t \in R_+$ where u_0 is nonnegative constant. Then

$$u(t) \le u_0[1 + \int_0^t p(s)\, \exp\{\int_0^s (p(\tau) + q(\tau))d\tau\}ds] \text{ for all } t \in R_+ .$$

Our main result is embodied in the following theorem.

Theorem: 2.3 Suppose that the hypotheses (H_1) - (H_6) hold. Then initial value problem (1.1)-(1.2) has a mild solution x defined on $[-r, T]$.

Remark: 2.1: In [11 and 12], authors have studied the problems such as existence, uniqueness and other properties of solution of a special form of the equations (1.1)-(1.2) by using different techniques and conditions. Here our conditions on the functions involved in equations (1.1)-(1.2) and the approach to the problem are different from those used in [11 and 12].

Remark: 2.2: Dauer and Balachandtran, in [8], have derived solution of equations like (1.1)-(1.2) by using Schaefer fixed point theorem. Further, we note that here in our Theorem, we are achieving same result with less hypotheses only by applying Pachpatte's inequality in addition to topological transversality theorem and semigroup theory.

3. PROOF OF THEOREM:

First we establish the priori bounds on the solutions to the initial value problem $(1.1)_\lambda - (1.2)$ for $\lambda \in (0, 1)$ where

$$\frac{d}{dt}\left[x(t) - \lambda h(t, x(t))\right] + Ax(t) = \lambda f(t, x_t, \int_0^t k(t,s)\, g(s, x_s)\, ds), \qquad t \in [0, T] \tag{1.1_λ}$$

Let $x(t)$ be a mild solution of the initial value problem $(1.1)_\lambda - (1.2)$ then it satisfies the equivalent integral equation

$$x(t) = T(t)[\phi(0) - \lambda h(0, \phi(0))] + \lambda T(t) h(t, x(t)) + \lambda \int_0^t T(t-s) f(s, x_s, \int_0^s k(s,\tau)\, g(\tau, x_\tau)\, d\tau) ds, \quad t \in [0, T] \tag{3.1}$$

$$x_0(t) = \phi(t), \qquad -r \le t \le 0 \tag{3.2}$$

Since k is continuous on the compact set $[0, T] \times [0, T]$, there is constant $L > 0$ such that

$$|\, k(s, t)\, | \le K, \text{ for } \quad 0 \le s \le t \le T \tag{3.3}$$

Using integral equation (3.1), hypotheses $(H_1) - (H_3)$, (H_6), condition (3.3) and the fact that $\lambda \in (0, 1)$, we obtain

$$\begin{aligned}
\| x(t) \| &\le \| T(t) \| \, \| \phi(0) \| + |\lambda| \|T(t)\| \|h(0, \phi(0)\| + |\lambda| \, \| T(t) \| \|h(t, x(t)\| \\
&\quad + |\, \lambda\, | \int_0^t \| T(t-s) \| \, \| f(s, x_s, \int_0^s k(s, \tau)\, g(\tau, x_\tau) d\tau) ds \| \\
&\le M \, \| \phi(0) \| + M\left[c_1 \, \| \phi(0) \| + c_2\right] + M\left[c_1 \, \| x(t) \| + c_2\right] + \int_0^t Mp(s)\left[\| x_s \|_C + \int_0^s Lq(\tau) \, \| x_\tau \|_C \, d\tau \right] ds \\
&= M\left[(1 + c_1) \, \| \phi(0) \| + 2c_2\right] + Mc_1 \, \| x(t) \| + \int_0^t Mp(s) \| x_s \|_C \, ds + \int_0^t Mp(s) \int_0^s Lq(\tau) \, \| x_\tau \|_C \, d\tau ds, \quad t \in [0, \ T]
\end{aligned}$$

$$\| x(t) \| = \frac{M}{1 - Mc_1}\left[(1 + c_1) \, \| \phi(0) \| + 2c_2\right] + \int_0^t \frac{M}{1 - Mc_1} p(s) \| x(s) \|_C ds + \int_0^t \frac{M}{1 - Mc_1} p(s) \int_0^s Lq(\tau) \, \| x_\tau \|_C d\tau ds \tag{3.4}$$

Case 1: Suppose $t \ge r$. Then for every $\theta \in [-r, 0]$, we have $t + \theta \ge 0$. For such $\theta^{,s}$, from (3.4), we have

$$\| x(t+\theta) \| \le \frac{M}{1 - Mc_1}\left[(1 + c_1) \, \| \phi(0) \| + 2c_2\right] + \int_0^{t+\theta} \frac{M}{1 - Mc_1} p(s) \| x(s) \|_C ds + \int_0^{t+\theta} \frac{M}{1 - Mc_1} p(s) \int_0^s Lq(\tau) \, \| x_\tau \|_C d\tau ds$$

$$\leq \frac{M}{1-Mc_1}\left[(1+c_1)\,\|\phi(0)\|+2c_2\right]+\int_0^t \frac{M}{1-Mc_1}\,p(s)\,\|x(s)\|_C\,ds+\int_0^t \frac{M}{1-Mc_1}\,p(s)\int_0^s Lq(\tau)\,\|x_\tau\|_C\,d\tau ds$$

which yields

$$\|x(t)\|_C \leq \frac{M}{1-Mc_1}\left[(1+c_1)\,\|\phi(0)\|+2c_2\right]+\int_0^t \frac{M}{1-Mc_1}\,p(s)\,\|x_s\|_C\,ds+\int_0^t \frac{M}{1-Mc_1}\,p(s)\int_0^s Lq(\tau)\,\|x_\tau\|_C\,d\tau ds \tag{3.5}$$

Case 2: Suppose $0 \leq t < r$. Then for every $\theta \in [-r, -t)$, we have $t+\theta < 0$. For such $\theta^{,s}$, we observe from the initial condition (3.2) that

$$\|x(t+\theta)\| = \|x_t(\theta)\| = \|\phi(t+\theta)\|$$

which yields

$$\|x_t\|_C = \|\phi\|_C \tag{3.6}$$

For $\theta \in [-t, 0]$, $t+\theta \geq 0$. Then, we get as in case 1,

$$\|x_s\|_C \leq \frac{M}{1-Mc_1}\left[(1+c_1)\,\|\phi(0)\|+2c_2\right]+\int_0^t \frac{M}{1-Mc_1}\,p(s)\,\|x_s\|_C\,ds+\int_0^t \frac{M}{1-Mc_1}\,p(s)\int_0^s Lq(\tau)\,\|x_s\|_C\,d\tau ds \tag{3.7}$$

Thus, for every $\theta \in [-r, 0]$, ($0 \leq t < r$), from (3.6) and (3.7) we obtain

$$\|x_t\|_C \leq \frac{M}{1-Mc_1}\left[(1+c_1)\,\|\phi\|_C+2c_2\right]+\int_0^t \frac{M}{1-Mc_1}\,p(s)\,\|x_s\|_C\,ds+\int_0^t \frac{M}{1-Mc_1}\,p(s)\int_0^s Lq(\tau)\,\|x_\tau\|_C\,d\tau ds \tag{3.8}$$

For every $t \in [0, T]$, from inequalities (3.5) and (3.8) we get

$$\|x_t\|_C \leq \frac{M}{1-Mc_1}\left[(1+c_1)\,\|\phi\|_C+2c_2\right]+\int_0^t \frac{M}{1-Mc_1}\,p(s)\,\|x_s\|_C\,ds+\int_0^t \frac{M}{1-Mc_1}\,p(s)\int_0^s Lq(\tau)\,\|x_\tau\|_C\,d\tau ds \tag{3.9}$$

Thanks to Pachpatte's inequality given in Theorem 2.2 and applying it with $u(t) = \|x_t\|_C$ and $u_0 = \frac{M}{1-Mc_1}\left[(1+c_1)\,\|\phi\|_C+2c_2\right]$ to the inequality (3.9), we obtain

$$\begin{aligned}\|x_t\|_C &\leq \frac{M}{1-Mc_1}\left[(1+c_1)\,\|\phi\|_C+2c_2\right]\left[1+\int_0^t \frac{M}{1-Mc_1}\,p(s)\exp\{\frac{M}{1-Mc_1}\,p(\tau)+Lq(\tau)d\tau\}ds\right]\\ &\leq \frac{M}{1-Mc_1}\left[(1+c_1)\,\|\phi\|_C+2c_2\right]\left[1+\frac{M}{1-Mc_1}P\exp\left\{\left(\frac{M}{1-Mc_1}P+LQ\right)T\right\}T\right]\\ &= \gamma\end{aligned}$$

for every $t \in [0, T]$, where $P = \sup\{p(t): 0 \leq t \leq T\}$ and $Q = \sup\{q(t): 0 \leq t \leq T\}$. This yields that $\|x(t)\| \leq \gamma$ for every $t \in [-r, T]$ which immediately follows of $\lambda \in (0, 1)$ such that $\|x\|_B \leq \gamma$.

Next, we rewrite initial value problem (1.1)-(1.2) as follows, see [Hale]. For $\phi \in C$, define $\hat{\phi} \in B = C([-r, T]\ X)$ by

$$\hat{\phi}(t) = \begin{cases} \phi(t), & -r \leq t \leq 0 \\ T(t)\phi(0), & 0 \leq t \leq T \end{cases}$$

If $y \in B$ and $x(t) = y(t) + \hat{\phi}(t)$, $t \in [-r, T]$ then it is easy to see that y satisfies

$$y(t) = y_0 = 0, \qquad -r \le t \le 0 \text{ and}$$

$$y(t) = -T(t)\,h\left(0,\ \phi(0) + T(t)h\left(t,\ y(t) + \hat{\phi}(t)\right)\right) + \int_0^t T(t-s)\ f(s,\ y_s + \hat{\phi}_s,\ \int_0^s k(s,\tau)\,g(\tau,\ y_\tau + \hat{\phi}_\tau)\ d\tau)\ ds$$

if and only if x satisfies integral equations (2.1) and (2.2).

Define the operator $F : B_0 \to B_0$, $B_0 = \{y \in B : \ y_0 = 0\}$ by

$$(Fy)(t) = \begin{cases} 0, & -r \le t \le 0 \\ -T(t)\,h(0,\ \phi(0)) + T(t)h\left(t,\ y(t) + \hat{\phi}(t)\right) & \\ \qquad + \int_0^t T(t-s)\ f(s,\ y_s + \hat{\phi}_s,\ \int_0^s k(s,\tau)\,g(\tau,\ y_\tau + \hat{\phi}_\tau)\ d\tau)\ ds, & t \in [0, T] \end{cases} \tag{3.10}$$

Then the integral equations (2.1) - (2.2) can be written as follows

$$y = F(y) \tag{3.11}$$

The integral equations (3.1)-(3.2) can also be written, by using the operator F, as follows

$$y = \lambda\ F(y) \tag{3.12}$$

Now, our intent is to apply Theorem 2.1 of section 2 to the equation (3.11) and obtain a solution of the initial value problem (1.1)-(1.2) via a fixed point of F. To this end we have to prove that $F : B_0 \to B_0$ is a completely continuous operator.

We, first, show that $F : B_0 \to B_0$ is continuous. Let $\{y_n\}$ be a sequence of elements of B_0 converging to y in B_0. Then, by making use of hypotheses (H_4) and (H_5), we have

$$f(s,\ y_{n_s} + \hat{\phi}_s,\ \int_0^s k(s,\tau)\,g(\tau,\ y_{n_\tau} + \hat{\phi}_\tau)\ d\tau) \ \to f(s,\ y_s + \hat{\phi}_s,\ \int_0^s k(s,\tau)\,g(\tau,\ y_\tau + \hat{\phi}_\tau)\ d\tau)$$

for each $s \in [0, T]$. By applying hypotheses (H_1), (H_4), (H_5) and the dominated convergence theorem, we obtain

$$\begin{aligned} \| F\ y_n(t) - Fy(t) \| \le \| T(t)\Big[h\left(t,\ y_n(t) + \hat{\phi}(t)\right) - h\left(t,\ y(t) + \hat{\phi}(t)\right)\Big] \\ + \int_0^t T(t-s)[f(s,\ y_{n_s} + \hat{\phi}_s,\ \int_0^s k(s,\tau)\,g(\tau,\ y_{n_\tau} + \hat{\phi}_\tau)\ d\tau) \\ - f(s,\ y_s + \hat{\phi}_s,\ \int_0^s k(s,\tau)\,g(\tau,\ y_\tau + \hat{\phi}_\tau)\ d\tau)\ ds \| \\ \to 0 \ \text{ as } n \to \infty, \qquad \forall\ t \in [0, T] \end{aligned}$$

and it follows that

$$\| F\ y_n - Fy \|_B \to 0 \ \text{ as } n \to \infty$$

i.e. $F\ y_n \to Fy$ in B_0 as $y_n \to y$ in B_0. Thus, F is continuous.

We, now, prove that F maps a bounded set of B_0 onto a precompact set of B_0. Set $B_m = \{ y \in B_0 : \| y \|_B \le m \}$ for $m \ge 1$. We show that F maps B_m into an equicontinuous family of functions with values in X. We consider the following cases. From the equation (3.10) and using hypotheses (H_1) - (H_5) and fact that $\| y \|_B \le m$, $y \in B_m$ implies $\| y_t \|_C \le m$, $t \in [0, T]$, we have

Case 1: Suppose $0 \le t_1 \le t_2 \le T$

$$\| (Fy)(t_2) - (Fy)(t_1) \| = \| \; -T(t_2)\, h\big(0, \phi(0)\big) + T(t_2) h\Big(t_2,\ y(t_2) + \hat{\phi}(t_2) \Big)$$
$$+ \int_0^{t_2} T(t_2 - s)\, f(s,\ y_s + \hat{\phi}_s,\ \int_0^s k(s,\tau)\, g(\tau,\ y_\tau + \hat{\phi}_\tau)\ d\tau)\, ds$$
$$+ T(t_1)\, h\big(0, \phi(0)\big) - T(t_1) h(t_1,\ y(t_1) + \hat{\phi}(t_1))$$
$$- \int_0^{t_1} T(t_1 - s) f(s,\ y_s + \hat{\phi}_s,\ \int_0^s k(s,\tau)\, g(\tau,\ y_\tau + \hat{\phi}_\tau)\ d\tau)\, ds \, \|$$

$$= \| T(t_2) - T(t_1) \| \| h\big(0, \phi(0)\big) \| + \| T(t_2) h\Big(t_2,\ y(t_2) + \hat{\phi}(t_2) \Big) - T(t_1) h\Big(t_1,\ y(t_1) + \hat{\phi}(t_1) \Big) \|$$
$$+ \| \int_0^{t_1} [T(t_2 - s) - T(t_1 - s)]\ f(s,\ y_s + \hat{\phi}_s,\ \int_0^s k(s,\tau)\, g(\tau,\ y_\tau + \hat{\phi}_\tau)\ d\tau)\, ds$$
$$+ \int_{t_1}^{t_2} T(t_2 - s) f(s,\ y_s + \hat{\phi}_s,\ \int_0^s k(s,\tau)\, g(\tau,\ y_\tau + \hat{\phi}_\tau)\ d\tau)\, ds \, \|$$

$$\le \| T(t_2) - T(t_1) \| \| h\big(0, \phi(0)\big) \| + \| T(t_2) h\Big(t_2,\ y(t_2) + \hat{\phi}(t_2) \Big) - T(t_1) h\Big(t_1,\ y(t_1) + \hat{\phi}(t_1) \Big) \|$$
$$+ \int_0^{t_1} \| T(t_2 - s) - T(t_1 - s) \|\ p(s)\ [\| y_s + \hat{\phi}_s \|_C + \int_0^s Lq(\tau) \| y_\tau + \hat{\phi}_\tau \|_C\, d\tau] ds$$
$$+ \int_{t_1}^{t_2} \| T(t_2 - s) \|\ p(s) [\| y_s + \hat{\phi}_s \|_C] + \int_0^s Lq(\tau) \| y_\tau + \hat{\phi}_\tau \|_C\, d\tau] ds$$

$$\le \| T(t_2) - T(t_1) \| \| h\big(0, \phi(0)\big) \| + \| T(t_2) h\Big(t_2,\ y(t_2) + \hat{\phi}(t_2) \Big) - T(t_1) h\Big(t_1,\ y(t_1) + \hat{\phi}(t_1) \Big) \|$$
$$+ \int_0^{t_1} \| T(t_2 - s) - T(t_1 - s) \|\, P[m + M_C + LQ(m + M_C)T] ds$$
$$+ \int_{t_1}^{t_2} \| T(t_2 - s) \|\, P[m + M_C + LQ(m + M_C)T] ds$$

where P and Q are defined as before and $c = \| \phi \|_C$.

Case 2: Suppose $-r \le t_1 \le 0 \le t_2 \le T$

$$\| (Fy)(t_2) - (Fy)(t_1) \| \le \| T(t_2) [h\Big(t_2,\ y(t_2) + \hat{\phi}(t_2) \Big) - h\big(0, \phi(0)\big)] \|$$
$$+ \int_0^{t_2} \| T(t_2 - s) \|\ p(s) [\| y_s \|_C + \| \hat{\phi}_s \|_C + \int_0^s Lq(\tau) \{ \| y_\tau \|_C + \| \hat{\phi}_\tau \|_C \} d\tau] ds$$

$$\leq \| T(t_2)[h\left(t_2,\, y(t_2)+\hat{\phi}(t_2)\right) - h\big(0,\, \phi(0)\big)] \| + \int_0^{t_2} \| T(t_2 - s) \| \, P[m + M_C + LQ(m + M_C)T]ds$$

$$\leq \| T(t_2)[h\left(t_2,\, y(t_2)+\hat{\phi}(t_2)\right) - h\big(0,\, \phi(0)\big)] \| MP[m + M_C + LQ(m + M_C)T]t_2$$

$$\leq \| T(t_2)[h\left(t_2,\, y(t_2)+\hat{\phi}(t_2)\right) - h\big(0,\, \phi(0)\big)] \| + MP[m + M_C + LQ(m + M_C)T](t_2 - t_1)$$

Case 3: Suppose $-r \leq t_1 \leq t_2 \leq 0$. Then

$$\| (Fy)(t_2) - (Fy)(t_1) \| = 0$$

Sine cases 1-3, imply that $\| (Fy)(t_2) - (Fy)(t_1) \| \leq \gamma \, |t_2 - t_1|$, with constant $\gamma > 0$ for every $t_1, t_2 \in [-r, T]$, we claim that FB_m is an equicontinuous family of functions with values in X.

Next, we show that FB_m is uniformly bounded. Using the equation (3.10) and the hypotheses (H_1) - (H_6) and fact that $\| y(t) \| \leq m$, $\forall \; t \in [-r, T]$, we obtain

$$\| (Fy)(t) \| \leq \| -T(t) \| \| h\big(0,\, \phi(0)\big) \| + \| T(t) \| \| h\left(t,\, y(t)+\hat{\phi}(t)\right) \|$$

$$+ \int_0^t \| T(t-s) \| \| f(s,\, y_s + \hat{\phi}_s,\, \int_0^s k(s,\tau)\, g(\tau,\, y_\tau + \hat{\phi}_\tau)\, d\tau) \| \, ds$$

$$\leq M[c_1 \| \phi(0) \| + c_2] + m[c_1 \| y(t) + \hat{\phi}(t) \| + c_2] + \int_0^t Mp(s)\, [\| y_s + \hat{\phi}_s \|_C + \int_0^s Lq(\tau) \| y_\tau + \hat{\phi}_\tau \|_C \, d\tau] ds$$

$$\leq M[c, c + c_2] + M[c, (m + Mc) + c_2] + MP[m + Mc + LQ(m + Mc)\frac{T}{2}]T,$$

which implies that the set $\{(Fy)(t) : \| y \|_B \leq m, \; -r \leq t \leq T\}$ is bounded in X and hence $\{FB_m\}$ is uniformly bounded.

We, next show $\overline{FB_m}$ is compact. Since we have shown FB_m is an equicontinuous and uniformly bounded collection, by the Arzela-Ascoli theorem, it is sufficiently to show that the set $\{(Fy)(t) :\, y \in B_m\}$ is precompact in X for each $t \in [-r, T]$. Since $(Fy)(t) = 0$ for $t \in [-r, 0]$ and $y \in B_m$, it suffices to show this for $0 < t \leq T$.

Let $0 < t \leq T$ be fixed and $\in$ a real number such that $0 < \in < t$. For $y \in B_m$, we define

$$(F_\in y)(t) = -T(t)\, h\big(0,\, \phi(0)\big) + T(t) h\left(t,\, y(t) + \hat{\phi}(t)\right) + \int_0^{t-\in} T(t-s)\, f(s,\, y_s + \hat{\phi}_s,\, \int_0^s k(s,\tau)\, g(\tau,\, y_\tau + \hat{\phi}_\tau)\, d\tau)\, ds \tag{3.13}$$

Since $T(t-s) = T(\in + t - s - \in)$ and $T(t)$ is a semigroup of bounded linear operators, we have $T(t-s) = T(\in) T(t - s - \in)$ then the equation (3.13) becomes

$$(F_\in y)(t) = -T(t)\, h\big(0,\, \phi(0)\big) + T(t) h\left(t,\, y(t) + \hat{\phi}(t)\right) + T(\in) \int_0^{t-\in} T(t - s - \in)\, f(s,\, y_s + \hat{\phi}_s,\, \int_0^s k(s,\tau)\, g(\tau,\, y_\tau + \hat{\phi}_\tau)\, d\tau)\, ds$$

An operator $T(t)$ being a compact operator, the set $Y_\in(t) = \{(F_\in y)(t) :\, y \in B_m\}$ is precompact in X for every $\in$, $0 < \in < t$. Moreover, for every $y \in B_m$ and using (3.10) and (3.13) we have

$$(Fy)(t) - (F_\in y)(t) = \int_{t-\in}^t T(t-s)\, f(s,\, y_s + \hat{\phi}_s,\, \int_0^s k(s,\tau)\, g(\tau,\, y_\tau + \hat{\phi}_\tau)\, d\tau)\, ds$$

Using hypotheses (H_1) - (H_3), the condition (3.3) and the fact that $\| y_t \|_C \le m$, we obtain

$$\| (Fy)(t) - (F_\in y)(t) \| \le \int_{t-\in}^{t} \| T(t-s) \| \| f(s,\, y_s + \hat{\phi}_s,\, \int_0^s k(s,\tau)\, g(\tau,\, y_\tau + \hat{\phi}_\tau)\, d\tau) \| \, ds$$

$$\le \int_{t-\in}^{t} \| T(t-s) \| \, p(s)[\| y_s \|_C + \| \hat{\phi}_s \|_C + \int_0^s Lq(\tau)\{\| y_\tau \|_C + \| \hat{\phi}_\tau \|_C\} d\tau] \, ds$$

$$\le \int_{t-\in}^{t} MP[m + Mc + \int_0^s LQ(m + Mc) d\tau] \, ds$$

This proves that there exists precompact sets arbitrarily close to the set $\{(Fy)(t) : y \in B_m\}$ and therefore the set $\{(Fy)(t) : y \in B_m\}$ is precompact in X. This completes the proof that the operator F is completely continuous.

Finally the set $\varepsilon(F) = \{ y \in B_0 : \; y = \lambda F y, \quad 0 < \lambda < 1\}$ is bounded in B, since every solution y in $\varepsilon(F)$ the function $x(t) = y(t) + \hat{\phi}(t)$ is a mild solution of initial value problem $(1.1)_\lambda - (1.2)$ for which we have shown that $\| x \|_B \le \gamma$ and therefore, $\| y \|_B \le \gamma + Mc$. Consequently, by Leray-Sohauder alternative, the operator F has a fixed point y in B_0 and hence the initial value problem $(1.1) - (1.2)$ has a mild solution. Thus the proof of the Theorem is complete.

4. APPLICATION:

This section exhibits an example to illustrate the application of our result established in the previous section. Consider the following nonlinear partial functional integrodifferential equation of the form

$$\frac{\partial}{\partial t}[w(u,t) - V(t, w(u,t))] - \frac{\partial^2}{\partial u^2} w(u,t) = G(t,\, w(u,\, t-r),\, \int_0^t k(t,\, s)\, N(s,\, w(u,\, s-r)\, ds),$$
$$0 \le u \le \pi, \qquad t \in [0,T] \tag{4.1}$$

with initial and boundary condition

$$w(0,\, t) = w(\pi,\, t) = 0, \qquad 0 \le t \le T \tag{4.2}$$
$$w(u,\, t) = \phi(u,\, t), \qquad 0 \le u \le \pi \qquad -r \le t \le 0 \tag{4.3}$$

where kernel function $k : [0,\, T] \times [0,\, T] \to R$, $G : [0,\, T] \times R \times R \to R$, $N : [0,\, T] \times R \to R$ and $V : [0,\, T] \times R \to R$, are continuous functions.

Assume that the functions K, G, N and V in (4.1) satisfy the following conditions.

1. There exists a constant $L_1 > 0$ such that

   $$|k(t,s)| \le L_1, \qquad \text{for} \quad 0 \le s \le t \le T.$$

2. There exist real continuous functions p_1 and q_1 defined on $[0,\, T]$ such that.

   i) $|G(t, y, z)| \le p_1(t)\,(|y| + |z|)$

   ii) $|N(t, y)| \le q_1(t)\,|y|$

   for $0 \le t \le T$ and $y,\, z \in R$.

3. There exist nonnegative L_2 and L_3 such that

   $$|V(t,\, y)| \le L_2 |y| + L_3$$

Let $X = L^2[0, \pi]$. We define an operator $A: X \to X$ by

$D(A) = \{v \in X : v,\ v'$ are absolutely continuous, $v'' \in X$ and $v(0) = v(\pi) = 0\ \}$.

Then, the operator A can be written as

$$Av = \sum_{n=1}^{\infty} n^2 (v,\ v_n)\ v_n, \qquad v \in D(A)$$

where $v_n(z) = \sqrt{\dfrac{2}{\pi}} \sin nz,\ n = 1, 2, 3, \ldots$ is the orthogonal set of eigenvectors of A, $-A$ is the infinitesimal generator of an analytic semigroup $T(t),\ t \geq 0$ in X given by

$$T(t)v = \sum_{n=1}^{\infty} \exp(-n^2 t)\ (v,\ v_n)\ v_n,\ v \in X.$$

Since an analytic semigroup $T(t)$ is compact, there exists constant $N_1 \geq 1$ such that $\| T(t) \| = N_1$ for each $t \geq 0$.

Now we define the functions $f: [0, T] \times C \times X \to X$ and $g: [0, T] \times C \to X$, $h: [0, T] \times X \to X$ as follows:

$$f(t, \psi,\ y)(u) = H(t, \psi(-r)u,\ y(u))$$
$$g(t, \psi)(u) = N(t, \psi(-r)u)$$
$$h(t,\ y)(u) = V(t,\ y(u))$$

for $t \in [0, T], \psi \in C,\ y \in X$ and $0 \leq u \leq \pi$. From the above choices of the functions and the operator A, the equations (4.1)-(4.3) can be formulated abstractly as

$$\frac{d}{dt}[x(t) - h(t, x(t))] + Ax(t) = f(t,\ x_t, \int_0^t k(t,s)\, g(s, x_s)\ ds), \qquad t \in [0, T] \tag{4.4}$$

$$x_0(t) = \phi(t), \qquad -r \leq t \leq 0 \tag{4.5}$$

Since all the hypotheses of the Theorem 2.3 are satisfied, the Theorem 2.3 can be applied to guarantee the existence of a mild solution

$$w(u,t) = x(\phi)(t)u$$

for $t \in [0, T],\ u \in [0, \pi]$, of the nonlinear partial functional integrodifferential equation (4.1)-(4.3).

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