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DEDUCTION IN POLYADIC ALGEBRA

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ABSTRACT

Certain polyadic filters and ultrafilters have been used to express deduction in polyadic algebra and functional polyadic algebra with terms.

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1. INTRODUCTION:

Polyadic algebra is an extension of Boolean algebra with operators corresponding to the usual existential and universal quantifiers over several variables together with endomorphisms to represent first order logic algebraically.

In this paper we study deduction $\Gamma \vdash a$ in a polyadic algebra B by using the polyadic filter $F(\Gamma)$ generated by Γ where $\Gamma \subseteq B$ and $a \in B$. The notion is extended to functional polyadic algebra with terms. Polyadic ultrafilter $UF(\Gamma)$ generated by Γ is also used to express the dichotomy either $\Gamma \vdash a$ or $\Gamma \vdash a'$ where a' is the complement of a in B

POLYADIC ALGEBRA:

Suppose that B is a complete Boolean algebra. An existential quantifier on B is a mapping $\exists : B \to B$ such that
i) $\exists(0) = 0$
ii) $a \le \exists(a)$ for any $a \in B$
iii) $\exists(a \wedge \exists(b)) = \exists(a) \wedge \exists(b)$ for any $a, b \in B$ $(B, \exists)$ is called a monadic algebra.

Let I be a set usually countable to index the variables . A mapping not necessarily one to one or onto from I into itself is called a transformation. Let $I^I = \{\tau | \tau : I \to I \ is\ a\ transformation\}$. Denote the set of all endomorphisms on B by $End(B)$ and the set of all quantifiers of B by $Qant(B)$.

A polyadic algebra is $(B, I, S, \exists)$ where $S : I^I \to End(B)$ and $\exists : 2^I \to Qant(B)$ such that for any $J, K \in 2^I$ and any $\sigma, \tau \in I^I$ we have:
1) $\exists(\phi) = id$
2) $\exists(J \cup K) = \exists(J)\exists(K)$
3) $S(id) = id$
4) $S(\sigma\tau) = S(\sigma)S(\tau)$
5) $S(\sigma)\exists(J) = S(\tau)\exists(J)\ if\ \ \sigma|_{I-J} = \tau|_{I-J}$
6) $\exists(J)S(\tau) = S(\tau)\exists(\tau^{-1}(J))\ if\ \ \tau\ is\ one\ to\ one\ on\ \tau^{-1}(J)$

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For $J \subseteq I$ we may define $\forall(J)$ by $\forall(J)(a) = (\exists(J)(a'))'$ for any $a \in B$.

3. POLYADIC IDEALS AND FILTERS:

A subset U of a Boolean algebra B is called a Boolean ideal if

i) $0 \in U$

ii) $a \vee b \in U$ for any $a, b \in U$

iii) If $a \in U$ and $b \leq a$ then $b \in U$

A subset F of a Boolean algebra B is called a Boolean filter if

i) $1 \in F$

ii) $a \wedge b \in F$ for any $a, b \in F$

iii) If $a \in F$ and $b \geq a$ then $b \in F$.

A subset U of a polyadic algebra B is called a polyadic ideal of B if

i) U is a Boolean ideal

ii) If $J \subseteq I$ and $a \in U$ then $\exists(J)(a) \in U$

iii) If $a \in U$ and $\sigma \in I^I$ then $S(\sigma)(a) \in U$

A subset F of a polyadic algebra B is called a polyadic filter of B if

i) F is a Boolean filter

ii) If $J \subseteq I$ and $a \in F$ then $\forall(J)(a) \in F$

iii) If $a \in F$ and $\sigma \in I^I$ then $S(\sigma)(a) \in F$.

Proposition: 3.1 [4] A subset U of a polyadic algebra B is an ideal of it if and only if U is an ideal of the Boolean algebra B and $\exists(I)(a) \in U$ for every $a \in U$. F is a filter of B if and only if F is a filter of the Boolean algebra B and $\forall(I)(a) \in F$ whenever $a \in F$.

Proof: Let U be an ideal of a polyadic algebra, by the definition U is an ideal of a Boolean algebra and $\exists(I)(a) \in U$.

Let U be an ideal of a Boolean algebra B and $\exists(I)(a) \in U$ for every $a \in U$.

If $J \subseteq I$, then $\exists(J)(a) < \exists(I)(a)$, so that $\exists(J)(a) \in U$. Now we verify that $S(\sigma)(a) \in U$, if $a \in U$ and $\sigma \in I^I$. We have $a < \exists(I)(a)$ and $S(\sigma)(a) < S(\sigma)\exists(I)(a)$, so it means to show that every $S(\sigma)$ acts as identity transformation from I^I since there is nothing out of I. Then $S(\sigma)\exists(I)(a) = (id)\exists(I)(a) = \exists(I)(a)$, therefore $S(\sigma)(a) \in U$.

A similar argument proves the second part.

Proposition: 3.2 [4] There is a one to one correspondence between ideals and filters : if U is an ideal , then the set $U' = F$ of all a' with $a \in U$ is a filter . Analogously, if F is a filter, then $U = F' = \{a' : a \in F\}$ is an ideal.

Proposition: 3.3 The set of all polyadic ideals and the set of all polyadic filters are closed under the arbitrary intersections.

Let B be a polyadic algebra and $\Gamma \subseteq B$. Let $U(\Gamma)$ denote the least polyadic ideal containing Γ and $F(\Gamma)$ denote the least polyadic filter containing Γ.We say that $U(\Gamma)$ and $F(\Gamma)$ are generated by Γ.

Proposition: 3.4 Let B be a polyadic algebra and $\Gamma \subseteq B$. Then

i) $U(\Gamma) = \{b \in B : b \leq x_1 \vee x_2 \vee ... \vee x_n \ \textit{for some} \ x_1, x_2, ..., x_n \in \Gamma\} \cup \{0\}$

ii) $F(\Gamma) = \{b \in B : b \geq x_1 \wedge x_2 \wedge ... \wedge x_n \ \textit{for some} \ x_1, x_2, ..., x_n \in \Gamma\} \cup \{1\}$

Proof:

i) Let $J = \{b \in B : b \leq x_1 \vee x_2 \vee ... \vee x_n \ \textit{for some} \ x_1, x_2, ..., x_n \in \Gamma\} \cup \{0\}$ $0 \in J$. Let $b_1, b_2 \in J$. Then $b_1 \leq x_1 \vee x_2 \vee ... \vee x_n$ and $b_2 \leq y_1 \wedge y_2 \wedge ... \wedge y_m$ for some $x_i, y_i \in \Gamma$.
$b_1 \vee b_2 \leq x_1 \vee x_2 \vee .... \vee x_n \vee y_1 \vee y_2 \vee ... \vee y_m$. Therefore $b_1 \vee b_2 \in J$. If $a \leq b \leq x_1 \vee x_2 \vee .... \vee x_n$, then $a \in J$. Then J is a Boolean ideal containing Γ. Therefore $U(\Gamma) \subseteq J$.

If $b \in J$, then $b_1 \leq x_1 \vee x_2 \vee ... \vee x_n$ where $x_i \in \Gamma$ i.e. $x_i \in U(\Gamma)$. Then $b \in U(\Gamma)$. Thus $U(\Gamma) = J$ as a Boolean ideal.

Let $a \in J$ then $a \leq x_1 \vee x_2 \vee ... \vee x_n$ $\exists(I)(a) \leq \exists(I)(x_1 \vee x_2 \vee ... \vee x_n) = \exists(I)(x_1) \vee \exists(I)(x_2) \vee ... \vee \exists(I)(x_n)$. Therefore $\exists(I)(a) \in U(\Gamma)$.

ii) A similar argument leads to (ii).

A filter F of a polyadic algebra is called ultrafilter if F is maximal with respect to the property that $0 \notin F$.

Ultrafilters satisfy the following important properties [3]

Proposition: 3.5 Let F be a filter of polyadic algebra B. Then

i) F is an ultrafilter of B iff for any $a \in F$ exactly one of a, a' belongs to F.

ii) F is ultrafilter of B iff $0 \in F$ and $a \vee b \in F$ iff $a \in F$ or $b \in F$ for any $a, b \in F$.

iii) If $a \in B - F$, then there is an ultrafilter L such that $F \subset L$ and $a \in L$.

Let $\Gamma \subseteq B$. The ultrafilter containing $F(\Gamma)$ is denoted by $UF(\Gamma)$.

A mapping $\mu : B_1 \to B_2$ between two Boolean algebras is called a Boolean homomorphism if

i) $\mu(a \wedge b) = \mu(a) \wedge \mu(b)$

ii) $\mu(a') = (\mu(a))'$

Obviously, $\mu(a \vee b) = \mu(a) \vee \mu(b)$, $\mu(0) = 0$, $\mu(1) = 1$.

A mapping $\mu : B_1 \to B_2$ between two polyadic algebras is called polyadic homomorphism if

i) μ is a Boolean homomorphism

ii) $\mu\exists = \exists\mu$

iii) $\mu\sigma = \sigma\mu$ for any $\sigma \in I^I$

Obviously, $\mu\forall = \forall\mu$.

4. DEDUCTION:

Notation: for $\Gamma \subseteq B$ and $b \in B$, $\Gamma \vdash b$ reads b is deduced from Γ in B. Now, we define $\Gamma \vdash b$ in a polyadic algebra B iff $b \in F(\Gamma)$.

By definitions of filter and deduction $\vdash$ we have:

Proposition: 4.6

i) $\Gamma \vdash 1$ and $\Gamma \nvdash 0$

ii) If $\Gamma \vdash x$, $\Gamma \vdash y$ then $\Gamma \vdash x \wedge y$.

General properties of deduction are given by the following:

Theorem: 4.7

i) If $b \in \Gamma$ then $\Gamma \vdash b$

ii) If $\Gamma \vdash b$ and $\Gamma \subseteq \Sigma$ then $\Sigma \vdash b$

iii) If $\Sigma \vdash a$ for any $a \in \Gamma$ and $\Gamma \vdash b$, then $\Sigma \vdash b$

iv) If $\Gamma \vdash b$, then $\sigma(\Gamma) \vdash \sigma(b)$ for any substitution σ

v) If $\Gamma \vdash b$, then $\Gamma_0 \vdash b$ for some finite $\Gamma_0 \subseteq \Gamma$.

Proof:

i) $b \in \Gamma \quad \therefore b \in F(\Gamma) \quad \therefore \Gamma \vdash b$

ii) $\Gamma \vdash b \quad \therefore b \in F(\Gamma) \quad \therefore b \geq x_1 \wedge x_2 \wedge \ldots \wedge x_n$ for some $x_i \in \Gamma$

$\Gamma \subseteq \Sigma \quad \therefore b \in F(\Sigma) \quad \therefore \Sigma \vdash b$

iii) Let $\Gamma \vdash b \quad \therefore b \in F(\Gamma)$

$\therefore b \geq x_1 \wedge x_2 \wedge \ldots \wedge x_n$ for some $x_i \in \Gamma$

$\Sigma \vdash x_1, \Sigma \vdash x_2, \ldots, \Sigma \vdash x_n$

$\Sigma \vdash x_1 \wedge x_2 \wedge \ldots \wedge x_n$ by proposition 4.6

$x_1 \wedge x_2 \wedge \ldots \wedge x_n \in F(\Sigma) \quad \therefore b \in F(\Sigma) \quad \therefore \Sigma \vdash b$

iv) $\Gamma \vdash b \quad \therefore b \in F(\Gamma) \quad \therefore b \geq x_1 \wedge x_2 \wedge \ldots \wedge x_n$ for some $x_i \in \Gamma$

$\sigma(b) \geq \sigma(x_1 \wedge x_2 \wedge \ldots \wedge x_n) = \sigma(x_1) \wedge \sigma(x_2) \wedge \ldots \wedge \sigma(x_n)$ for some $\sigma(x_i) \in F(\sigma(\Gamma))$

$\therefore \sigma(b) \in F(\sigma(\Gamma)) \quad \therefore \sigma(\Gamma) \vdash \sigma(b)$

v) By proposition 3.4(ii) and proposition 4.6

5. FUNCTIONAL POLYADIC ALGEBRA:

Let $(B, \vee, \wedge, ', 0, 1)$ be a complete Boolean algebra, $B^A = \{p \mid p : A \to B$ is a function$\}$ where A is an algebra of type F.

For $p, q \in B^A$ define $p \vee q$, $p \wedge q$, p' and $0,1$ pointwise as follows:

$(p \vee q)(a) = p(a) \vee q(a)$, $(p \wedge q)(a) = p(a) \wedge q(a)$, $p'(a) = (p(a))'$, $0(a) = 0$, $1(a) = 1$ for any $a \in A$.

We have:

Proposition: 5.8 [2] $(B^A, \vee, \wedge, ', 0, 1)$ is a functional Boolean algebra.

Proposition: 5.9 [2] $(B^A, \exists)$ is a functional monadic algebra, where $\exists : B^A \to B^A$ is given by $\exists(p)(a) = \sup\{p(a) : a \in A\}$.

Proposition: 5.10 $(B^A, I, S, \exists)$ is a functional polyadic algebra, where $S : I^I \to End(B^A)$ and $\exists : 2^I \to Qant(B^A)$ are given by $S(\tau)(p)(a) = p(\tau(a))$ for any $\tau \in I^I$ and $\exists(J)p(a) = \sup_J\{p(a) : a \in A\}$ for any $J \subseteq I$.

Proof: For any $J, M \in 2^I$ and any $\sigma, \tau \in I^I$ we have:

i) $\exists(\phi)p(a) = \sup_{\phi}\{p(a)\} = p(a) \qquad \therefore \exists(\phi) = id$

ii) $\exists(J \cup M)p(a) = \sup_{J\cup M}\{p(a)\} = \sup_{J}\{p(a)\}\sup_{M}\{p(a)\} = \exists(J)p(a)\exists(M)p(a) \quad \therefore \exists(J \cup M) = \exists(J)\exists(M)$.

iii) $S(id)p(a) = p(id(a)) = p(a) \qquad \therefore S(id) = id$

iv) $S(\sigma\tau)p(a) = p(\sigma(\tau(a))) = S(\sigma)p(\tau(a)) = (S(\sigma)S(\tau))p(a) \quad \therefore S(\sigma\tau) = S(\sigma)S(\tau)$

v) $S(\sigma)\exists(J)p(a) = S(\sigma)\sup_{J}\{P(a)\} = S(\tau)\sup_{J}\{p(a)\}$ if $\sigma|_{I-J} = \tau|_{I-J}$

$= S(\tau)\exists(J)p(a) \qquad \therefore S(\sigma)\exists(\tau) = S(\tau)\exists(J)$ if $\sigma|_{I-J} = \tau|_{I-J}$

vi) $\exists(J)S(\tau)p(a) = \exists(J)p(\tau(a))\sup_{J}\{p(\tau(a))\} = \sup_{\tau^{-1}(J)}\{S(\tau)p(a)\}$ if τ is $1-1$ on $\tau^{-1}(J)$

$= S(\tau)\sup_{\tau^{-1}(J)}\{p(a)\} = S(\tau)\exists(\tau^{-1}(J))p(a) \qquad \therefore \exists(J)S(\tau) = S(\tau)\exists(\tau^{-1}(J))$.

Natural inference rules of polyadic logic are now transferred into certain algebraic rules in B^A governing the algebraic deduction in $F(\Gamma)$ where $\Gamma \subseteq B^A$. These are given as follows:

Theorem: 5.11 Let $\Gamma \subseteq B^A$. Then

i) $\Gamma \vdash 1$ and $\Gamma \nvdash 0$.

ii) $\Gamma \vdash p \wedge q$ iff $\Gamma \vdash p$ and $\Gamma \vdash q$.

iii) $\Gamma \vdash p$ iff $\Gamma \nvdash p'$.

iv) $\Gamma \vdash p \vee q$ iff $\Gamma \vdash p$ or $\Gamma \vdash q$.

v) $\Gamma \vdash p$ iff $\Gamma \vdash \forall(J)p$.

vi) $F(\Gamma) \vdash p(a_0)$ iff $F(\Gamma) \vdash \exists(J)p(a)$.

Proof:

i) By proposition 5.6(i)

ii) Follows from definition of filter and proposition 3.1

iii) This is so because if both $\Gamma \vdash p$ and $\Gamma \vdash p'$ we get $p(a)$ and $(p(a))'$ in $F(\Gamma)$. Thus we get 0, which is a contradiction.

iv) Let $\Gamma \vdash p \vee q \qquad \therefore p(a) \vee q(a) \in F(\Gamma) \qquad \therefore (p(a) \vee q(a))' \notin F(\Gamma)$

$(p(a))' \wedge (q(a))' \notin F(\Gamma)$

$(p(a))' \notin F(\Gamma)$ or $(q(a))' \notin F(\Gamma) \qquad \therefore p(a) \in F(\Gamma)$ or $q(a) \in F(\Gamma)$. Thus $\Gamma \vdash p$ or $\Gamma \vdash q$.

Let $\Gamma \vdash p$ or $\Gamma \vdash q \qquad \therefore p(a) \in F(\Gamma)$ or $q(a) \in F(\Gamma) \qquad \therefore p(a) \vee q(a) \geq p(a)$

$\therefore p(a) \vee q(a) \in F(\Gamma)$. Therefore $\Gamma \vdash p \vee q$.

v) This is by $\forall p \leq p$ and $\forall(F(\Gamma)) \subseteq F(\Gamma)$.

vi) This is so by the definition $\exists(J)p(a) = \sup_{J}\{p(a) : a \in A\}$.

If we alter the definition of deduction to $\Gamma \vdash p$ if $p \in UF(\Gamma)$ we get the following dichotomy by proposition 3.5

Theorem: 5.12 Either $\Gamma \vdash p$ or $\Gamma \vdash p'$.

6. TERMS:

Let X be a set of variables and $\mathcal{F}$ be a type of algebra. The set $T(X)$ of terms of type $\mathcal{F}$ over X is the smallest set such that

i) $X \cup \mathcal{F}_0 \subseteq T(X)$

ii) If $t_1,...,t_n \in T(X)$ and $f \in \mathcal{F}_n$ then $f(t_1,...,t_n) \in T(X)$.[3]

For $t \in T(X)$ we often write t as $t(x_1,...,x_n)$ to indicate that the variables occurring in t are among $x_1,...,x_n$ A term t is an n-ary if the number of variables appearing explicitly is less than or equal to n.

The term t defines a function $t_A : A^n \to A$ as follows $t_A(a_1,...a_n) = t(x_i / a_i)$ for $a_1,...,a_n \in A$. The term algebra $T(X)$ of type F over X has as its universe the set $T(X)$ and the fundamental operations satisfy:

$$f^{T(X)} : (t_1,...t_n) \mapsto f(t_1,...,t_n) \text{ for } f \in \mathrm{F}_n \text{ and } t_i \in T(X) \ , 1 \le t \le n .$$

Now, consider the functional polyadic algebra $(B^A, I, S, \exists)$. A term t defines a function $t_* : B^A \to B^A$ as follows: $t_*(p)(a) = pt_A(a) = pt(x/a)$, where $x = (x_1,...,x_n)$ and $a = (a_1,...,a_n)$.

Proposition: 6.13 [1]

i) $t_* \in End(B^A)$

ii) $\exists(J)\, t_* = t_* \exists(J)$ for any $J \subseteq I$.

Note that $t_*(F(\Gamma)) = F(t_*(\Gamma))$ and $t_*(UF(\Gamma)) = UF(t_*(\Gamma))$. Define $\Gamma \vdash pt$ if $pt \in F(t_*(\Gamma))$.Then theorem 5.10 can be generalized with respect to terms as follows:

Theorem: 6.14

i) $\Gamma \vdash pt \wedge qt$ iff $\Gamma \vdash pt$ and $\Gamma \vdash qt$.

ii) $\Gamma \vdash pt$ iff $\Gamma \nvdash p't$

iii) $\Gamma \vdash pt \vee qt$ iff $\Gamma \vdash pt$ or $\Gamma \vdash qt$.

iv) $\Gamma \vdash pt$ iff $\Gamma \vdash \forall(J)pt$.

v) $\Gamma \vdash pt(a_0)$ iff $\Gamma \vdash \exists(J)pt(a)$.

Also we have:

Theorem: 6.15 With respect to $UF(\Gamma)$, either $\Gamma \vdash pt$ or $\Gamma \vdash p't$.

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