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SOME ALGORITHMS FOR SOLVING PRIMAL-DUAL LINEAR PROGRAMMING USING BARRIER METHODS

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ABSTRACT

There are many literatures in the field of interior point methods for exploring the properties of linear programming and its dual linear programming. Most of them use advanced mathematical arguments to introduce the field of linear programming. The purpose of this paper is to show that the interior point methods by barrier function approach can be used to introduce the field of linear programming. By using only elementary analytically results, we have proved that for every value of the barrier parameters, the primal logarithmic barrier function for the primal problem has a unique minimizer, and then the sequence of the minimizers converges to a minimizer of the primal problem. From this, we present an algorithm for solving primal problem using barrier function methods.

Similarly, we have also proved that for every value of the barrier parameters, the dual logarithmic barrier function for the dual problem has a unique maximizer, and then the sequence of the maximizers converges to a maximizer of the dual problem. From this, we also present an algorithm for solving dual problem using barrier function methods. Finally, by introducing a system called centering conditions, we also present a characterization of existence of optimal solution for primal-dual problem using barrier function methods.

Keywords: *Barrier Function, Boundedness, Centering Conditions, Linear Programming, and Interior Points*

1. INTRODUCTION

Since it is an efficient and reliable solution technique the simplex method becomes a favorite method for solving the linear programming problem (LP).

In 1984 Karmarkar proposed a new interior point method for LP. He showed that the algorithm has a polynomial-time complexity bound and claimed that it outperforms the simplex method on many problems. In very influential papers, Guler [4] and Mehotra [5] emphasized the role of the central path in the design and the analysis of interior point method. A variation of Karmarkar's algorithm for solving LP problems has been introduced by Barnes [1]. Recently, Corradi [2] use higher-order derivatives in LP for solving LP problems. Wolfe [6] and Fletcher [3] had introduced a barrier function for analyzing a LP problem.

In this paper, we use only elementary analytically results to analyze a LP problem and present an algorithm for solving the LP Problem.

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2. LINEAR PROGRAMMING PROBLEMS

In this paper, we consider the (primal) linear programming problems in the symmetric form

$$\text{(P)}\quad \text{Min } \{c^T x : Ax \geq b,\ x \geq 0\} \tag{1}$$

where $A = (a_{ij})_{m\times n}$, $c, x \in R^n$ and $b \in R^m$.

The feasible region of (P) is defined by

$$F_P = \{x : Ax \geq b,\ x \geq 0\}. \tag{2}$$

and the set of the feasible interior points of (P) is given by

$$F_P^0 = \{x :\ Ax > b,\ x > 0\}. \tag{3}$$

The optimal set of (P) is given by

$$F_P^* = \{x^* \in F_P : c^T x^* \leq\ c^T x,\ x \in F_p\}. \tag{4}$$

If $F_p^0 \neq \phi$, then F_p and F_p^* is nonempty and bounded.

3. THE PRIMAL LOGARITHMIC BARRIER FUNCTION

In this paper, the primal logarithmic barrier functions is defined by

$$\text{(PLB)}\quad g(x, \mu) = c^T x - \mu \sum_{i=1}^{m} \ln(A_i x - b_i)\ ,\ x \in F_p^0 \tag{5}$$

where $A_i = (a_{i1}, a_{i2}, \ldots, a_{in})$, $b = (b_1, b_2, \ldots, b_m)^T$ and $\mu > 0$ is called barrier parameter. It is straightforward from the definition of the function $g(x, \mu)$ we can see that $c^T x$ and $-\ln(A_i x - b_i)$ are both convex; therefore we have the following theorem.

Theorem 1 The primal logarithm barrier function $g(x, \mu)$ is convex in F_p^0.•

The compactness property of the function $g(x, \mu)$ is written in the following theorem.

Theorem 2 (Compactness) Suppose that $F_p^0 \neq \phi$. The level set of the primal logarithmic barrier function defined by

$$S(\sigma) = \{x \in F_p^0 : g(x, \mu) \leq \sigma\} \tag{6}$$

is bounded and closed, for any $\mu > 0$.

Proof: Let $x_0 \in F_p^0$ and given any $\varepsilon > 0$. It is easy to show that the level set

$$D_0 = \{x \in F_p : c^T x \leq c^T x_0 + \varepsilon\} \tag{7}$$

is bounded and closed.

We now can prove boundedness of $S(\sigma)$ by contradiction. Assume that the contrary of the desired result, namely that for some $\mu_k > 0$, there is an unbounded sequence $\{y_j\}$ of points F_p^0 for which the barrier function values $g(x, \mu_k)$ remain bounded above.

For such a sequence, let j be sufficiently large so that y_j lies outside D_0. Therefore by the definition of D_0 we have

$$c^T y_j > c^T x_0 + \varepsilon.$$

Let z_j be the point on the boundary of D_0 where the line connecting x_0 and y_j intersects the boundary. Since D_0 is convex, z_j is unique. Let λ_j be the scalar satisfying $0 < \lambda_j < 1$ such that

$$z_j = (1 - \lambda_j)x_0 + \lambda_j y_j . \tag{8}$$

We have assumed that $\{y_j\}$ is unbounded for sufficiently large j. Since $\{z_j\}$ is finite, (8) shows that

$$\lambda_j \to 0 \text{ as } j \to \infty. \tag{9}$$

Because x_0 and y_j are both in F_p^0, we know that $x_0 > 0$ and $y_j > 0$; and thus

$$z_j = (1 - \lambda_j)x_0 + \lambda_j y_j > 0, \tag{10}$$

which shows that $z_j \in F_p^0$. Since z_j is by definition in bnd(D_0), we conclude from (7) that $c^T z_j = c^T x_0 + \varepsilon$.

By multiplying two sides which c^T to (10) implies

$$c^T z_j = (1 - \lambda_j)c^T x_0 + \lambda_j c^T y_j.$$

Dividing by λ_j and substituting $c^T z_j = c^T x_0 + \varepsilon$, we obtain

$$c^T y_j = c^T x_0 + \frac{\varepsilon}{\lambda_j} \tag{11}$$

so that the objective function values at $\{y_j\}$ become unbounded.

Turning back to the constraint functions, positivity of λ_j means that the first equality in (10) can be written as

$$y_j = x_0 + \frac{z_j - x_0}{\lambda_j}. \tag{12}$$

Since the set bnd(D_0) is compact, the function $x - x_0$ achieves its maximum for some $x \in$ bnd(D_0). Let d denote

$$d = \max\{ x - x_0 : x \in \text{bnd}(D_0)\}.$$

We now wish to demonstrate that $d \geq 0$. Because $z_j \in$ bnd(D_0) and $y_j > 0$, we apply the definition of d and relation (12) to show that

$$x_0 + \frac{d}{\lambda_j} \geq y_j > 0. \tag{13}$$

If d were negative, the first expression in (13) would eventually become negative as $\lambda_j \to 0$, which impossible. It follows that $d \geq 0$.

Finally, the barrier function $g(y_j, \mu_k)$ is formed. Using (11), (13), monotonicity of the logarithm function, and positivity of μ_k, we have

$$\begin{aligned} g(y_j, \mu_k) &= c^T y_j - \mu_k \sum_{i=1}^{m} \ln(A_i y_j - b_i) \\ &\geq c^T x_0 + \frac{\varepsilon}{\lambda_j} - \mu_k \sum_{i=1}^{m} \ln[A_i(x_0 + \frac{d}{\lambda_j}) - b_i] \\ &= c^T x_0 + \frac{\varepsilon - \mu_k \lambda_j \sum_{i=1}^{n} \ln\left[(A_i x_0 - b_i) + \frac{A_i d}{\lambda_j}\right]}{\lambda_j}. \end{aligned} \tag{14}$$

The logarithm function has property that for a positive constant ν and $\lambda > 0$,

$$\lim_{\lambda \to 0^+} \lambda \ln\left(\nu + \frac{A_i d}{\lambda}\right) = 0.$$

Thus the limit of the numerator in (14) is ε, and the quotient in (14) is unbounded above as $\lambda_j \to 0$. It follows that $g(y_j, \mu_k)$ is unbounded above as $j \to \infty$, thereby contradicting our assumption that the barrier function values $\{g(y_j, \mu_k)\}$ are bounded above for an unbounded sequence $\{y_j\}$. This proves that $S(\sigma)$ is bounded.

To show that $S(\sigma)$ is closed, we will prove that it contains all its limit point. Let $\{x_j\}$ be a convergent sequence in $S(\sigma)$, with limit point x. It follows from the continuity of $g(x_j, \mu_k)$ that x must satisfy $g(x, \mu_k) \leq \sigma$. Furthermore, x must either be in F_p^0 or else have the property that $x_i = 0$ for at least one index i.

If x is in F_p^0, by definition x is in $S(\sigma)$. Suppose that x is not in F_p^0. Then, since $x_i = 0$ for some index i, then the unboundedness of the logarithm for a zero argument and convergence of $\{x_j\}$ to x together imply that, for sufficientlylarge j, the barrier term $-\sum_{i=1}^{m} \ln(A_i x_j - b_i)$ cannot be bounded above. In particular, for any constant γ and sufficiently large *j*,

$$-\sum_{i=1}^{m}\ln(A_i x_j - b_i) > \gamma. \quad (15)$$

We now define $\gamma = (\sigma - f^*)/\mu_k$, where f^* be the objective value of problem (*P*); the value of γ is finite because f^* is finite. Since x_j lies in F_p^0, we know that $c^T x_j \geq f^*$, which means that $-f^* \geq -c^T x_j$. Applying this inequality and definition of γ in (15), we obtain

$$-\sum_{i=1}^{m}\ln(A_i x_j - b_i) > \gamma > \frac{\sigma - c^T x_j}{\mu_k}. \quad (16)$$

After rearrangement, this relation implies that $g(x_j, \mu_k) > \sigma$, i.e., $x_j \notin S(\sigma)$, a contradiction. We conclude that any limit point of a sequence in S(σ) must lies in S(σ), which means that S(σ) is closed. We have shown that S(σ) is bounded and closed; its compactness is immediate.

The following theorem explains the existence of the global unconstrained minimizer of the barrier methods for primal linear programming.

Theorem 3 Suppose that f* denotes the optimal value of (P). If $F_p^0 \neq \phi$ and let $\{\mu_k\}$ be a decreasing sequence of positive barrier parameters such that $\mu_k \to 0$ as $k \to \infty$.
Then

(i) $g(x, \mu_k)$ has a finite unconstrained minimizer in F_p^0 for every $\mu_k > 0$, and the set M_k of unconstrained minimizers of $g(x, \mu_k)$ in F_p^0 is compact for every k,
(ii) any unconstrained local minimizers of $g(x, \mu_k)$ in F_p^0 is also a global unconstrained minimizer of $g(x, \mu_k)$,
(iii) for all k ≥1,

$$c^T y_{k+1} \leq c^T y_k \quad \text{and} \quad -\sum_i \ln(A_i y_k - b_i) \leq -\sum_i \ln(A_i y_{k+1} - b_i)$$

where y_k denotes an unconstrained minimizer of $g(x, \mu_k)$ in F_p^0 and
there exists a compact set S such that for all k ≥1, every minimizer y_k of $g(x, \mu_k)$ lies in $S \cap F_p^0$.

Proof: Let $x_0 \in F_p^0$. For the barrier parameter μ_k and ε > 0, we define the set S_0 as

$$S_0 = \{x \in F_p^0 : g(x, \mu_k) \leq g(x_0, \mu_k) + \varepsilon\}.$$

Theorem 2 implies that S_0 is compact for all $\mu_k > 0$. It follows that the smooth function $g(x, \mu_k)$ achieves its minimum in S_0, necessarily in F_p^0. We then conclude that $g(x, \mu_k)$ has at least one finite unconstrained minimizer.

Since $g(x, \mu_k)$ is convex, any local minimizer is also a global minimizer, so that every unconstrained minimizers of $g(x, \mu_k)$ must be in S_0. Thus the set M_k of unconstrained minimizers of $g(x, \mu_k)$ is bounded. The set M_k is closed because the minimum value of $g(x, \mu_k)$ is unique, and it follows that M_k is compact. Therefore, the result (i) and (ii) have been verified.

To show validity of (iii), let y_k and y_{k+1} denote global minimizers of barrier function corresponding to the barrier parameters μ_k and μ_{k+1}, respectively. By definition of y_k and y_{k+1}, we have

$$c^T y_k - \mu_k \sum_i \ln(A_i y_k - b_i) \leq c^T y_{k+1} - \mu_k \sum_i \ln(A_i y_{k+1} - b_i); \quad (17)$$

$$c^T y_{k+1} - \mu_{k+1} \sum_i \ln(A_i y_{k+1} - b_i) \leq c^T y_k - \mu_{k+1} \sum_i \ln(A_i y_k - b_i). \quad (18)$$

Through multiply the inequality (17) by the ratio μ_{k+1}/μ_k, adding the inequality (18) and cancel the terms involving logarithms, we obtain

$$c^T y_{k+1}\left(1 - \frac{\mu_{k+1}}{\mu_k}\right) \leq c^T y_k \left(1 - \frac{\mu_{k+1}}{\mu_k}\right).$$

Since $0 < \mu_{k+1} < \mu_k$, it follows that $c^T y_{k+1} \leq c^T y_k$. Applying this result in (17) and dividing by the positive number μ_k to give

$$-\sum_i \ln(A_i y_k - b_i) \leq -\sum_i \ln(A_i y_{k+1} - b_i), \quad (19)$$

as required for the second part of (iii).

To verify existence of the set S in (iv), we use result (iii). Since $c^T y_{k+1} \leq c^T y_k$ for each *k*, the compact convex level set

$$\{x \in F_p : c^T x \leq c^T y_k\}$$

not only contains all minimizers of $g(x, \mu_k)$, but also contains all minimizers of $g(x, \mu_{k+1})$, because

$$\{x \in F_p : c^T x \leq c^T y_k\} \supseteq \{x \in F_p^0 : g(x, \mu_k) \leq c^T y_k\}$$

The compact level set *S* defined by the strictly feasible point y_0,

$$\{x \in F_p^0 : g(x, \mu_k) \leq c^T y_0\} = S. \quad (20)$$

Accordingly contains F_p^* as well as minimizers of $g(x, \mu_k)$ for all *k*.

The following theorem explains the convergence of the barrier methods for primal linear programming.

Theorem 4 Suppose that f* denotes the optimal value of (P). If $F_p^0 \neq \phi$, Let $\{\mu_k\}$ be a decreasing sequence of positive barrier parameters such that $\mu_k \to 0$ as $k \to \infty$ and $\{y_k\}$denotes global minimizers sequence of barrier function corresponding to the barrier parameters μ_k .
Then

(i) $\{y_k\}$ has at least a subsequence converge $\{x_k\}$ and its limit point is a local constrained minimizer of problem (P),

(ii) $c^T x_k \to f^*$ as $k \to \infty$ where $\{x_k\}$ denotes the subsequence of an unconstrained minimizers of $g(x, \mu_k)$ and

(iii) $g(x_k, \mu_k) \to f^*$ as $k \to \infty$.

Proof: We now show validity of (i). It follows from the Theorem 3 that every minimizer of y_k must lie in the compact set *S* defined by (20). We conclude that the sequence $\{y_k\}$ is uniformly bounded, and hence contains at least convergent subsequence, say with limit point $\underline{x}$. Because y_k lies in S for all k, $\underline{x}$ must be feasible.

In order to prove that $\underline{x}$ is a local constrained minimizer of the primal problem (P), we assume that $\underline{x} \notin F_p^*$. Since every local solution of a primal problem is a global solution, this would imply that $c^T\underline{x} > f^*$. A contradiction is now established from this inequality and the definition of $\underline{x}$ as a limit point of a convergent subsequence of minimizers of $g(x, \mu_k)$.

Let $\{x_k\}$ denote a subsequence of $\{y_k\}$ converging to $\underline{x}$. Continuity of c^Tx and the relation $c^Ty_k \geq c^Ty_{k+1}$ imply that, for all k,

$$c^Tx_k \geq c^T\underline{x}. \tag{21}$$

We next show that there must exist a strictly feasible point x_{int} such that

$$c^T\underline{x} > c^Tx_{\text{int}}.$$

Let x^* denote any point in the set of F_p^* of constrained minimizers, so that $c^Tx^* = f^*$ and x^* is in the set of S defined by (20). If x^* itself is strictly feasible, we simply take $x_{\text{int}} = x^*$, since our initial assumption was that $c^T\underline{x} > c^Tx^*$. If x^* is not strictly feasible, x_{int} is found as follows. By assumption, F_p^0 is nonempty, and hence contains at least one point, sat z; the definition and uniqueness of f^* imply that $c^Tz \geq c^Tx^*$. If $c^Tz < c^T\underline{x}$, z may be taken as x_{int}. If $c^Tz \geq c^T\underline{x}$, consider a generic point w on the line segment joining x^* and z, defined by $w = (1- \lambda)x^* + \lambda z$ for λ satisfying $0 < \lambda < 1$. So we have $w > 0$, i.e., w is strictly feasible.

Multiply by c^T to two sides, implies that $c^Tw = (1- \lambda)c^Tx^* + \lambda c^Tz$, where $c^Tz \geq c^T\underline{x} > c^Tx^*$. Using continuing of c^Tx, we see that that $c^Tw < c^T\underline{x}$ for some suitably small λ, namely λ such that

$$\lambda < \frac{c^T\underline{x} - c^Tx^*}{c^Tz - c^Tx^*} \leq 1. \tag{22}$$

For any λ satisfying (22), w may be taken as x_{int}.
Thus far we have shown that, if $\underline{x}$ is not in the minimizing set F_p^*, then a strictly feasible point x_{int} exists such that

$$c^Tx_k \geq c^T\underline{x} > c^Tx_{\text{int}}. \tag{23}$$

Since x_k is a global minimizer of $g(x, \mu_k)$,

$$c^Tx_k - \mu_k\sum_i \ln(A_ix_k - b_i) \leq c^Tx_{\text{int}} - \mu_k\sum_i \ln(A_ix_{\text{int}} - b_i). \tag{24}$$

Taking $k \to \infty$, we have

$$\lim_{k\to\infty} g(x_k, \mu_k) = c^T\underline{x} \text{ and } \lim_{k\to\infty} g(x_{\text{int}}, \mu_k) = c^Tx_{\text{int}};$$

thus, we obtain inequality $c^T\underline{x} \leq c^Tx_{\text{int}}$, which contradicts with (23).

Suppose that $\underline{x}$ is not in F_p^0, so that $(\underline{x})_i = 0$ for at least one index i. Adding a barrier term involving x_{int} to both sides of the inequality $c^Tx_k > c^Tx_{\text{int}}$, we have

$$c^Tx_{\text{int}} - \mu_k\sum_i \ln(A_ix_{\text{int}} - b_i) < c^Tx_k - \mu_k\sum_i \ln(A_ix_{\text{int}} - b_i).$$

Combining this inequality with (24), we obtain

$$c^Tx_k - \mu_k\sum_i \ln(A_ix_k - b_i) < c^Tx_k - \mu_k\sum_i \ln(A_ix_{\text{int}} - b_i).$$

Canceling c^Tx_k from both sides and dividing by μ_k, then gives

$$-\sum_i \ln(A_ix_k - b_i) < -\sum_i \ln(A_ix_{\text{int}} - b_i).$$

The sum on the right hand side involving x_{int} is finite. However, $\underline{x}$ is not stricly feasible,

$-\ln(A_ix_k - b_i)$ approaches infinity for at least one i. The left hand side is therefore unbounded above, and we again have a contradiction.

The conclusion is that $\underline{x}$ lies in F_p^*, the set of minimizers. Since $\underline{x}$ is the limit point of $\{x_k\}$, we have obtained the crucial result (i). For the remainder of the proof x^* will denote the limit point of $\{x_k\}$.

Part (ii) of the theorem follows immediately from the fact that $\lim_{k\to\infty} x_k = x^*$, and c^Tx continue, then we have

$$\lim_{k\to\infty} c^Tx_k = c^Tx^* = f^*.$$

To show validity of (iii), we note first that the optimal value of $g(x, \mu_k)$ is unique, and is equal to $g(x_k, \mu_k)$. We distinguish two cases, depending on whether or not x^* (the limit point of $\{x_k\}$) is strictly feasible.

If x^* is strictly feasible, the sum of logarithm of $(A_ix_k - b_i)$ remains finite as $k \to \infty$. It is easy to see that in this case

$$\lim_{k\to\infty} g(x_k, \mu_k) = c^Tx^* = f^*.$$

Consider the other possibility, that x^* is not strictly feasible. Since at least one $(x_k)_i$ converging to zero, the barrier term $-\sum_i \ln(A_ix_k - b_i)$, must be *positive* for all sufficiently large k. Combining this property with (19),
we have

$$0 < -\sum_i \ln(A_ix_k - b_i) \leq -\sum_i \ln(A_ix_{k+1} - b_i). \tag{25}$$

One implication of this result is that, for sufficiently large k,

$$g(x_k, \mu_k) > c^Tx_k. \tag{26}$$

In addition, the minimizing property of x_{k+1}, the first inequality in (25), and the relation $\mu_{k+1} < \mu_k$ together give

$$c^Tx_{k+1} - \mu_{k+1}\sum_i \ln(A_ix_{k+1} - b_i) \leq c^Tx_k - \mu_{k+1}\sum_i \ln(A_ix_k - b_i)$$

$$c^T x_k - \mu_{k+1} \sum_i \ln(A_i x_k - b_i) \le c^T x_k - \mu_k \sum_i \ln(A_i x_k - b_i),$$

which show that, for sufficiently large k,

$$g(x_{k+1}, \mu_{k+1}) \le g(x_k, \mu_k). \tag{27}$$

It follows from (26) and (27) that, for sufficiently large k,

$$f^* \le \ldots \le g_{k+1} \le g_k, \tag{28}$$

where g_k denotes $g(x_k, \mu_k)$. The sequence of $\{g_k\}$ of barrier function values is consequently non increasing and bounded from below, and must converge monotonically from above to a limit, say g^*, where $g^* \ge f^*$.

Suppose that $g^* > f^*$. In this case, we define a positive number $\delta := \frac{1}{2}(g^* - f^*)$. It follows from continuity of $c^T x$ that there must be a neighborhood of x^* in which

$$c^T x \le g^* - \delta, \tag{29}$$

for all x in the neighborhood. Consider a particular strictly feasible point x_0 in this neighborhood. Strictly feasibility of x_0 implies that the quantity $\sum_i \ln(x_0)_i$ is finite. Because $\mu_k > 0$ and $\mu_k \to 0$, there must be an integer K such that, for $k \ge K$,

$$-\mu_k \sum_i \ln(Ax_0 - b_i) < \tfrac{1}{2}\delta. \tag{30}$$

Since x_k is a global minimizer of $g(x_k, \mu_k)$, we know that

$$g(x_k, \mu_k) \le g(x_0, \mu_k) = c^T x_0 - \mu_k \sum_i \ln(Ax_0 - b_i).$$

If we apply (29) and (30), the result is

$$g(x_k, \mu_k) \le g^* - \delta + \tfrac{1}{2}\delta = g^* - \tfrac{1}{2}\delta,$$

which contradicts the monotonically convergence of $\{g_k\}$ to g^* from above. We conclude $g^* = f^*$, which gives result (viii).

The implications of this theorem are remarkable strong. For any primal linear programming with nonempty set of interior region, the barrier function has a finite unconstrained minimizer for every value of the barrier parameter, and every limit point of a minimizing sequence for the barrier function is a constrained minimizer of a primal problem.

Algorithm 1. Data $Ax \ge b$, $\mu_1 > 0$ and $\varepsilon > 0$.
1. Choose $x_1 \in R^n$ such that $Ax_1 = b$ and $x_1 > 0$.
2. If the optimality conditions are satisfied for problem (P) at x_1, then stop.
3. Compute y_1 the minimum of $g(x, \mu_1)$.
4. Compute $y_k := \min_x g(x, \mu_k)$ and $\mu_k := \dfrac{\mu_{k-1}}{10^k}$ for $k = 2$.
5. If $| y_k - y_{k-1} | < \varepsilon$ then stop, else $k := k + 1$ and go to step 4.

Consider the barrier subproblem involves minimizing $g(x, \mu)$ subject to linear equality constraints:

$$\text{minimize } c^T x - \mu \sum_{i=1}^{n} \ln x_i \quad \text{subject to} \quad Ax = b \tag{7}$$

The associated Lagrangian function for subproblem (7) is

$$L(x, y) = c^T x - \mu \sum_{i=1}^{n} \ln x_i - y^T(Ax - b),$$

where y is an m-vector. The optimality conditions for linear equality constraints imply the existence of y such that the solution of (7) satisfies

$$c - \mu X^{-1} e - A^T y = 0 \text{ and } Ax = b,$$

where $X = \text{diag}(x_1, \ldots, x_n)$ and $e^T = (1, \ldots, 1)$.

Defining $s = \mu X^{-1} e$, we may write

$$A^T y + s = c,\ Ax = b, \text{ and } Xs = \mu e.$$

So, if $g(x,\mu)$ has a minimizer in F_p^0, the following system of equations:

$$\begin{aligned} Ax &= b,\ x \ge 0 \\ A^T y + s &= c,\ s \ge 0 \\ Xs &= \mu e, \end{aligned} \tag{8}$$

have a (unique) solution. Conversely, assume that the system (8) have a solution. Since $\mu > 0$, x and s must be positive at a solution (x, y, s) of the system. This means that F^0 is nonempty especially F_p^0 is nonempty. For reasons that will become clear later, we shall designate the equations as the *centering conditions*. Thus, we have the following properties.

Theorem 3. Let $\mu > 0$. The following statements are equivalent:

(1) F_p^0 is nonempty;
(2) there exists a (unique) minimizer of $g(x, \mu)$ on F_p^0;
(3) the centering conditions have a (unique) solution.

4. THE DUAL LOGARITHMIC BARRIER FUNCTION

We consider the dual of the primal linear programming as follows

1. $\max\{b^T y : A^T y + s = c, s \ge 0\}$

The dual logarithmic barrier functions is given by

$$h(y, \mu) = -b^T y - \mu \sum_{i=1}^{n} \ln s_i,\ s \in F_d^0,$$

where $\mu > 0$ and $F_d^0 = \{s: A^T y + s = c, s > 0\}$ be interior point set of the feasible region of (D). If $F_p^0 \ne \phi$, then we have that F_d^* be optimal set of problem (D) is nonempty and bounded [4].

The following theorem is similar with Theorem 1 where its proof also can be omitted.

Theorem 4

$$S(\sigma) = \{y \in F_d^0 : h(y, \mu_k) \le \sigma\}$$

the level set of the dual logarithmic barrier function under the assumption that $F_d^0 \ne \phi$ is bounded and closed, for any $\mu_k > 0$.

We are now ready to give the main theorem concerning barrier methods for dual linear programming and its proof is also similar with the proof of Theorem 2.

Theorem 5 Consider the dual linear programming. Let F_d denote the feasible for this problem, and assume that F_d^0 is nonempty. Let $\{\mu_k\}$ be a decreasing sequence of positive barrier parameters such that $\lim_{k\to\infty}\mu_k = 0$. Assume that F_d^* of constrained local minimizers of the dual linear programming is nonempty and bounded, and let -f* denote the optimal value of (D). Then

(i) the dual logarithmic barrier function $h(y, \mu_k)$ is convex in F_d^0;

(ii) $h(y, \mu_k)$ has a finite unconstrained minimizer in F_d^0 for every $\mu_k > 0$, and the set M_k of unconstrained minimizers of $h(y, \mu_k)$ in F_d^0 is convex and compact for every k;

(iii) any unconstrained local minimizers of $h(y, \mu_k)$ in F_d^0 is also a global unconstrained the minimizer of $h(y, \mu_k)$;

(iv) let y_k denote an unconstrained minimizer of $h(y, \mu_k)$ in F_d^0; then, for all k,

$$-c^Ty_k+1 \le -c^Ty_k \text{ and } -\sum_i \ln(s_k)_i \le -\sum_i \ln(s_{k+1})_i ;$$

(v) there exists a compact set S such that, for all k, every minimizing point y_k of $h(y, \mu_k)$ lies in $S\cap F_d^0$;

(vi) any sequence $\{y_k\}$ of unconstrained minimizers of $h(y, \mu_k)$ has at least one convergence, and every limit point of $\{y_k\}$ is a local constrained minimizer of the dual problem (D);

(vii) let $\{x_k\}$ denote a subsequence of an unconstrained minimizers of $h(y, \mu_k)$ then

$b^T x_k \to f^*$ as $k \to \infty$;and

(viii) $h_k \to -f^*$ as $k \to \infty$ where h_k denotes $h(y_k, \mu_k)$.

The implications of this theorem are remarkable strong. For any dual linear programming with nonempty set of interior region, the barrier function has a finite unconstrained minimizer for every value of the barrier parameter, and every limit point of a minimizing sequence for the barrier function is a constrained minimizer of a dual problem.

Algorithm 2

Data $A^Ty + s = b$, $\mu_1 > 0$ and $\varepsilon > 0$.

1. Choose $s_1 \in R^n$ such that $A^Ty + s_1 = b$ and $s_1 > 0$.
2. If the optimality conditions are satisfied for problem (D) at s_1, then stop.
3. Compute y_1 the minimum of $h(y, \mu_1)$
4. Compute $y_k := \min_x h(x, \mu_k)$ and $\mu_k := \dfrac{\mu_{k-1}}{10^k}$ for k = 2.
5. If $| y_k - y_{k-1} | < \varepsilon$ then stop, else k := k + 1and go to step 4.

Consider the barrier subproblem involves minimizing $h(y, \mu)$ subject to linear equality constraints:

$$\text{minimize } -b^Ty - \mu\sum_{i=1}^{n}\ln s_i \quad \text{subject to} \quad A^Ty + s = c. \tag{9}$$

The associated Lagrangian function for subproblem (9) is

$$L(y, s, x) = -b^Ty - \mu\sum_{i=1}^{n}\ln s_i + x^T(A^Ty + s - c),$$

where x is an n-vector. The optimality conditions for linear equality constraints imply the existence of x such that the solution of (9) satisfies

$$-b + Ax = 0,\ -\mu S^{-1}e + x = 0, \text{ and } A^Ty + s = c.$$

We may write

$$Ax = b,\ A^Ty + s = c, \text{ and } Xs = \mu e.$$

So, if $h(y, \mu)$ has a minimizer in F_d^0, then the centering conditions have a solution. Conversely, the centering conditions have a solution.. Because $\mu > 0$, x and s must be positive at a solution (x, y, s) of the system. This means that F^0 is nonempty, especially F_d^0 is nonempty. Thus, we have the following properties.

Theorem 6 Let $\mu > 0$. The following statements are equivalent:

(1) F_d^0 is nonempty;

(2) there exists a (unique) minimizer of $h(y, \mu)$ on F_d^0;

(3) the centering conditions have a (unique) solution.

5. THE PRIMAL-DUAL BARRIER FUNCTION

Let $\mu > 0$ be the barrier parameter, then the primal-dual barrier function is defined by $f_\mu : F^0 \to R$ as

$$f_\mu(x, s) = x^Ts - \mu\sum_{i=1}^{n}\ln x_i s_i , \tag{10}$$

where $F^0 = \{(x, s): Ax = b, A^Ty + s = c, x > 0, s > 0\}$.

One may easily verify that the first term in (10) is linear on the domain of f_μ and that the Hessian of the second term is positive definite. It follows that f_μ is strictly convex. We consider the barrier problem

$$\min \{ f_\mu(x,s) : (x,s) \in F^0\}$$

and study the properties of its solution. It is convenient to give an alternative expression of the barrier function. Letting $g : R^+ \to R$ be the function in the variable t defined by

$$g(t) = t - \mu\ln t,$$

we may write

$$f_\mu(x, s) = \sum_{i=1}^{n} g(x_i s_i). \tag{11}$$

The function g has some properties that will be of great help in proving that f_μ achieves its minimum value on F^0. These properties are collected in the following lemma, whose elementary proof is omitted.

Lemma 1 g is strictly convex on its domain R^+. Furthermore, if $t \in R^+$ then $g(t) \ge \mu(1 - \ln\mu)$ with equality if and only if $t = \mu$. Finally, $g(t) \to \infty$ as $t \to 0$ or $t \to \infty$.

Note that it is immediate from this lemma that the barrier function $f_\mu(x, s)$ is bounded from below by $n\mu(1 - \ln\mu)$, and hence it has a finite infimum. The next lemma essentially shows that the infimum is located in a compact set, hence that it is achieved by some point.

Lemma 2 Suppose that F^0 is nonempty hold and let σ be any real number. Then the level set $\{(x,s) \in F^0 : f_\mu(x,s) \le \sigma\}$ is bounded, and uniformly bounded away from zero.

Proof: See [4].

The above lemma asserts that each level set of f_μ is included in a compact subset of its domain F^0. Hence f_μ achieves its minimum value on the domain of definition. Since f_μ is strictly convex on F^0, the solution is unique.

Conversely, if f_μ achieves its minimum value on F^0, we get

$$s - \mu X^{-1}e = 0; \text{ that is } Xs = \mu e;$$

that is the centering conditions have a solution. Thus, we have the following properties.

Theorem 7 Let $\mu > 0$. The following statements are equivalent:

(1) F^0 is nonempty;
(2) $f_\mu(x,s)$ has a (unique) minimizer on F^0;
(3) the centering conditions have a (unique) solution.

Combining Theorem 3, 6, and 7, we get the following theorem.

Theorem 8 Let $\mu > 0$. The following statements are equivalent:
(1) F^0 is nonempty;
(2) $g(x,\mu)$ has a (unique) minimizer on F_p^0;
(3) $h(y,\mu)$ has a (unique) minimizer on F_d^0;
(4) $f_\mu(x,s)$ has a (unique) minimizer on F^0;
(5) the centering conditions have a (unique) solution.

6. CONCLUSIONS

From the above discussion, we have shown that the linear programming and its dual can be solved by using barrier function methods given in Algorithm 1 and Algorithm 2 respectively.

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