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ON PRODUCT SUMMABILITY OF CONJUGATE SERIES OF FOURIER SERIES

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ABSTRACT

In this paper, a theorem on $A(E,z)$ product summability of conjugate series of Fourier series is proved.

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Mathematics subject classification: *42B05, 42B08.*

1. INTRODUCTION:

Let $\sum a_n$ be a given infinite series with the sequence of partial sums $\{s_n\}$. Let $A = (a_{mn})_{\infty\times\infty}$ be a triangular matrix .Then the sequence –to-sequence transformation

(1.1) $$t_m = \sum_{\upsilon=0}^{m} a_{m\nu}\, s_\upsilon \;, m = 1,2,\cdots$$

defines the sequence $\{t_m\}$ of the A -mean of the sequence $\{s_n\}$. If

(1.2) $$t_m \to s \quad , \text{as} \quad m \to \infty \;,$$

then the series $\sum a_n$ is said to be A summable to s.

The conditions for regularity of A -summability are easily seen to be[**3**]

(i) $\sup_m \sum_{n=0}^{\infty} |a_{mn}| < H$ where H is an absolute constant.

(ii) $\lim_{m\to\infty} a_{mn} = 0$

(iii) $\lim_{m\to\infty} \sum_{n=0}^{\infty} a_{mn} = 1$

Let

(1.3) $$(E,z) = E_n^z = \frac{1}{(1+z)^n} \sum_{\upsilon=0}^{n} \binom{n}{\upsilon} z^{n-\upsilon} s_\upsilon \to s, \text{ as } \; n \to \infty \,.$$

Then the series $\sum a_n$ is said to be summable (E,z) to a definite number s.

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Let

(1.4) $$T_n = \sum_{k=0}^{n} \frac{a_{nk}}{(1+z)^k} \sum_{\upsilon=0}^{k} \binom{k}{\nu} z^{k-\upsilon} s_\upsilon \to s \text{ as } n \to \infty .$$

Then the series $\sum a_n$ is said to be summable to s by the $A(E,z)$ method .
It is known [**1**] that (E,z) is regular. It is supposed that the method $A(E,z)$ is regular through out this paper.

Let $f(t)$ be a periodic function with period 2π, integrable in the sense of Lebesgue over (-π,π) then

(1.5) $$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt) = \sum_{n=0}^{\infty} A_n(t)$$

where a_n and b_n are the Euler-Fourier constants, is the Fourier series associated with f and the conjugate series of the Fourier series (1.5) is

(1.6) $$\sum_{n=1}^{\infty} (a_n \sin nx - b_n \cos nx) \equiv \sum_{n=1}^{\infty} B_n(x)$$

We use the following notation through out this paper

(1.7) $$\psi(t) = \frac{1}{2}\{f(x+t) - f(x-t)\},$$

(1.8) $$\overline{K}_n(t) = \frac{1}{\pi} \sum_{k=0}^{n} \frac{a_{nk}}{(1+z)^k} \sum_{\nu=0}^{k} \binom{k}{\nu} z^{k-\upsilon} \frac{\cos\frac{t}{2} - \cos\left(\upsilon + \frac{1}{2}\right)t}{\sin\frac{t}{2}} .$$

2. KNOWN THEOREM:

:

Dealing with $(N,p_n)(E,z)$ method of a Fourier series, Nigam,et.al[**2**] proved the following theorem:

Theorem: 2.1 Let $\{p_n\}$ be a positive, monotonic, non-increasing sequence of real constants such that

$$P_n = \sum_{\upsilon=0}^{n} p_\upsilon \to \infty \text{ as } n \to \infty .$$

If

(2.1) $$\Phi(t) = \int_0^t |\phi(u)|\, du = O\left\{\frac{t}{\alpha\left(\frac{1}{t}\right)}\right\} , \text{ as } t \to +0$$

and

(2.2) $$\alpha(n) \to \infty \text{ as } n \to \infty$$

where $\alpha(t)$ be a positive, non-increasing function of t , then the Fourier series $\sum_{n=0}^{\infty} A_n(t)$ is summable $(N,p_n)(E,z)$ to $f(x)$ at the point $t = x$.

In this paper, we have generalized it to $A(E,z)$ summability of conjugate series of Fourier series (1.6).

3. MAIN THEOREM:

Theorem: 3.1 Let $A = (a_{mn})_{\infty\times\infty}$ be a regular triangular matrix and

(3.1) $$\Psi(t) = \int_0^t |\psi(u)|\, du = O\left\{\frac{t}{\alpha\left(\frac{1}{t}\right)}\right\} , \text{ as } t \to +0$$

where $\alpha(t)$ is positive, non-increasing function of t and

(3.2) $\alpha(n) \to \infty$ as $n \to \infty$,

then the conjugate Fourier series $\sum_{n=0}^{\infty} B_n(t)$ is summable $A(E,z)$ at the point t.

4. REQUIRED LEMMAS:

We require the following Lemmas to prove the theorem.

Lemma: 4.1 If $\overline{K_n}(t)$ is as defined in (1.8), then

$$\left|\overline{K_n}(t)\right| = \begin{cases} O(n) \quad , & 0 \le t \le \dfrac{1}{n+1} \\ O\left(\dfrac{1}{t}\right), & \dfrac{1}{n+1} \le t \le \pi \end{cases}$$

Proof: For $0 \le t \le \dfrac{1}{n+1}$, we have sinnt ≤ n sint then

$$\begin{aligned}
\left|\overline{K}_n(t)\right| &= \frac{1}{\pi}\left|\sum_{k=0}^{n} \frac{a_{nk}}{(1+z)^k} \sum_{\nu=0}^{k} \binom{k}{\nu} z^{k-\upsilon} \frac{\cos\frac{t}{2} - \cos\left(\upsilon + \frac{1}{2}\right)t}{\sin\frac{t}{2}}\right| \\
&\le \frac{1}{\pi}\left|\sum_{k=0}^{n} \frac{a_{nk}}{(1+z)^k} \sum_{\nu=0}^{k} \binom{k}{\nu} z^{k-\upsilon} \frac{\cos\frac{t}{2} - \cos\upsilon t.\cos\frac{t}{2} + \sin\upsilon t.\sin\frac{t}{2}}{\sin\frac{t}{2}}\right| \\
&\le \frac{1}{\pi}\left|\sum_{k=0}^{n} \frac{a_{nk}}{(1+z)^k} \sum_{\nu=0}^{k} \binom{k}{\nu} z^{k-\upsilon} \left(\frac{\cos\frac{t}{2}\left(2\sin^2 \upsilon\frac{t}{2}\right)}{\sin\frac{t}{2}} + \sin\upsilon t\right)\right| \\
&\le \frac{1}{\pi}\left|\sum_{k=0}^{n} \frac{a_{nk}}{(1+z)^k} \sum_{\nu=0}^{k} \binom{k}{\nu} z^{k-\upsilon} \left(O\left(2\sin\upsilon\frac{t}{2}\,\sin\upsilon\frac{t}{2}\right) + \upsilon\sin t\right)\right| \\
&\le \frac{1}{\pi}\left|\sum_{k=0}^{n} \frac{a_{nk}}{(1+z)^k} O(k) \sum_{\nu=0}^{k} \binom{k}{\nu} z^{k-\upsilon}\right| \\
&= \frac{1}{\pi}\left|\sum_{k=0}^{n} O(k) \frac{a_{nk}}{(1+z)^k} (1+z)^k\right| \\
&= O(n).
\end{aligned}$$

For $\dfrac{1}{n+1} \le t \le \pi$, we have by Jordan's lemma, $\sin\left(\dfrac{t}{2}\right) \ge \dfrac{t}{\pi}$, then

$$\left|\overline{K}_n(t)\right| = \frac{1}{\pi}\sum_{k=0}^{n} \frac{a_{nk}}{(1+z)^k}\left|\sum_{\nu=0}^{k} \binom{k}{\nu} z^{k-\upsilon} \frac{\cos\frac{t}{2} - \cos\left(\upsilon + \frac{1}{2}\right)t}{\sin\frac{t}{2}}\right|$$

$$\leq \frac{1}{\pi}\sum_{k=0}^{n}\frac{a_{nk}}{(1+z)^k}\left|\sum_{\nu=0}^{k}\binom{k}{\nu}z^{k-\upsilon}\frac{\cos\frac{t}{2}-\cos\upsilon\frac{t}{2}.\cos\frac{t}{2}+\sin\upsilon\frac{t}{2}.\sin\frac{t}{2}}{\sin\frac{t}{2}}\right|$$

$$\leq \frac{1}{\pi}\sum_{k=0}^{n}\frac{a_{nk}}{(1+z)^k}\left|\sum_{\nu=0}^{k}\frac{\pi}{2t}\binom{k}{\nu}z^{k-\upsilon}\left(\cos\frac{t}{2}\left(2\sin^2\upsilon\frac{t}{2}\right)+\sin\upsilon\frac{t}{2}\sin\frac{t}{2}\right)\right|$$

$$\leq \frac{1}{2t}\sum_{k=0}^{n}\frac{a_{nk}}{(1+z)^k}\left|\sum_{\nu=0}^{k}\binom{k}{\nu}z^{k-\upsilon}\right|$$

$$= \frac{1}{2t}\sum_{k=0}^{n}\frac{a_{nk}}{(1+z)^k}(1+z)^k\ .$$

$$= O\left(\frac{1}{t}\right).$$

5. PROOF OF THE THEOREM 3.1:

If $\overline{s_n}(f;x)$ is the n-th partial sum of the conjugate of Fourier series given by (1.6), then by using Riemann-Lebesgue theorem, following Titchmarch [**4**] we have

$$\overline{s_n}(f;x)\text{-}f(x)=\frac{2}{\pi}\int_0^{\pi}\psi(t)\frac{\cos\frac{t}{2}-\sin\left(\upsilon+\frac{1}{2}\right)t}{2\sin\left(\frac{t}{2}\right)}\,dt$$

Thus, the (E,z) transform E_n^z of $\overline{s_n}$ is given by

$$E_n^z-f(x)=\frac{2}{\pi(1+z)^n}\int_0^{\pi}\frac{\psi(t)}{2\sin\left(\frac{t}{2}\right)}\left\{\sum_{k=0}^{n}\binom{k}{\upsilon}z^{n-k}\left\{\cos\frac{t}{2}-\sin\left(k+\frac{1}{2}\right)t\right\}\right\}dt$$

If T_n denote the $A(E,z)$ transform of $\overline{s_n}$ **,** we then have

$$T_n-f(x)=\frac{2}{\pi}\sum_{k=0}^{n}\frac{a_{nk}}{(1+z)^k}\int_0^{\pi}\frac{\psi(t)}{2\sin\left(\frac{t}{2}\right)}\left\{\sum_{\upsilon=0}^{k}\binom{k}{\upsilon}z^{k--\upsilon}\left\{\cos\frac{t}{2}-\sin\left(\upsilon+\frac{1}{2}\right)t\right\}\right\}dt$$

$$=\int_0^{\pi}\psi(t)\,\overline{K_n}(t)dt$$

In order to prove the theorem, under an assumption, it is sufficient to show that

$$\int_0^{\pi}\psi(t)\,\overline{K_n}(t)\,dt=O(1)\quad\text{as}\quad n\to\infty$$

For $0<\delta<\pi$, we have

$$T_n-f(x)=\int_0^{\pi}\psi(t)\,\overline{K_n}(t)\,dt$$

$$= \left\{ \int_0^{\frac{1}{n+1}} + \int_{\frac{1}{n+1}}^{\pi} \right\} \psi(t)\, \overline{K_n}(t)\, dt$$

$$= I_1 + I_2 \text{, say}$$

Now

$$|I_1| = \left| \int_0^{1/n+1} \psi(t)\, \overline{K_n}(t)\, dt \right| \le \int_0^{1/n+1} |\psi(t)|\, \left|\overline{K_n}(t)\right| dt.$$

$$\le O(n) \int_0^{1/n+1} |\psi(t)|\, dt \text{, Using Lemma -1}$$

$$= O(n)\left\{ O\left(\frac{1}{n\alpha(n)}\right)\right\} \text{, using (3.1).}$$

$$= O\left(\frac{1}{\alpha(n)}\right) \text{, as } n \to \infty.$$

$$= O(1) \text{, as } n \to \infty \text{, using (3.2).}$$

Next

$$|I_2| \le \left| \int_{1/n+1}^{\pi} |\psi(t)| \left|\overline{K_n}(t)\right| dt \right|$$

$$= O\left\{ \int_{1/n+1}^{\pi} \frac{|\psi(t)|}{t} dt \right\} \text{, using lemma -2}$$

$$= O\left\{ \left[\frac{\Psi(t)}{t}\right]_{1/n+1}^{\pi} + \int_{1/n+1}^{\pi} \frac{\Psi(t)}{t^2} dt \right\}.$$

$$= O\left\{ O\left[\frac{1}{\alpha\left(\frac{1}{t}\right)}\right]_{1/n+1}^{\pi} + \int_{1/n+1}^{\pi} O\left(\frac{1}{u\alpha(u)}\right) du \right\} \text{, where u=1/t}$$

$$= O\left(\frac{1}{\alpha(n)}\right) + O\left(\frac{1}{n\alpha(n)}\right) \int_{1/n+1}^{n} du \text{, using second mean-}$$

Value theorem for the integral in the 2nd term as $\alpha(n)$ is monotonic.

$$= O(1) + O(1) \text{, as } n \to \infty \text{, using (3.2)}$$

$$= O(1) \text{, as } n \to \infty.$$

Thus,

$$T_n - f(x) = O(1) \text{, as } n \to \infty.$$

This completes the proof of the theorem.

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