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A STUDY OF GENERALIZATION OF DOUBLE RIEMANN ZETA FUNCTION

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ABSTRACT

I*n the present paper, we introduce and investigate a new generalized double zeta function which is a generalization of Hurwitz-Lerch zeta function and includes their various particular cases. We study its recurrence relations, integral representations, fractional derivatives and series expansion also.*

Keywords and Phrases: *Riemann zeta function, Hurwitz-Lerch zeta function, double zeta function, Gauss hypergeometric function*

Subject Classification: *11M06, 11M35, 33C05.*

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1. INTRODUCTION:

A generalized Hurwitz-Lerch Zeta function $\phi(y,z,a)$ [3, p.27, Eq.(1)] is defined in the series form as

$$\phi(y,z,a)=\sum_{n=0}^{\infty}\frac{y^n}{(n+a)^z} \tag{1}$$

where

$\left(a\in C/\{0,-1,-2,...\},z\in C \text{ when } |y|<1 \text{ and Re(z)>1 when } |y|=1\right)$

This function has many special cases such as Riemann-Zeta [3], Hurwitz-Zeta [3].

Lin and Srivastava [7] investigated the Hurwitz-Lerch zeta function in the following form

$$\phi_{\mu,\upsilon}^{(\rho,\sigma)}(y,z,a)=\sum_{n=0}^{\infty}\frac{(\mu)_{\rho n}}{(\upsilon)_{\sigma n}}\frac{y^n}{(n+a)^z} \tag{2}$$

where $\mu\in C;a,\upsilon\in C/\{0,-1,-2,...\};\rho,\sigma\in R^+;\rho<\sigma$ when $y,z\in C$: $\rho=\sigma$ for $y\in C$; $\rho=\sigma$, $z\in C$ for $|y|<1$; $\rho=\sigma$, $\text{Re}(z-\mu+\upsilon)>1$ for $|z|=1$.

If we take $\rho=\sigma=\upsilon=1$ in the above equation, we get

$$\phi_{\mu,1}^{(1,1)}(y,z,a)=\phi_{\mu}^{*}(y,z,a)\equiv\sum_{n=0}^{\infty}\frac{(\mu)_n y^n}{(a+n)^z n!} \tag{3}$$

Here $\phi_{\mu}^{*}(y,z,a)$ stands for generalized Hurwitz-Lerch zeta function defined by Goyal and Laddha [5].

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Mridula Garg et al. [4, p.27, Eq.(1.4)] introduced a generalization of the general Hurwitz-Lerch zeta function in the following form given as

$$\phi_{\alpha,\beta,\gamma}(z,s,a)=\sum_{n=0}^{\infty}\frac{(\alpha)_n(\beta)_n}{(\gamma)_n\, n!}\frac{z^n}{(n+a)^s} \tag{4}$$

where $\gamma, a\neq 0,-1,-2,\ldots s\in C$, when $|z|<1$ and $\operatorname{Re}(s+\gamma-\alpha-\beta)>0$ when $|z|=1$.

The double zeta function of Barnes [1] is defined by

$$\zeta_2(z;a,\omega)=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(a+n+\omega m)^{-z}\ ; \tag{5}$$

$\{\operatorname{Re}(z)>2, a\neq 0, \omega$ is a non-zero complex number $\}$.

Bin Saad [2] recently gave a generalization of double zeta function as

$$\zeta_\lambda^\mu(x,y;z,a)=\sum_{m=0}^{\infty}(\mu)_m\phi(y,z,a+\lambda m)\frac{x^m}{m!} \tag{6}$$

where $|x|<1,\ |y|<1\ ;\mu\in C/\{0,-1,-2,\ldots\},\ \lambda\in C/\{0\};\ a\in C/\{-(n+\lambda m)\}$,
$\{n,\ m\}\in N\cup\{0\}$.

In this paper, we present here another generalization of the double zeta function defined by

$$\zeta_\lambda^{\alpha,\beta,\gamma}(x;y,z,a)=\sum_{m=0}^{\infty}\frac{(\alpha)_m(\beta)_m}{(\gamma)_m}\phi(y,z,a+\lambda m)\frac{x^m}{m!} \tag{7}$$

or

$$\zeta_\lambda^{\alpha,\beta,\gamma}(x;y,z,a)=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\alpha)_m(\beta)_m}{(\gamma)_m}\frac{y^n}{(a+n+\lambda m)^z}\frac{x^m}{m!} \tag{8}$$

$\{\alpha,\beta\in C,\gamma\in C/\{0,-1,-2,\ldots\}$ for $|x|<1$ and $\operatorname{Re}(\gamma-\alpha-\beta)>0$ for $|x|=1$, $\lambda\in C/\{0\}$;
$a\in C/\{-(n+\lambda m)\}, z\in C$ for $|y|<1$ and Re(z)>1 for $|y|=1\}$.

2. RELATIONSHIP OF $\zeta_\lambda^{\alpha,\beta,\gamma}(x;y,z,a)$ WITH KNOWN FUNCTIONS:

(i) If we take y=0 and $\lambda=1$ in (8), it yields the general Hurwitz-Lerch zeta function $\phi_{\alpha,\beta,\gamma}(x,z,a)$.

$$\zeta_1^{\alpha,\beta,\gamma}(x;0,z,a)=\phi_{\alpha,\beta,\gamma}(x,z,a) \tag{9}$$

(ii) Further on taking $\beta=\gamma=1$ in (9), we obtain

$$\zeta_1^{\alpha,1,1}(x;0,z,a)=\phi_\alpha^*(x,z,a) \tag{10}$$

(iii) If we take x=0 in (8), we arrive at a generalization of the Hurwitz-Lerch zeta function $\phi(y,z,a)$ defined by (1) as follows:

$$\zeta_\lambda^{\alpha,\beta,\gamma}(0;y,z,a)=\phi(y,z,a) \tag{11}$$

(iv) For y=1, equation (11) yields the connection between generalized double zeta function and Hurwitz (or generalized) zeta function as:

$$\zeta_\lambda^{\alpha,\beta,\gamma}(0;1,z,a)=\zeta(z,a) \tag{12}$$

which, further for a =1, gives the relation with the well-known Riemann zeta function $\zeta(z)$.

(v) It is not difficult to see from the definition (8), in connection with (5) and (6) that

$$\zeta_\lambda^{\alpha,1,1}(x;y,z,a)=\zeta_\lambda^{\alpha}(x;y,z,a) \tag{13}$$

and

$$\zeta_\lambda^{1,1,1}(1;1,z,a)=\zeta_2(z;a,\lambda) \tag{14}$$

3. BASIC PROPERTIES OF THE FUNCTION $\zeta_\lambda^{\alpha,\beta,\gamma}(x;y,z,a)$:

As a consequence of the definition (7), the following recurrence relations hold:

Theorem: 1 If $\alpha,\beta\in C,\ \gamma\in C/\{0,-1,-2,...\}$ for $|x|<1$ then

$$\beta\zeta_\lambda^{\alpha,\beta+1,\gamma}(x;y,z,a)+(\gamma-\beta-1)\zeta_\lambda^{\alpha,\beta,\gamma}(x;y,z,a)=(\gamma-1)\zeta_\lambda^{\alpha,\beta,\gamma-1}(x;y,z,a), \tag{15}$$

$$\gamma\zeta_\lambda^{\alpha,\beta,\gamma}(x;y,z,a)-(\gamma-\beta)\zeta_\lambda^{\alpha,\beta,\gamma+1}(x;y,z,a)=\beta\zeta_\lambda^{\alpha,\beta+1,\gamma+1}(x;y,z,a) \tag{16}$$

and

$$\alpha\zeta_\lambda^{\alpha+1,\beta,\gamma}(x;y,z,a)-\alpha\zeta_\lambda^{\alpha,\beta,\gamma}(x;y,z,a)=x\frac{d}{dx}\zeta_\lambda^{\alpha,\beta,\gamma}(x;y,z,a) \tag{17}$$

Proof: Taking L.H.S. of equation (15), we have

$$\beta\zeta_\lambda^{\alpha,\beta+1,\gamma}(x;y,z,a)+(\gamma-\beta-1)\zeta_\lambda^{\alpha,\beta,\gamma}(x;y,z,a)$$
$$=\beta\sum_{m=0}^{\infty}\frac{(\alpha)_m(\beta+1)_m}{(\gamma)_m}\phi(y,z,a+\lambda m)\frac{x^m}{m!}+(\gamma-\beta-1)\sum_{m=0}^{\infty}\frac{(\alpha)_m(\beta)_m}{(\gamma)_m}\phi(y,z,a+\lambda m)\frac{x^m}{m!}$$

$$=(\beta+m)\sum_{m=0}^{\infty}\frac{(\alpha)_m(\beta)_m}{(\gamma)_m}\phi(y,z,a+\lambda m)\frac{x^m}{m!}+(\gamma-\beta-1)\sum_{m=0}^{\infty}\frac{(\alpha)_m(\beta)_m}{(\gamma)_m}\phi(y,z,a+\lambda m)\frac{x^m}{m!}$$

$$=(\gamma-1)\sum_{m=0}^{\infty}\frac{(\alpha)_m(\beta)_m}{(\gamma-1)_m}\phi(y,z,a+\lambda m)\frac{x^m}{m!}$$
$$=(\gamma-1)\zeta_\lambda^{\alpha,\beta,\gamma-1}(x;y,z,a)$$

which is a proof of (15).

Similarly other two results can be derived in a similar manner.

The following result, which gives the differential relations for $\zeta_\lambda^{\alpha,\beta,\gamma}(x;y,z,a)$, are proved by using the following familiar derivative formula [8]:

$$D_x^m\left(x^n\right)=\frac{\Gamma(n+1)}{\Gamma(n-m+1)}x^{n-m}\;;n-m\geq 0,\quad D_x\equiv\frac{d}{dx};m\in N \tag{18}$$

Theorem: 2 Let $k\in N$, then

$$D_x^k\left[\zeta_\lambda^{\alpha,\beta,\gamma}(x;y,z,a)\right]=\frac{(\alpha)_k(\beta)_k}{(\gamma)_k}\zeta_\lambda^{\alpha+k,\beta+k,\gamma+k}(x;y,z,a+\lambda k) \tag{19}$$

$$D_x^k\left[x^{\alpha+k-1}\zeta_\lambda^{\alpha,\beta,\gamma}(x;y,z,a)\right]=x^{\alpha-1}(\alpha)_k\,\zeta_\lambda^{\alpha+k,\beta,\gamma}(x;y,z,a) \tag{20}$$

$$D_y^k\left[\zeta_\lambda^{\alpha,\beta,\gamma}(x;y,z,a)\right]=\Gamma(k+1)\sum_{m=0}^{\infty}\frac{(\alpha)_m(\beta)_m}{(\gamma)_m}\phi_{k+1}^{*}(y,z,a+k+\lambda m)\frac{x^m}{m!} \tag{21}$$

and

$$D_a^k\left[\zeta_\lambda^{\alpha,\beta,\gamma}(x;y,z,a)\right]=(-1)^k(z)_k\,\zeta_\lambda^{\alpha,\beta,\gamma}(x;y,z+k,a) \tag{22}$$

Proof: First of all using (18), we have

$$D_x^k\left[\zeta_\lambda^{\alpha,\beta,\gamma}(x;y,z,a)\right]=\sum_{m=k}^{\infty}\sum_{n=0}^{\infty}\frac{(\alpha)_m(\beta)_m}{(\gamma)_m}\frac{y^n}{(a+n+\lambda m)^z}\frac{x^{m-k}}{(m-k)!} \tag{23}$$

Now, letting $m\to m+k$ in the above result and considering the definition (8), we get the right hand side of (19). Similarly, we can obtain the formula (20), (21) and (22).

Next, we obtain the fractional derivatives of generalized double zeta function. For this, we give the definitions of fractional derivatives used in analysis as follows:

The Riemann-Liouville fractional derivative of order υ is defined by

$${}_0D_t^{\upsilon}f(t)=\frac{1}{\Gamma(n-\upsilon)}\frac{d^n}{dt^n}\int_0^t(t-u)^{n-\upsilon-1}f(u)du,\;\; n=\left[\mathrm{Re}(\upsilon)\right]+1 \tag{24}$$

and the Weyl fractional derivative is given as

$${}_{-\infty}D_t^{\upsilon}f(t)=\frac{1}{\Gamma(n-\upsilon)}\frac{d^n}{dt^n}\int_{-\infty}^t(t-u)^{n-\upsilon-1}f(u)du,\;\; n=\left[\mathrm{Re}(\upsilon)\right]+1 \tag{25}$$

Theorem: 3 For $\mathrm{Re}(\upsilon)>0$, we have

$${}_0D_t^{\upsilon}\left[t^{\gamma-1}\zeta_\lambda^{\alpha,\beta,\gamma}(wt;y,z,a)\right]=(-1)^{\upsilon}t^{\gamma-\upsilon-1}(1-\gamma)_{\upsilon}\,\zeta_\lambda^{\alpha,\beta,\gamma-\upsilon}(wt;y,z,a) \tag{26}$$

$${}_0D_t^{\upsilon}\left[t^{\beta+\upsilon-1}\zeta_\lambda^{\alpha,\beta,\gamma}(wt;y,z,a)\right]=t^{\beta-1}(\beta)_{\upsilon}\,\zeta_\lambda^{\alpha,\beta+\upsilon,\gamma}(wt;y,z,a) \tag{27}$$

and

$${}_{-\infty}D_z^{\upsilon}\left[\zeta_\lambda^{\alpha,\beta,\gamma}(x;y,z,a)\right]=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\alpha)_m(\beta)_m x^m y^n}{(\gamma)_m(a+n+\lambda m)^z\, m!}\left[\log(a+n+\lambda m)\right]^{\upsilon} \tag{28}$$

Proof: Making use of definition (24), it readily follows that for $f(t)=t^{\rho}$ [8, p.36, Eq.(6.5)]

$$ {}_0D_t^{\upsilon}\, t^{\rho}=\frac{\Gamma(\rho+1)}{\Gamma(\rho-\upsilon+1)}t^{\rho-\upsilon},\ \ \mathrm{Re}(\rho)>-1, \mathrm{Re}(\upsilon)>0 \tag{29}$$

Now the result (26) and (27) directly follow from (7) and (29).

Further, the Weyl fractional derivative of the exponential function $e^{-at}, a>0$ is given as [8, p.248, Eq.(7.4)]

$$ {}_{-\infty}D_t^{\upsilon}\, e^{-at}=a^{\upsilon}e^{-at},\ \ \mathrm{Re}(\upsilon)>0 \tag{30}$$

Since

$$(a+n+\lambda m)^{-z}=e^{-z\log(a+n+\lambda m)},$$

we have

$$\zeta_{\lambda}^{\alpha,\beta,\gamma}(x;y,z,a)=\sum_{m=n=0}^{\infty}\frac{(\alpha)_m(\beta)_m x^m y^n}{(\gamma)_m\, m!}e^{-z\log(a+n+\lambda m)}$$

Finally on applying the formula (30), we get the desired result.

4. INTEGRAL REPRESENTATIONS:

In this section, we present some integral representations for the generalized double zeta function.

First of all, we establish an integral representation for $\zeta_{\lambda}^{\alpha,\beta,\gamma}(x;y,z,a)$ which is derived directly from the corresponding integral representation of the Hurwitz-Lerch zeta function ϕ
[3, p.27, Eq.(3)]

$$\phi(y,z,a)=\frac{1}{\Gamma(z)}\int_0^{\infty}t^{z-1}e^{-at}\left(1-ye^{-t}\right)^{-1}dt \tag{31}$$

where Re(a)>0 and either $|y|\le 1,\ y\ne 1,\ \mathrm{Re}(z)>0$ or y = 1, Re(z)>1.

Theorem: 4 For $\mathrm{Re}(a)>0, \mathrm{Re}(\lambda)>0; \gamma\ne 0,-1,-2,\ldots$ and either $|x|\le 1, |y|\le 1, x\ne 1, y\ne 1,$ $\mathrm{Re}(z)>0$ or x = 1, $\mathrm{Re}(\gamma-\alpha-\beta)>0$, y = 1, Re(z)>1, we have

$$\zeta_{\lambda}^{\alpha,\beta,\gamma}(x;y,z,a)=\frac{1}{\Gamma(z)}\int_0^{\infty}t^{z-1}e^{-at}\left(1-ye^{-t}\right)^{-1}{}_2F_1\left(\alpha,\beta;\gamma;xe^{-\lambda t}\right)dt \tag{32}$$

Proof: From (7) and (31), we have

$$\zeta_{\lambda}^{\alpha,\beta,\gamma}(x;y,z,a)=\sum_{m=0}^{\infty}\frac{(\alpha)_m(\beta)_m}{(\gamma)_m}\left[\frac{1}{\Gamma(z)}\int_0^{\infty}t^{z-1}e^{-(a+\lambda m)t}\left(1-ye^{-t}\right)^{-1}\right]\frac{x^m}{m!}$$

The desired result (32) can be easily obtained by changing the order of summation and integration and using the definition of Gauss hypergeometric function [3].

Next, we establish two double integral representations for the generalized double zeta function.

Theorem: 5 For $\mathrm{Re}(a)>0, \mathrm{Re}(\beta)>0, \mathrm{Re}(\lambda)>0; \gamma\neq 0,-1,-2,\ldots$ and either $|x|\leq 1$, $|y|\leq 1,\ x\neq 1,\ y\neq 1,\ \mathrm{Re}(z)>0$ or x = 1, $\mathrm{Re}(\gamma-\alpha)>0$, y = 1, Re(z)>1, we have

$$\zeta_{\lambda}^{\alpha,\beta,\gamma}(x;y,z,a)=\frac{1}{\Gamma(z)\Gamma(\beta)}\int_0^{\infty}\int_0^{\infty} e^{-u}u^{\beta-1}t^{z-1}e^{-at}\left(1-ye^{-t}\right)^{-1}{}_1F_1\left(\alpha;\gamma;xue^{-\lambda t}\right)dudt \qquad (33)$$

Proof: Using

$$\left(a+n+\lambda m\right)^{-z}=\frac{1}{\Gamma(z)}\int_0^{\infty} e^{-(a+n+\lambda m)t}\ t^{z-1}dt \qquad (34)$$

and

$$(\beta)_m=\frac{1}{\Gamma(\beta)}\int_0^{\infty} u^{\beta+m-1}e^{-u}du \qquad (35)$$

in the definition (8), we readily obtain the required result.

Theorem: 6 For $\mathrm{Re}(a)>0,\ \mathrm{Re}(z)>0$ and $\mathrm{Re}(\upsilon)<1$, we have

$$\zeta_{\lambda}^{\alpha,\beta,\gamma}(x;y,z,a)=\frac{1}{\Gamma(z)\Gamma(1-\upsilon)}\int_0^{\infty}\int_0^{\infty} e^{-a(t+u)}u^{-\upsilon}\ t^{z-1}\ \phi\left(ye^{-(t+u)},\upsilon-1,a\right)\ {}_2F_1\left(\alpha,\beta;\gamma;xe^{-\lambda t}\right)dt \qquad (36)$$

Proof: On applying equation (34) in (8), we obtain

$$\zeta_{\lambda}^{\alpha,\beta,\gamma}(x;y,z,a)=\frac{1}{\Gamma(z)}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\alpha)_m(\beta)_m}{(\gamma)_m}\frac{x^m y^n}{m!}\int_0^{\infty} e^{-(a+n+\lambda m)t}\ t^{z-1}dt$$

which on doing simple manipulations and using the definition of ${}_2F_1$ [3] and $\phi(y,z,a)$ leads us to the result (36).

The Hurwitz-Lerch zeta function has the following contour integral representation [3, p.28, Eq.(5)]:

$$\phi(y,z,a)=\frac{-\Gamma(1-z)}{2\pi i}\int_{\infty}^{0+}(-t)^{z-1}e^{-at}\left(1-ye^{-t}\right)^{-1}dt \qquad (37)$$

valid for $\mathrm{Re}(a)>0, z\in C$ and $|\arg(-t)|, \pi$. Also the contour does not enclose any of the points $t=\log z\pm 2n\pi i,\ (n=0,1,2,\ldots)$ which are the poles of the integrand of (37).

For the function $\zeta_{\lambda}^{\alpha,\beta,\gamma}(x;y,z,a)$ we have the following contour integral representation.

Theorem: 7 Let $\mathrm{Re}(a)>0, \mathrm{Re}(\lambda)>0$ and $|\arg(-t)|<\pi$, then

$$\zeta_{\lambda}^{\alpha,\beta,\gamma}(x;y,z,a)=\frac{-\Gamma(1-z)}{2\pi i}\int_{\infty}^{0+}(-t)^{z-1}e^{-at}\left(1-ye^{-t}\right)^{-1}{}_2F_1\left(\alpha,\beta;\gamma;xe^{-\lambda t}\right)dt \qquad (38)$$

Proof: It follows from (7) and (37) that

$$\zeta_{\lambda}^{\alpha,\beta,\gamma}(x;y,z,a)=\sum_{m=0}^{\infty}\frac{(\alpha)_m(\beta)_m}{(\gamma)_m}\left[\frac{-\Gamma(1-z)}{2\pi i}\int_{\infty}^{0+}(-t)^{z-1}e^{-(a+\lambda m)t}\left(1-ye^{-t}\right)^{-1}dt\right]\frac{x^m}{m!}$$

On interchanging the order of summation and integration and using the definition of ${}_2F_1$, we easily arrive at the result (38).

5. SERIES EXPANSIONS:

Series expansions play an important role in the investigation of various useful properties of the sequences which they expand. In this section, we obtain certain series relations for the generalized double zeta function based on following two forms of Taylor's theorem [9, p.21-22]:

$$f(x+y)=\sum_{m=0}^{\infty} f^{m}(x)\frac{y^{m}}{m!} \tag{39}$$

and

$$f(xy)=\sum_{m=0}^{\infty} f^{m}(x)\frac{\left[(y-1)x\right]^{m}}{m!} \tag{40}$$

where $|y|<\rho,\ \rho$ being the radius of convergence of the analytic function f(x).

Theorem: 8 Let $|\omega|<1$, then

$$\zeta_{\lambda}^{\alpha,\beta,\gamma}(x+\omega;y,z,a)=\sum_{k=0}^{\infty}\frac{(\alpha)_k(\beta)_k}{(\gamma)_k}\zeta_{\lambda}^{\alpha+k,\beta+k,\gamma+k}(x;y,z,a+\lambda k)\frac{\omega^{k}}{k!} \tag{41}$$

$$\zeta_{\lambda}^{\alpha,\beta,\gamma}(x;y+\omega,z,a)=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\alpha)_m(\beta)_m}{(\gamma)_m}\phi_{k+1}^{*}(\omega,z,a+n+\lambda m)\frac{x^{m}y^{n}}{m!} \tag{42}$$

$$\zeta_{\lambda}^{\alpha,\beta,\gamma}(x\omega;y,z,a)=\sum_{k=0}^{\infty}\frac{(\alpha)_k(\beta)_k}{(\gamma)_k}\zeta_{\lambda}^{\alpha+k,\beta+k,\gamma+k}(x;y,z,a+\lambda k)\frac{(\omega-1)^{k}x^{k}}{k!} \tag{43}$$

and

$$\zeta_{\lambda}^{\alpha,\beta,\gamma}(x;y\omega,z,a)=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\alpha)_m(\beta)_m}{(\gamma)_m}\phi_{k+1}^{*}(y,z,a+n+\lambda m)\frac{x^{m}(\omega-1)^{n}y^{n}}{m!} \tag{44}$$

Proof: The proof is a direct application of the formulas (39), (40) and the results (19) and (21).

Another expansion function for $\zeta_{\lambda}^{\alpha,\beta,\gamma}$ can be derived by using the result [6, p.374, exercise 9.4(7)]:

$${}_2F_1\left[a,a+\frac{1}{2};\frac{1}{2};x\right]=\frac{1}{2}\left(1+\sqrt{x}\right)^{-2a}+\frac{1}{2}\left(1-\sqrt{x}\right)^{-2a} \tag{45}$$

Theorem: 9 Let $\operatorname{Re}(a)>0,\quad |x|<1,\ |y|<1$ and $|\omega|<|a|$. Then

$$\sum_{k=0}^{\infty}\binom{z+2k-1}{2k}\zeta_{\lambda}^{\alpha,\beta,\lambda}(x;y,z+2k,a)\omega^{2k}=\frac{1}{2}\left[\zeta_{\lambda}^{\alpha,\beta,\gamma}(x;y,z,a+\omega)+\zeta_{\lambda}^{\alpha,\beta,\gamma}(x;y,z,a-\omega)\right] \tag{46}$$

Proof: we have

$$\sum_{k=0}^{\infty}\binom{z+2k-1}{2k}\zeta_{\lambda}^{\alpha,\beta,\gamma}(x;y,z+2k,a)\omega^{2k}$$

$$=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\alpha)_m(\beta)_m x^m y^n}{(\gamma)_m m!(a+n+\lambda m)^z}\sum_{k=0}^{\infty}\frac{(z)_{2k}\,\omega^k}{(2k)!(a+n+\lambda m)^{2k}} \tag{47}$$

By applying the formula (45) to the last summation in the right-hand side of equation (47), we arrive at the desired result.

Next, we derive a series expansion for the function $\zeta_{\lambda}^{\alpha,\beta,\gamma}$ involving Appell's function of two variables defined by the series [10, p.23 (3)]

$$F_2[a,b,b';c,c';x,y]=\sum_{m,n=0}^{\infty}\frac{(a)_{m+n}(b)_m(b')_n x^m y^n}{(c)_m(c')_n m!n!} \tag{48}$$

Theorem: 10 Let $\max\{|x/b|,|y/b|\}<1,\quad |b|<\operatorname{Re} a$ and $\lambda\neq 0$. Then

$$\sum_{k=0}^{\infty}(\upsilon)_k\zeta_{\lambda}^{\alpha,\beta,\gamma}(x;y,z+k,a+b)\frac{\omega^k}{k!}=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\alpha)_m(\beta)_m x^m y^n}{(\gamma)_m m!}$$
$$\times F_2\left[z,\upsilon,1;z,1;\frac{\omega}{a+n+\lambda m},\frac{-b}{a+n+\lambda m}\right](a+n+\lambda m)^{-z} \tag{49}$$

Proof: Since

$$(a+n+\lambda m+b)^{-(z+k)}=(a+n+\lambda m)^{-(z+k)}\left(1+\frac{b}{a+n+\lambda m}\right)^{-(z+k)}$$

It follows that

$$\sum_{k=0}^{\infty}(\upsilon)_k\zeta_{\lambda}^{\alpha,\beta,\gamma}(x;y,z+k,a+b)\frac{\omega^k}{k!}=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\alpha)_m(\beta)_m x^m y^n}{(\gamma)_m m!(a+n+\lambda m)^z}$$
$$\times\sum_{k=0}^{\infty}\sum_{s=0}^{\infty}\frac{(z)_{k+s}(\upsilon)_k}{k!s!(z)_k}\left(\frac{\omega}{a+n+\lambda m}\right)^k\left(\frac{-b}{a+n+\lambda m}\right)^s \tag{50}$$

The result (50) now follows from the definition (48).

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