International Journal of Mathematical Archive-3(1), 2012, Page: 24-30
IJMA Available online through www.ijma.info ISSN 2229 – 5046

ALEKSANDROV-URYSON G_δ COMPACTNESS CRITERION IN MAXIMAL CENTERED SYSTEMS

G. Vasuki*, E. Roja and M. K. Uma

Department of Mathematics, Sri Sarada College for Women, Salem - 636 016, Tamil Nadu, India
E-mail: vasukigj@yahoo.in

(Received on: 19-12-11; Accepted on: 09-01-12)

ABSTRACT

In this paper the concepts of G_δ-Hausdorff space, G_δ-extremally disconnected spaces, G_δ-θ continuous mappings are introduced. In this connection,G_δ-Hausdorff extension of spaces and the Aleksandrov-Uryson $_\delta$-compactness criterion are established.

Keywords: *G_δ-Hausdorff space, G_δ-extremally disconnected spaces, G_δ-θ continuous mappings.*

Mathematics Subject classification (2000): *54A05, 54A10, 54A20.*

1. INTRODUCTION:

The method of centered systems was introduced and established by S. Illiadis and S. Fomin [1]. In this paper making use of G_δ-sets, we introduce the concept of G_δ-Hausdorff space, G_δ-extremally disconnected spaces, G_δ-θ-continuous mappings, and Aleksandrov-Uryson G_δ-compactness in the centered systems.

2. PRELIMINARIES:

Definition: 2.1 [3] A set $A \subset X$ in a topological space (X, T) is called a G_δ-set if $A = \bigcap_{n=1}^{\infty} A_n$ where each $A_n \in T$. The complement of G_δ-set is called a F_σ set.

Definition: 2.2 For any set A in (X, T), define the σ-closure of A denoted by σ-cl A, to be the intersection of all F_σ-sets containing A.

That is $\sigma\text{-cl } A = \bigcap \{U: U \text{ is a } F_\sigma\text{-set and } U \supseteq A\}$

Definition: 2.3 For any set A in (X, T), define the σ-interior of A denoted by σ-int A, to be the union of all G_δ - sets contained in A.

That is $\sigma\text{-int } A = \bigcup \{U : U \text{ is a } G_\delta\text{- set and } U \subseteq A\}$

Definition: 2.4 [2] A topological space is a Hausdorff space iff whenever x and y are distinct points of the space there exists disjoint neighbourhoods of x and y.

Definition: 2.5 [1] Let R be a Hausdorff space. A system $p = \{U_\alpha\}$ of open sets of R is called centered if any finite collection of sets of the system has a non-empty intersection. The system p is called a maximal centered system or briefly an end if it cannot be included in any larger centered system of open sts.

Definition: 2.6 [1] Let f be a mapping of a space X into a space Y with f(x) = y. The f is called θ-continuous at x if for every neighbourhood O_y of y there exists neighbourhood O_x of x such that $f(\overline{O_x}) \subset \overline{O_y}$. The mapping is called θ - continuous if it is θ-continuous at every point of X. A mapping that is one-to-one and θ-continuous in both directions is called a θ-homeomorphism.

***Corresponding author: G. Vasuki*,*E-mail: vasukigj@yahoo.in**

It is clear that a continous mapping is θ-continuous. An example of a θ-continuous mapping that is not continuous. Let I be the interval [0, 1] with the usual topology, and I′ the same interval with the following topology: the neighbourhoods of every point $x \neq 0$ are the same as those in the half-open interval (0, 1], but the neighbourhoods of $x = 0$ are the sets of the form $[0, \varepsilon)\backslash D$, where D is the set of all points 1/n (n=1, 2, …;$0 < \varepsilon < 1$). It is easy to see that the space obtained is not regular at 0. Let f be the identity mapping of [0,1] onto itself. It is easy to verify that this mapping of I onto I′ is θ-continuous, we have also obtained a θ-homeomorphism that is not a homeomorphism. It is essential here that the space I′ is not regular, since it is easy to show that if the image is regular, then a θ-continuous mapping is automatically continuous.

Remark: 2.1 The canonical open sets (sets of the form I ($\bar{U}$) where U is open) form a base.

3. THE SPACES OF MAXIMAL CENTERED SYSTEM:

Definition: 3.1 A topological space (X, T) is said to be G_δ-Hausdorff iff for any two distinct points $x_1, x_2 \in X$, there exist G_δ sets U and V with $x_1 \in U$ and $x_2 \in V$ such that $U \cap V = \phi$.

Notation: 3.1 G_δ-Hausdorff space is denoted by R.

Definition: 3.2 Let R be a G_δ-Hausdorff space. A system $p^* = \{S_\alpha\}$ of G_δ sets of R is called centered if any finite collection of sets of the system has a non-empty intersection. The system p^* is called a maximal centered system, or briefly, an end if it cannot be included in any larger centered system of G_δ sets.

The following are the properties of maximal centered systems:

1. If $S_i \in p^*$ (i = 1, 2 … n) then $\bigcap_{i=1}^{n} S_i \in p^*$.
2. If $S \subset H$, $S \in p^*$ and H is G_δ- set then $H \in p^*$.
3. If H is G_δ- set, then $H \notin p^*$, iff there exists $S \in p^*$ such that $S \cap H$ is empty.
4. If $S_1 \cup S_2 = S_3 \in p^*$, S_1 and S_2 are G_δ- sets and $S_1 \cap S_2 = \phi$, then either $S_1 \in p^*$ or $S_2 \in p^*$.
5. If σ-cl (S) = R, then $S \in p^*$ for any end p^*.

Remark: 2.1 Every centered system of G_δ - sets can be extended in atleast one way to a maximal one.

4. MAXIMAL STRUCTURE IN θ (R):

Definition: 4.1 A set U in a topological space (X, T) is a G_δ- neighbourhood of a point x iff U contains a G_δ- set to which x belongs.

Definition: 4.2 A family $\mathcal{A}$ is a G_δ- cover of a G_δ- set B iff each member of B belongs to some member of $\mathcal{A}$

Definition: 4.3 A topological space is G_δ-compact iff each G_δ- cover has a finite subcover.

Notation: 4.1 Let θ (R) denote the collection of all end belonging to a given space R. **We introduce maximal structure θ (R) in the following way:**

Let O_S be the set of all ends that contain S as an element, where S is a G_δ-set of R. Now O_S is to be a G_δ- neighbourhood of each end contained in O_S. Thus to each G_δ- set $S \subset R$ there corresponds a G_δ- neighbourhood O_S in θ (R).

Proposition: 4.1 If S and T are two G_δ-sets, then
(a) $O_{S \cup T} = O_S \cup O_T$
(b) $O_S = \theta(R) \setminus O_{R\setminus \sigma\text{-cl}(S)}$

Proof: (a) Let $p^* \in O_S$, ie., $S \in p^*$. Then by property (2), $S \cup T \in p^*$,

ie., $p^* \in O_{S \cup T}$. Hence $O_S \cup O_T \subset O_{S \cup T}$. Now, let $p^* \in O_{S \cup T}$, ie., $S \cup T \in p^*$. If $p^* \notin O_S$, ie., $S \notin p^*$, then R\σ-cl (S) $\in p^*$ and hence, $(R\setminus\sigma\text{-cl}(S)) \cap (S \cup T) \in p^*$. But $(R\setminus\sigma\text{-cl}(S)) \cap (S \cup T) \subset T$. Hence $T \in p^*$, that is, $p^* \in O_T$. Thus $O_{S \cup T} \subset O_S \cup O_T$. Hence, $O_{S \cup T} = O_S \cup O_T$.

(b) put T = R\σ-cl (S) in (a) then we have $O_{S \cup R \setminus \sigma\text{-cl}(S)} = O_S \cup O_{R \setminus \sigma\text{-cl}(S)}$. By using, $O_{S \cup R \setminus \sigma\text{-cl}(S)} = \theta(R)$. We have $\theta(R) = O_S \cup O_{R \setminus \sigma\text{-cl}(S)}$

Hence $O_S = \theta(R) \setminus O_{R \setminus \sigma\text{-cl}(S)}$.

Notation: 4.2 $G_\delta F_\sigma$ denote a set which is both G_δ and F_σ

Definition: 4.4 A topological space (X, T) is said to be zero dimensional if X has a base of G_δ - neighbourhoods that are both G_δ and F_σ.

Definition: 4.6 A topological space is a G_δ - T_1 space if for given any two distinct points a and b of X, each has a G_δ - neighbourhood not containing the other.

Proposition: 4.2 The maximal structure θ (R) described above is a G_δ - compact, G_δ - Hausdorff space and has a base of G_δ - neighbourhoods that are $G_\delta F_\sigma$.

Proof: Each set O_S is G_δ - by definition and by equation (b), of Proposition 4.1 it is also F_σ. Thus θ (R) has a base of G_δ - neighbourhoods that are $G_\delta F_\sigma$, that is, θ (R) is zero dimensional. Since θ (R) has a base of G_δ -neighbourhoods that are $G_\delta F_\sigma$ and G_δ -T_1 space it follows that it is G_δ -Hausdorff. Finally to prove that θ (R) is G_δ-compact. Suppose that there is a G_δ- covering of θ (R). By replacing each element of the covering by the union of the appropriate sets O_S, we may assume that the covering has the from $\left(O_{S_\alpha}\right)$. If it is impossible to take a finite subcovering from this G_δ-covering, then no set of the form $R \setminus \bigcup_{i=1}^{n} \sigma\text{-cl}\left(S_{\alpha_i}\right)$ is empty. Since otherwise the G_δ - sets $O_{S_{\alpha_i}}$ would form a finite G_δ-covering of θ (R). Hence the G_δ-sets $R \setminus \bigcup_{i=1}^{n} \sigma\text{-cl}\left(S_{\alpha_i}\right)$ form a centered system. It may be extended to a maximal system p*. This maximal system is not contained in any O_{s_α} , since it contains, in particular all the R\σ-cl (s_α). This contradiction proves that θ (R) is G_δ-compact. Thus with each G_δ-Hausdorff space R we have associated a G_δ-Hausdorff space θ (R)-the space of maximal centered systems of G_δ-sets.

5. G_δ-EXTREMALLY DISCONNECTED SPACES:

Definition: 5.1 A **G_δ**-Hausdorff space R is called G_δ-extremally disconnected if the σ-closure of any **G_δ**-set is **G_δ**.
It is clear that a space is G_δ-extremally disconnected iff two disjoint G_δ- sets have disjoint σ-closures.

Proposition: 5.1 An everywhere G_δ-dense subset R′ of G_δ-extremally disconnected space R is itself G_δ-extremally disconnected.

Proof: We prove this by contradiction. Suppose that there exists two G_δ-sets S_1 and S_2 in R′ such that $S_1 \cap S_2 = \phi$. But $\sigma\text{-cl}(S_1) \cap \sigma\text{-cl}(S_2) \neq \phi$. Let T_1 and T_2 be any two G_δ-sets in R such that $T_1 \cap R' = S_1$ and $T_2 \cap R' = S_2$. Then $T_1 \cap T_2 = \phi$ for if $T \subset T_1 \cap T_2$, then $T \cap R' \neq \phi$ and T is contained in $S_1 \cap S_2$ which is impossible. On the otherhand, $\sigma\text{-cl}(T_1) \cap_1 \sigma\text{-cl}(T_2) \supset \sigma\text{-cl}(S_1) \cap \sigma\text{-cl}(S_2) \neq \phi$, contradicting the fact that R is G_δ-extremally disconnected. Hence the Lemma.

Proposition: 5.2 The space θ (R) of maximal centered systems of an arbitrary G_δ-Hausdorff space R is G_δ-extremally disconnected.

Proof: The proof of this theorem follows from the following equation: $O_{\bigcup_\alpha S_\alpha} = \sigma\text{-cl}\left(\bigcup_\alpha O_{S_\alpha}\right)$. To verify this, if

$S \subset T$, it follows that $O_S \subset O_T$ and therefore $\bigcup_\alpha O_{S_\alpha} \subset O_{\bigcup_\alpha S_\alpha}$, and since $O_{\bigcup_\alpha S_\alpha}$ is F_σ, σ-cl $\left(\bigcup_\alpha O_{S_\alpha}\right) \subset O_{\bigcup_\alpha S_\alpha}$.

To prove the opposite inclusion, let q be an arbitrary element of $O_{\bigcup_\alpha S_\alpha}$, ie., $\bigcup_\alpha S_\alpha \in q$ and let S be an arbitrary G_δ-set of q. Then $S \cap \bigcup_\alpha S_\alpha \neq \phi$, and hence there exists α such that $S \cap S_\alpha \neq \phi$. But then $O_S \cap O_{S_\alpha} \neq \phi$, and since $S \in q$ is arbitrary, This means that $q \in$ σ-cl $\left(\bigcup_\alpha O_{S_\alpha}\right)$. That is, $O_{\bigcup_\alpha S_\alpha} \subset$ σ-cl $\left(\bigcup_\alpha O_{S_\alpha}\right)$. Hence $O_{\bigcup_\alpha S_\alpha} =$ σ-cl $\left(\bigcup_\alpha O_{S_\alpha}\right)$. Hence the theorem.

Proposition: 5.3 The equation R = θ (R) holds iff R is a G_δ-compact, G_δ-extremally disconnected and G_δ-Hausdorff space.

Proof: The necessary condition follows from Proposition 5.1 and Proposition 5.2. To prove sufficiency, let R satisfy the condition of the theorem. Now, we construct a homeomorphism π of θ (R) onto R. Let $p = \{ S_\alpha \} \in \theta(R)$. Then the system of F_σ-sets σ-cl $\{ S_\alpha \}$ is centered and has a non-empty intersection. This intersection consists of a single point. For suppose that there are two distinct points r_1 and r_2 in $\bigcap$ σ-cl (S_α). Let O_{r_1} and O_{r_2} be two disjoint G_δ-neighbourhoods of these points. Since $O_{r_1} \cap S_\alpha \neq \phi$ and $O_{r_2} \cap S_\alpha \neq \phi$ for all $S_\alpha \in p$, which gives that $O_{r_1} \in p$ and $O_{r_2} \in p$ which is impossible. Thus $\bigcap$ σ-cl (S_α) consists of a single point r. Let π (p) = r. We shall prove that the mapping π is one-one and continuous. Since θ(R) is G_δ-compact, this will prove the theorem. The mapping is onto. For let $r \in R$ and let $\{V_\alpha\}$ be the system of all G_δ-neighbourhoods of r in R. This system can be extended uniquely to a maximal one. For, if $\{V_\alpha\}$ is contained in two different maximal systems then there would be two G_δ-sets S_1 and S_2 in R such that $S_1 \cap S_2 = \phi$, each of them would intersect every V_α, that is, $r \in \left((\sigma\text{-cl}(S_1)) \cap (\sigma\text{-cl}(S_2))\right)$ but which contradicts the fact that R is extreamally G_δ-disconnected. Extending the system $\{V_\alpha\}$ to a maximal one, there is a point $p = \{S_\alpha\}$ in θ (R). But π (p) = r. Already we have proved that π is one-one. Hence from the definition of π it follows that π (O_S) = σ-cl (S).

Let O_r' be any G_δ-neighbourhood of r. Let S be a G_δ-neighbourhood such that σ-cl(S) $\subset O_r'$. Then O_S is a G_δ F_σ neighbourhood of p such that π (O_S) $\subset$ σ-cl (S) $\subset O_r'$. Thus π is continuous and hence the proof.

6. G_δ-θ CONTINUOUS MAPPINGS:

Definition: 6.1 Let f be a mapping of a space X into a space Y with f(x) = y. Then f is called G_δ-θ continuous at x iff for every G_δ-neighbourhood O_y of y there exists a G_δ-neighbourhood Ox of x such that f (σ-cl (O_x)) $\subset$ σ-cl (O_y). The mapping is called G_δ-θ continuous if it is G_δ-θ continuous at every point of X. A mapping that is one-one and G_δ-θ continuous in both directions is called a G_δ-θ homeomorphism. It is clear that a continuous mapping is G_δ-θ continuous.

The Realization of R in θ (R):

Consider a G_δ-Hausdorff space R and its space θ (R). Let $r \in R$ and x(r) denote the set of all ends p^* of R that contain all the G_δ-neighbourhoods of r. Now, the set x(r) is G_δ F_σ in θ (R). Since θ (R) is G_δ-compact, x (r) is G_δ -compact.

Now define a space R^* constructed as follows: Its points are the F_σ-sets x (r) and its structure is defined as, let V be a G_δ-set of θ (R). Let V^* denote the set of all F_σ-sets x(r) that are completely contained in V. By definition, the set of all V^* is to form a base of R^*.

Definition: 6.2 A topological space is G_δ-regular iff for each point x and each G_δ-neighbourhood U and x there is a F_σ-neighbourhood V of x such that $V \subset U$.

Definition: 6.3 If there exists a G_δ-θ homeomorphism of one space onto another, the two spaces are said to be G_δ-θ-homeomorphic.

Proposition: 6.1 R^* is G_δ-θ homeomorphic to R. If R is G_δ-regular, then R^* is homeomorphic to R.

Proof: Let π be the mapping of R^* onto R in which π (x(r)) = r. We shall show that π is the required G_δ-θ homeomorphism. To prove this the equivalence of the following inclusions are established.

Now, $x(r) \subset O_H = O_{\sigma\text{-int }(\sigma\text{-cl}(H))}$ and $r \in \sigma$-int (σ-cl (H)).

If If $r \in \sigma$-int (σ-cl (H)), it is clear that $x(r) \subset O_H$. If $x (r) \subset O_{\sigma\text{-int }(\sigma\text{-cl }(H))}$, but $r \notin O_{\sigma\text{-int }(\sigma\text{-cl }(H))}$ then there would be end p^* in x(r) not containing σ-int (σ-cl (H)). But then $p^* \notin O_{\sigma\text{-int }(\sigma\text{-cl }(H))}$, which is impossible. From this equivalence it follows that π^{-1} is continuous. For let V^* be a G_δ-neighbourhood of the set x(r) in R^*. Since x(r) is G_δ-compact, assume that V has the form O_H where H is G_δ-in R. Then π^{-1} (σ-int (σ-cl (H))) $\subset O_H = V^*$. This proves the continuity of π^{-1}. To prove that π is G_δ-θ continuous, it is easy to see that if $x(r') \bigcap O_H \neq \phi$. then $r' \in \sigma$-cl (H). From the construction of V^*, it is clear that if $x(r') \in (\sigma\text{-cl}(V^*)) = \sigma\text{-cl}(O_H)$ then $x(r') \bigcap O_H \neq \phi$. Let H be an arbitrary G_δ-neighbourhood of r, and let V = O_H. Then π (σ-cl $(V^*)) \subset \sigma$-cl (H), which proves that π is G_δ-θ continuous, since V^* is a G_δ-neighbourhood of x(r) in R^*. Thus the spaces R^* and R are G_δ-θ homeomorphic. If R is G_δ-regular, then π is G_δ-θ continuous and so π a homeomorphism. Hence the lemma.

The absolute ω^*(R) of a space R:

In ω^*(R) each point $r \in R$ is represented by ends containing all G_δ-neighbourhoods of R. It is obvious that $\omega^*(R) = \bigcup_{r\in R} x(r)$ where x(r) are the sets defined above. The subset ω^*(R) is mapped in a natural way onto R.
If $p \in \omega^*(R)$, then by definition $\pi_R(p) = r$, where r is the point whose G_δ-neighbourhoods all belong to p. π_R is called the natural mapping of ω^*(R) onto R.

Proposition: 6.2 ω^*(R) is everywhere G_δ-dense in θ (R).

Proof: Let p be an arbitrary end of R and O_U be a G_δ-neighbourhood of it. Then O_U contains the sets x(r) corresponding to any point $r \in U$ and so has a non-empty intersection with ω^*(R),

Proposition: 6.3 ω^*(R) is G_δ-extremally disconnnected.

Proof: From Proposition 6.2 ω^*(R) is everywhere G_δ-dense in θ(R). And also from Proposition 5.1 and Proposition 5.2, ω^*(R) is G_δ-extremally disconnected.

Proposition: 6.4 ω^*(R) is G_δ-θ homeomorphic to R iff R is G_δ-extremally disconnected.

Proof: Let ω^*(R) be G_δ-θ homeomorphic to R and from Proposition 6.3, ω^*(R) is G_δ-extremally disconnected. Now to prove the sufficiency, let $\{U_\alpha(r)\}$ be the collection of all G_δ-sets in R containing r. The system $\{U_\alpha(r)\}$ can be extended to a maximal one in a unique way, for otherwise there exist G_δ-disjoint sets G_1 and G_2 meeting $U_\alpha(r)$, that is, $r \in$ (σ-cl $(G_1)) \bigcap$ (σ-cl $(G_2)) \neq \phi$, which is impossible for G_δ-extremally disconnected space. Thus, for each point $r \in R$ the set x(r) consists of a single point. But then the space R constructed above coincides with ω^*(R). Hence R is G_δ-θ homeomorphic to ω^*(R).

Proposition: 6.5 If R is a G_δ-regular, G_δ-extremally disconnected space, then R is a G_δ-homeomorphic to ω^*(R).

Proof: From Proposition 6.1, if R is a G_δ-regular, G_δ-extremally disconnected space, then it is G_δ-homeomorphic to R and hence to ω^*(R).

7. G_δ-HAUSDORFF EXTENSION OF SPACES:

Definition: 7.1A G_δ-Hausdorff space δ (R) is called an extension of G_δ-Hausdorff space R if R is contained in δ (R) as an everywhere G_δ-dense subset. R is called G_δ-H closed if every extension δ (R) coincides with R itself. An extension δ (R) is called G_δ-H-closed if δ (R) is G_δ-H-closed and G_δ-compact if δ (R) is G_δ-ompact.

Proposition: 7.1 The space R is G_δ-H-closed if and only if any centered system $\{U_\alpha\}$ of G_δ-sets of R satisfies the condition $\bigcap_\alpha \sigma - \text{cl}\ (U_\alpha) \neq \phi$.

Proof: Necessary: If $p = \{U_\alpha\}$ were a centered system with $\bigcap$ σ-cl $(U_\alpha) = \phi$, then we would construct the extension δ (R) which does not coincide with R itself. The points of σ (R) are those of R and a new point p. The G_δ-neighbourhoods of each point $r \in R$ in δ (R) are the same as in R. Any set U_α together with the point is a G_δ-neighbourhood of p. Because of the condition $\bigcap$ σ-cl $(U_\alpha) = \phi$, the space δ (R) is G_δ-Hausdorff and because $\{U_\alpha\}$ is a centered system, it contains R as an everywhere G_δ-dense subset, that is, R is not G_δ-H closed.

Sufficiency: Let R be a proper everywhere G_δ-dense subset of δ (R). Consider in δ (R) all the G_δ-neighbourhoods of some point $p \in$ δ (R)/R. Let this be the system $\{U_\alpha\}$. This is centered, for otherwise p would be an isolated point in δ (R) and R would not be everywhere G_δ-dense in δ (R). Since δ (R) is a G_δ-Hausdorff, we have $\bigcap$ σ-cl (U_α) = p. But then the system $\{V_\alpha = U_\alpha \bigcap R\}$ is centered and $\bigcap$ σ–cl $(V_\alpha) = \phi$, which contradicts the condition of the lemma.

8. THE ALEKSANDROV – URYSON G_δ-COMPACTNESS CRITERION:

Let R be a G_δ-Hausdorff space, $\omega^*(R)$ its absolute and π_R the natural mapping of $\omega^*(R)$ onto R. Also Let F be any subset of R^*. We associate it with a certain subset $\tilde{F}$ of $\omega^*(R)$, defined by saying that the point $p \in \pi_R^{-1}(x)$, $x \in R$, belongs to $\tilde{F}$ if $p \in O_U$ for every U satisfying the condition $x \in$ σ-int (σ-cl(U $\bigcap$ F). By construction, $\tilde{F}$ is contained in the complete inverse image π_R^{-1} (F) of F in $\omega^*(R)$. Then we call $\tilde{F}$ the G_δ-reduced inverse image of F in ω (R).

Proposition: 8.1 (Alexsandrov-Uryson G_δ-compactness) A G_δ-Hausdorff space R is G_δ-compact iff each of its F_σ-subsets is G_δ-H closed.

Proof: Since in a G_δ-compact space every F_σ-subset is G_δ-compact and hence G_δ-H closed. The proof of sufficiency, based on the following properties of G_δ-reduced inverse images.

Property: I If $F_1 \subset F_2 \subset \ldots \subset F_n = R$, with F_1 non-empty, then $\bigcap_{i=1}^{n} \tilde{F}_i \neq \phi$.

Let $x \in F_1^*$ and let $q' = \{G^1\}$ be a end of F_1 containing a centered system of G_δ-sets G^1 in F such that $x \in$ σ-int (σ-cl(G^1)). Assume that we have constructed systems $q^i = \{G^i\}$ of F_i such that q^i contains all the G_δ-sets $G^i \subset F_i$ for which $x \in$ σ-int (σ-cl(G^i)) and all the sets whose intersection with F_{i-1} is some G^{i-1}. By definition q^{i+1} is to consist of all sets $G^{i+1} \subset F_{i+1}$ for which $x \in$ σ-int (σ-cl (G^{i+1}) and of all sets whose intersection with F_i is some G^i. Clearly q^{i+1} is a centered system. Thus, for each i, we construct a centred system q^i. Let p = {H} denote the end of R containing q^n. We have to prove that $p \in \bigcap_{i=1}^{n} \tilde{F}_i$. It follows from the construction of p, that if we have $H \bigcap F_i \in q^i$ for some i and some G_δ-set H in R, then $H \in p$. We prove that $p \in \tilde{F}_i$. Let H be a G_δ-set of R such that $x \in$ σ-int (σ-cl ($H \bigcap F_i$)). Then $H \bigcap F_i \in q^i$ and hence $H \in p$, that is, $p \in \tilde{F}_i$ which proves property I.

Remark: 8.1 If O_H is a G_δ-neighbourhood of $\pi_R^{-1}(x) \bigcap$ σ-cl F, where H is the largest of the G_δ-sets H′ with the property $O_{H'} = O_H$ then $x \in$ σ-int (σ-cl ($H \bigcap F$) . For, otherwise R\σ-cl (H) = V ≠ ϕ, with $x \in$ σ-cl ($V \bigcap F$) . If some set G, G_δ in R, has the property $x \in$ σ-int (σ-cl ($G \bigcap F$)), then $V \bigcap G$ is non-empty. Hence we may consider system q consisting of all G_δ-neighbourhoods. But, on the otherhand, since $p \in O_V$ and $O_V \bigcap O_H = \phi$ since $V \bigcap H = \phi$, then $p \in \tilde{F} \subset O_H$, which is a contradiction. Hence $x \in$ σ-int (σ-cl ($H \bigcap F$).

We now prove that $\pi_R^{-1}(x) \cap \tilde{F}$ is G_δ-compact. Let q be the system of all the G_δ-sets G in R such that x ∈ σ-int (σ-cl (G ∩ F) and all the G_δ- neighbourhoods of x in r. It is clear that $\pi_R^{-1}(x) \cap \tilde{F}$, consists of all ends p^* containing q. If p′ is an end belonging to $\pi_R^{-1}(x)$ and such that any of its G_δ-neighbourhoods O_H contains some point p∈ $\pi_R^{-1}(x) \cap \tilde{F}$ then any H ∈ p′ meets an arbitrary element of q, and hence p′∈ $\pi_R^{-1}(x) \cap \tilde{F}$, that is the latter set F_σ in $\pi_R^{-1}(x)$ and so is G_δ-compact.

Property: II If F is G_δ-H closed, then $\tilde{F}$ is G_δ-compact.

Proof: Let $\{H_\alpha\}$ be any G_δ-covering of $\tilde{F}$ by G_δ-sets in $\tilde{F}$. They may be extended to G_δ-sets in $\omega^*(R)$. Assume that each of the extended sets has the form O_U, where U is a G_δ-set in R. Otherwise $\{H_\alpha\}$ may be replaced by a finer G_δ-covering for which this condition holds. So we may assume that $\{H_\alpha\}$ is a G_δ- covering of F by G_δ-sets in $\omega^*(R)$ of the form O_{U_α} , where U_α is G_δ-in R. Let x ∈ F. Let H_β^x denote the union of a finite number of sets H_α G_δ-covering the G_δ-compact set $\pi_R^{-1}(x)$. Clearly H_β^x has the form $O_{U_\beta}^x$, where U_β^x is G_δ-set in R and is maximal among the sets H for which $O_H = O_{U_\beta}^x$. Hence it follows that the system σ-int $\{ U_\beta^x \cap F \}$ is a G_δ-covering of F.

Since F is G_δ-H closed, choose a finite number of elements of this G_δ-covering such that $\bigcup_{i=1}^{n}$ σ-cl(σ-int(σ-cl($U_{\beta_i}^x \cap F$))) = F. We prove that $\bigcup_{i=1}^{n} O_{U_{\beta_i}}^x \supset \tilde{F}$. Since the union $\bigcup_{i=1}^{n} U_{\beta_i}^x$ = U has the property that x ∈ σ-int(σ-cl(F ∩ U) for any x, then an arbitrary end $p^* \in \tilde{F}$ contains U, and hence belongs to some $O_{U_\beta}^x$. Thus, if we choose only those H_α that make $O_{U_{\beta_i}}^x$ and take their intersections with $\tilde{F}$, we obtain the required finite covering. Hence property II.

Proposition: 8.2 The G_δ-Hausdorff space R is G_δ-compact iff every well-ordered decreasing sequence of non-empty F_σ-sets has a non-empty intersection.

Proof: Suppose that the conditions of the theorem are satisfied and that $\{F_\alpha\}$ is a well-ordered decreasing system of F_σ-sets of R. Then by property I, the G_δ-set $\tilde{F}$ form a centered system in $\omega^*(R)$. Also since all the F_α are G_δ-H closed, by property II, $\tilde{F}$ are G_δ-compact. Hence $\bigcap \tilde{F}_\alpha \neq \phi$. Let y ∈ $\tilde{F}_\alpha$. Then $\pi_R(y) \in F_\alpha$ for every α, that is $\bigcap_\alpha F_\alpha \neq \phi$.

REFERENCES:

[1] S. Illiadis and S. Fomier, The method of centered systems in the theory of topological spaces.

[2] James R. Munkres, Topology A first course.

[3] Richard R. Goldberg, Methods of Real Analysis.

****

