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ON b-FRAMES IN BANACH SPACES

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ABSTRACT

In the paper the generalization of the notion b-frame in Banach spaces is given by means of bilinear mappings and the notion of b-convergence and b-representation systems are introduced. The relations between these notions are established. The criterion when the system in Banach space forms b-frame, is given. The projection description of b-frame is constructed. Using a linear algorithm of expansion in b-frame, the extremality estimation for the coefficients of expansion in b-frame are established.

Key words: *b-frame, expansion in b-frame, b-basis, b-completeness, CB-space.*

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1. INTRODUCTION:

Note that, the notion of frame was determined by Duffin and Schaeffer [1] in Hilbert spaces in the following way. Let H be a separable Hilbert space the system of non-zero elements $\{\varphi_n\}_{n\in N} \subset H$ be called a frame in H if there exist the constants $0 < A \le B < \infty$ such that for each $h \in H$ it is valid

$$A\,\|h\|_H^2 \le \sum_{n=1}^{\infty} |(h,\varphi_n)|^2 \le B\,\|h\|_H^2\,, \tag{1}$$

where $\|\cdot\|_H$ and $(\cdot,\cdot)$ is a norm and scalar product in H. The constants A and B in (1) are called lower and upper bounds the number $k = A/B$ is called a condition coefficient of the frame $\{\varphi_n\}_{n\in N}$. In the case, when $k=1$ $\{\varphi_n\}_{n\in N}$ is an tight frame.

Development in theory of a frame in Hilbert spaces reduced to obtaining the analogues of the known results for Banach case. By theory of frame [2]-[5] and others have Banach extensions. The notion of X-frame generalizing the notion of p-frame studied in [5] was introduced in the paper [6]. In [7] gives the dual notion of a frame in Banach space, for which in [8] projection description is studied, a criterion on the existence of a linear expansion algorithm in frame is proved, the analogue of the extremality properties for the coefficients of expansion in frame is obtained. A frame is a special case of Bessel sequence introduced and studied [9] in Hilbert spaces. Banach analogies these results were investigated in the paper [10]-[14].

Cite the known result [15] from the quant theory of information for projection description of frame.

Theorem: 1.1 *Let $\{\varphi_n\}_{n\in N}$ be a tight frame in Hilbert space H. Then there exists a Hilbert space $H' \supset H$ and orthonormed basis $\{\psi_n\}_{n\in N} \subset H'$ such that*

$$\pi(\psi_n) = \varphi_n\,,\ n \in N\,,$$

where π is an operator of orthogonal projection from H' onto H.

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The results [16], [17] and [18] in this direction are known. The frames in Hilbert spaces have close relations with Riesz bases. This matter was considered [19] and is in the following.

Theorem: 1.2 *For the system* $\{\varphi_n\}_{n\in N}$ *of Hilbert space* H *to be a frame with boundaries* A *and* B, $A \le B$, *it is necessary and sufficient to find a Hilbert space* H' *containing* H *and the Riesz basis* $\{\psi_n\}_{n\in N} \subset H'$ *such that*

$$A^{1/2}\left(\sum_{n=1}^{\infty}|c_n|^2\right)^{1/2} \le \left\|\sum_{n=1}^{\infty} c_n \psi_n\right\|_{H'} \le B^{1/2}\left(\sum_{n=1}^{\infty}|c_n|^2\right)^{1/2}, \tag{2}$$

where $\pi(\psi_n) = \varphi_n$, $n \in N$, π *is an operator of orthogonal projection from* H' *onto* H.

The goal of our paper is extension of the results of [8] to $b_{\tilde{X}}$-frames for CB-space $\tilde{X}$. In the paper, necessary and sufficient condition $b_{\tilde{X}}$-frame of the system in Banach space is proved, projection description of $b_{\tilde{X}}$-frame and criterion of the existence of linear expansion algorithm in $b_{\tilde{X}}$-frame is given.

2. b -FRAME IN BANACH SPACE:

Let X be a Banach space with the norm $\|\cdot\|_X$. Denote by $\tilde{X}$ B-space of sequences $\tilde{x} = \{x_n\}_{n\in N}$, $x_n \in X$, with coordinate-wise linear operations. If the convergence in $\tilde{X}$ is coordinate-wise and the subspaces $\tilde{E}_n = \left\{\tilde{x} \in \tilde{X} : \tilde{x} = \{\delta_{in} x\}_{i\in N}, x \in X\right\}$ form a basis, then $\tilde{X}$ called CB-space with a canonical basis.

Find the general form of a linear continuous functional in $\tilde{X}$. Let $\tilde{t}$ be an arbitrary linear continuous functional in $\tilde{X}$. Since for each $\tilde{x} \in \tilde{X}$ ($\tilde{x} = \{x_n\}_{n\in N}$) the equality $\tilde{x} = \sum_{n=1}^{\infty}\{\delta_{in} x_n\}_{i\in N}$ is valid, then $\tilde{t}(\tilde{x}) = \sum_{n=1}^{\infty}\tilde{t}(\{\delta_{in} x_n\}_{i\in N})$. Consider the functional $t_n : X \to C$ such that $t_n(x) = \tilde{t}(\{\delta_{in} x\}_{i\in N})$ for $\forall x \in X$. Then we get that $\tilde{t}(\tilde{x}) = \sum_{n=1}^{\infty} t_n(x_n)$. From the coordinate-wise convergence in $\tilde{X}$ it follows that $t_n \in X^*$. Consequently, identifying $\tilde{t}$ with $\{t_n\}_{n\in N} \subset X^*$, $\tilde{X}^*$ is isometrically isomorphic to some B-space of sequences of elements from X^* and

$$\tilde{t}(\tilde{x}) = \sum_{n=1}^{\infty} t_n(x_n)$$

is a general form of a linear continuous functional in $\tilde{X}$, with the norm

$$\|\tilde{t}\|_{\tilde{X}^*} = \sup_{\|\tilde{x}\|_{\tilde{X}} \le 1}\left|\sum_{n=1}^{\infty} t_n(x_n)\right|.$$

Let Y and Z be B-spaces with appropriate norms $\|\ \|_Y$ and $\|\ \|_Z$. Consider a bilinear mapping $b(x, y) : X \times Y \to Z$ satisfying the condition:

$$\exists\, M \rangle 0 : \quad \|xy\|_Z \le M\|x\|_X\|y\|_Y \quad \forall x \in X,\ y \in Y. \tag{3}$$

where $xy = b(x, y)$, $x \in X$, $y \in Y$. Denote by $\hat{b}(f, y)$ the mapping $\hat{b}(f, y) : Z^* \times Y \to X^*$, determined by the relation $\hat{b}(f, y)(x) = f(xy)$ $\forall f \in Z^*$, $y \in Y$, $x \in X$. Further, for brevity we assume $< f, y > = \hat{b}(f, y)$. It is easy to get from (3) the validity of the inequality

$$\|< f, y >\|_{X^*} \le M\|f\|_{Z^*}\|y\|_Y \quad \forall f \in Z^*,\ y \in Y. \tag{4}$$

Let $\tilde{X}$ be a CB -space with a canonical basis. Introduce the following denotation.

Definition: 2.1 *The system $\{\varphi_n\}_{n\in N} \subset Y$ is said to be $b_{\tilde{X}}$-frame in Z if there exist the constants A, B, $0 < A \le B < \infty$ such that for any $g \in Z^*$ the sequence $\{< g, \varphi_n >\}_{n\in N}$ belongs to $\tilde{X}^*$ and satisfies the inequality*

$$A \, \|g\|_{Z^*} \le \left\| \{< g, \varphi_n >\}_{n\in N} \right\|_{\tilde{X}^*} \le B \, \|g\|_{Z^*} . \tag{5}$$

Definition: 2.2 *The system $\{\varphi_n\}_{n\in N} \subset Y$ is said to be a system of $b_{\tilde{X}}$ -representation if for any $z \in Z$ there exists $\{x_n\}_{n\in N} \in Z$ such that $z = \sum_{n=1}^{\infty} x_n \varphi_n$ and a system of $b_{\tilde{X}}$ -convergence in Z if for any $\tilde{x} \in \tilde{X}$ ($\tilde{x} = \{x_n\}_{n\in N}$) the series $\sum_{n=1}^{\infty} x_n \varphi_n$ converges in Z.*

Cite the criteria when the system $\{\varphi_n\}_{n\in N} \subset Y$ forms a $b_{\tilde{X}}$ -frame in Z.

Theorem: 2.3 *The system $\{\varphi_n\}_{n\in N} \subset Y$ forms a $b_{\tilde{X}}$ -frame in Z iff it is simultaneously both a system of $b_{\tilde{X}}$ -representation and a system of $b_{\tilde{X}}$ -convergence in Z.*

Proof: Necessary: Let $\{\varphi_n\}_{n\in N}$ be a $b_{\tilde{X}}$ -frame in Z. Determine the operator S (of synthesis) in linearity on finite sequences $\{x_n\}$ by the equality $S(\{x_n\}) = \sum_n x_n \varphi_n$. For an arbitrary $g \in Z^*$ we get

$$g(S(\{x_n\})) = g(\sum_n x_n \varphi_n) = \sum_n g(x_n \varphi_n) = \sum_n < g, \varphi_n > (x_n) = \{< g, \varphi_n >\}_{n\in N} \, (\{x_n\}) .$$

Then

$$\left| g(S(\{x_n\})) \right| = | \{< g, \varphi_n >\}_{n\in N} \, (\{x_n\}) | \le \left\| \{< g, \varphi_n >\}_{n\in N} \right\|_{\tilde{X}^*} \, \left\| \{x_n\} \right\|_{\tilde{X}} \le B \, \|g\|_{Z^*} \, \left\| \{x_n\} \right\|_{\tilde{X}} .$$

So, $\left\| S(\{x_n\}) \right\|_Z \le B \, \left\| \{x_n\} \right\|_{\tilde{X}}$. Continuing S in continuity on all $\tilde{X}$, we get $S \in L(\tilde{X}, Z)$ and $S(\tilde{x}) = \sum_{n=1}^{\infty} x_n \varphi_n$, $\tilde{x} \in \tilde{X}$ ($\tilde{x} = \{x_n\}_{n\in N}$). Consequently, $\{\varphi_n\}_{n\in N}$ is a system of $b_{\tilde{X}}$ -convergence in Z. Consider an operator (of analysis) $R : Z^* \to \tilde{X}^*$ in the equality $R(g) = \{< g, \varphi_n >\}_{n\in N}$, $g \in Z^*$. It is clear that it holds the equality $R = S^*$. Since for any $g \in Z^*$ it holds $A \, \|g\|_{Z^*} \le \left\| S^*(g) \right\|_{\tilde{X}^*}$, then S maps $\tilde{X}$ on all the space Z (see [20], theorem 4.15). Thus, $\{\varphi_n\}_{n\in N}$ is a system of $b_{\tilde{X}}$ -representation in Z.

Sufficiency: Let $\{\varphi_n\}_{n\in N}$ be a system of $b_{\tilde{X}}$ -representation and $b_{\tilde{X}}$ -convergence in Z. Consider (linear) operators $S_n : \tilde{X} \to Z$ by the equalities $S_n(\tilde{x}) = \sum_{k=1}^{n} x_k \varphi_k$, $n \in N$. From the coordinate-wise convergence in $\tilde{X}$ it follows, that $S_n \in L(\tilde{X}, Z)$. Since $\{\varphi_n\}_{n\in N}$ is a system of $b_{\tilde{X}}$ -convergence in Z, then $\exists \lim_{n\to\infty} S_n(\tilde{x})$ and by the theorem on uniform boundedness $\|S_n\|_{L(\tilde{X},Z)} \le B$ for any $n \in N$. Consequently, the operator S determined by the equality $S(\tilde{x}) = \lim_{n\to\infty} S_n(\tilde{x})$ belongs to $L(\tilde{X}, Z)$. Further, consider an operator $R \in L(Z^*, \tilde{X}^*)$ such that $R = S^*$. Then for any $g \in Z^*$ and $\tilde{x} \in \tilde{X}$ we get

$$R(g) \, (\tilde{x}) = S^*(g) \, (\tilde{x}) = g(S(\tilde{x})) = \sum_{n=1}^{\infty} g(x_n \varphi_n) = \{< g, \varphi_n >\}_{n\in N} \, (\tilde{x}) .$$

Hence we get $R(g) = \{< g, \varphi_n >\}_{n\in N}$. Since $\{\varphi_n\}_{n\in N}$ is a system of $b_{\tilde{X}}$-reresentation then $S(\tilde{X}) = Z$. Therefore there exists $A > 0$: $A\,\|g\|_{Z^*} \le \|S^*(g)\|_{\tilde{X}^*}$ (see. [20], theorem 4.15) and $\{\varphi_n\}_{n\in N}$ forms a $b_{\tilde{X}}$-frame in Z. The theorem is proved.

3. PROJECTIVE b -FRAMES:

Let X, Y and Z be B-spaces , a $\tilde{X}$ be a CB-space with a canonical basis. Cite some notion from [14].

The system $\{\varphi_n\}_{n\in N} \subset Y$ is said to be b-complete in Z if the aggregate $L_b(\{\varphi_n\}_{n\in N})$ of all possible finite sums $\sum x_n\varphi_n$ is dense in Z.

The system $\{\varphi_n\}_{n\in N} \subset Y$ is said to be $b_{\tilde{X}}$-basis in Z if for $\forall z \in Z$ there exists a unique sequence $\{x_n\}_{n\in N} \subset X$: $z = \sum_{n=1}^{\infty} x_n\varphi_n$, moreover $\tilde{X} = \{\tilde{x} = \{x_n\}_{n\in N} \subset X : \sum_{n=1}^{\infty} x_n y_n \in Z\}$.

As the analogue of the projective frame (see [8]) introduce the following definition.

Definition: 3.1 *b-frame $\{\varphi_n\}_{n\in N} \subset Y$ in Z is said to be projective if:*

(à) there exist B-space $\tilde{Z} \supset Z$, including the space Z as a closed subspace;

(b) there exist $\tilde{b}_{\tilde{X}}$-basis $\{\psi_n\}_{n\in N} \subset L(X,\tilde{Z})$ in $\tilde{Z}$, where $\tilde{b}(x,\psi) \equiv (x,\psi) = \psi(x)$, $x \in X$, $\psi \in L(X,\tilde{Z})$;

(c) there exist a continuous projector $P \in L(\tilde{Z}, Z)$ from $\tilde{Z}$ onto Z such that $P(x,\psi_n) = x\varphi_n$ for $\forall x \in X$, $n \in N$.

Definition: 3.2 *The space of coefficients of zero-series of the system $\{\varphi_n\}_{n\in N} \subset Y$ is called a closed in $\tilde{X}$ subspace $\tilde{N} = \{\tilde{x} = \{x_n\}_{n\in N} \in \tilde{X} : \sum_{n=1}^{\infty} x_n\varphi_n = 0\}$.*

Theorem: 3.3 *Let $\{\varphi_n\}_{n\in N} \subset Y$ be a $b_{\tilde{X}}$-frame in Z. Then $\{\varphi_n\}_{n\in N}$ is a projective $b_{\tilde{X}}$-frame in Z iff $\tilde{N}$ is complemented in $\tilde{X}$.*

Proof: Necessity: Let $\{\varphi_n\}_{n\in N}$ be a projective $b_{\tilde{X}}$-frame in Z. By theorem 2.3 there exists a synthesis operator $S \in L(\tilde{X}, Z)$: $S(\tilde{x}) = \sum_{n=1}^{\infty} x_n\varphi_n$, $\tilde{x} \in \tilde{X}$ ($\tilde{x} = \{x_n\}_{n\in N}$). Let J be a natural isomorphism between $\tilde{X}$ and $\tilde{Z}$, i.e. $J(\{x_n\}_{n\in N}) = \sum_{n=1}^{\infty}(x_n,\psi_n)$ and P be a projector from $\tilde{Z}$ onto Z. Then the following equality holds: $S = PJ$. Denote by $\tilde{M} = J^{-1}(Z)$ a closed subset in $\tilde{X}$. Show that $\tilde{X} = \tilde{M} \oplus \tilde{N}$. Take an arbitrary $\tilde{x}$ from $\tilde{X}$. Denote by $z = S(\tilde{x})$ and $\tilde{x}_{\tilde{M}} = J^{-1}(z)$. Consider $\tilde{x}_{\tilde{N}} = \tilde{x} - \tilde{x}_{\tilde{M}}$. Since $S(\tilde{x}_{\tilde{N}}) = S(\tilde{x}) - S(\tilde{x}_{\tilde{M}}) = 0$, then $\tilde{x}_{\tilde{N}} \in \tilde{N}$, consequently, $\tilde{x} = \tilde{x}_{\tilde{M}} + \tilde{x}_{\tilde{N}}$. Now let $\tilde{x} \in \tilde{M} \cap \tilde{N}$. From $\tilde{x} \in \tilde{N}$ it follows that $S(\tilde{x}) = 0$. Since $\tilde{x} \in \tilde{M}$, there exists z from Z, such that $J^{-1}(z) = \tilde{x}$. Consequently, $z = P(z) = PJ(\tilde{x}) = S(\tilde{x}) = 0$ Therefore $\tilde{x} = J^{-1}(z) = 0$. Thus, $\tilde{X} = \tilde{M} \oplus \tilde{N}$.

Sufficiency: Let $\tilde{N}$ be complemented in $\tilde{X}$, i.e. $\tilde{X} = \tilde{M} \oplus \tilde{N}$. Denote by $\tilde{Z}$ the direct sum of $Z \oplus \tilde{N}$. Obviously, $\tilde{Z}$ is a B-space by the norm $\|\{z,\tilde{x}\}\|_{\tilde{Z}} = \max\left(\|z\|_Z, \|\tilde{x}\|_{\tilde{X}}\right)$, $\{z,\tilde{x}\} \in \tilde{Z}$. Let P be a projection operator from $\tilde{Z}$

onto Z, i.e. $P(\{z,\tilde{x}\}) = z$, and Q a projection operator from $\tilde{X}$ onto $\tilde{M}$ along $\tilde{N}$. Define the operator $J:\tilde{X}\to\tilde{Z}$ from the formula $J(\tilde{x}) = \{S(\tilde{x}),\tilde{x}-Q(\tilde{x})\}$. Obviously, $J\in L(\tilde{X},\tilde{Z})$. Show that J is an isomorphism between $\tilde{X}$ and $\tilde{Z}$. If $J(\tilde{x}) = 0$, so $S(\tilde{x}) = 0$ and $\tilde{x} = Q(\tilde{x})$. Then $\tilde{x}\in\tilde{M}\cap\tilde{N}$, therefore $\tilde{x} = 0$. Further, we have $J(\tilde{X}) = \tilde{Z}$. Indeed, if $\{z,\tilde{x}_0\}\in\tilde{Z}$, then there exists $\tilde{\xi}\in\tilde{X}$ such that $z = S(\tilde{\xi})$. Denote $\tilde{x} = \tilde{x}_0 + Q(\tilde{\xi})$. It is clear that $Q(\tilde{x}) = Q(\tilde{\xi})$. So, $\tilde{x}_0 = \tilde{x} - Q(\tilde{x})$ and $z = S(\tilde{\xi}) = S(Q(\tilde{\xi})) = S(Q(\tilde{x})) = S(\tilde{x})$,
i.e. $\{z,\tilde{x}_0\} = \{S(\tilde{x}),\tilde{x}-Q(\tilde{x})\}$. Using the Banach theorem on inverse operator, we get that J is an isomorphism. Since $\tilde{X}$ is a CB - space with a canonical basis, the system $\{\psi_n\}_{n\in N}$: $J(\{\delta_{in}x\}_{i\in N}) = (x,\psi_n)$ forms $b_{\tilde{X}}$-basis in $\tilde{Z}$. On the other hand for $\forall x\in X$, $n\in N$ it is valid $P(x,\psi_n) = x\varphi_n$.

The theorem is proved.

4. LINEAR EXPANSION ALGORITHM IN $b_{\tilde{X}}$ -FRAME:

Let X, Y and Z be B-spaces, $\tilde{X}$ be a CB-space with a canonical basis.

Definition: 4.1 *We say that it holds a linear expansion algorithm in $b_{\tilde{X}}$-frame $\{\varphi_n\}_{n\in N}\subset Y$ in Z, if there exists a system $\{\varphi_n^*\}_{n\in N}\subset L(Z,X)$ such that for any $z\in Z$ the sequence $\{\varphi_n^*(z)\}_{n\in N}$ belongs to $\tilde{X}$ and $z = \sum_{n=1}^{\infty}\varphi_n^*(z)\varphi_n$.*

Theorem: 4.2 *Let $\{\varphi_n\}_{n\in N}\subset Y$ be $b_{\tilde{X}}$-frame in Z. For the existence of a linear expansion algorithm in $b_{\tilde{X}}$-frame $\{\varphi_n\}_{n\in N}$ it is necessary and sufficient that $\{\varphi_n\}_{n\in N}$ be a projective $b_{\tilde{X}}$-frame.*

Proof: Necessity: Let there exists a linear expansion algorithm in $b_{\tilde{X}}$-frame $\{\varphi_n\}_{n\in N}$. As in the proof of theorem 2.1 we denote by $S\in L(\tilde{X},Z)$: $S(\tilde{x}) = \sum_{n=1}^{\infty}x_n\varphi_n$, $\tilde{x}\in\tilde{X}$ ($\tilde{x} = \{x_n\}_{n\in N}$) the synthesis operator in $\tilde{X}$. Determine the operator $T:Z\to\tilde{X}$ from the formula $T(z) = \{\varphi_n^*(z)\}_{n\in N}$. Show that $T\in L(Z,\tilde{X})$. Consider the sequence of the operators $T_m\in L(Z,\tilde{X})$: $T_m z = \sum_{n=1}^{m}\{\delta_{in}\varphi_n^*(z)\}_{i\in N}$. Since the series $\sum_{n=1}^{\infty}\{\delta_{in}\varphi_n^*(z)\}_{i\in N}$ converges, there exists $\lim_{m\to\infty}T_m z$ and by the Banach Steinhous theorem the sequence $\{\|T_n\|\}_{n\in N}$ is bounded. So, $T\in L(Z,\tilde{X})$, since $Tz = \lim_{m\to\infty}T_m z$. Obviously $(ST)(z) = \sum_{n=1}^{\infty}\varphi_n^*(z)\,\varphi_n = z$, i.e. $ST = I$, where I is an identity operator in Z. Further, let $\tilde{M} = T(Z)$. We have $\tilde{X} = \tilde{M}\oplus\tilde{N}$, where $\tilde{N} = KerS$. Indeed, let $\tilde{x}$ be an arbitrary element in $\tilde{X}$. Denote $z = S(\tilde{x})$ and $\tilde{x}_{\tilde{M}} = T(z)$. Then $S(\tilde{x}-\tilde{x}_{\tilde{M}}) = S(\tilde{x}) - S(\tilde{x}_{\tilde{M}}) = 0$. Therefore $\tilde{x}_{\tilde{N}} = \tilde{x}-\tilde{x}_{\tilde{M}}\in\tilde{N}$. Show that $\tilde{M}\cap\tilde{N} = \{0\}$. If $\tilde{x}\in\tilde{M}\cap\tilde{N}$, then $S(\tilde{x}) = 0$. Suppose that z from Z is such that $T(z) = \tilde{x}$. We get $z = (ST)(z) = S(\tilde{x}) = 0$. Consequently, $\tilde{x} = 0$ and $\tilde{X} = \tilde{M}\oplus\tilde{N}$. It remains to apply theorem 3.3.

Sufficiency: Let $\{\varphi_n\}_{n\in N}$ be a projective $b_{\tilde{X}}$-frame in Z. Then by theorem 3.3 $\tilde{N}$ is complemented in $\tilde{X}$, i.e. $\tilde{X} = \tilde{M}\oplus\tilde{N}$. Denote by S a synthesis operator, J is a natural isomorphism between $\tilde{X}$ and $\tilde{Z}$, $T\in L(Z,\tilde{X})$:

$T(z) = J^{-1}(z)$, $z \in Z$, and operator e_n ($\tilde{X} \to X$): $e_n(\tilde{x}) = x_n$, $\tilde{x} = \{x_n\}_{n\in N}$. Determine the operator φ_n^* : $Z \to X$ from the expression $\varphi_n^*(z) = e_n(T(z))$. So, $Tz = \{\varphi_n^*(z)\}_{n\in N}$. Take an arbitrary $z \in Z$. Let $\tilde{x} = T(z)$ ($\tilde{x} = \{x_n\}_{n\in N}$). Then we have

$$z = (ST)(z) = S(\tilde{x}) = \sum_{n=1}^{\infty} x_n \varphi_n = \sum_{n=1}^{\infty} \varphi_n^*(z)\ \varphi_n .$$

The theorem is proved.

Theorem: 4.3 *Let* $\{\varphi_n\}_{n\in N} \subset Y$ *be a* $b_{\tilde{X}}$ *-frame in* Z *and* $\tilde{N}$ *be a space of coefficients of zero-series of the system* $\{\varphi_n\}_{n\in N}$. *Then the following properties are equivalent:*

(i) there exists a continuous projector $P : \tilde{X} \to \tilde{N}$, *such that* $\|I - P\| = 1$;

(ii) it holds a linear expansion algorithm in $\{\varphi_n\}_{n\in N}$, *moreover if for* $z \in Z$, *there exists the sequence* $\{x_n\}_{n\in N}$ *from* $\tilde{X}$ *:* $z = \sum_{n=1}^{\infty} x_n \varphi_n$ *, then* $\left\|\{x_n\}_{n\in N}\right\|_{\tilde{X}} \ge \left\|\{\varphi_n^*(z)\}_{n\in N}\right\|_{\tilde{X}}$.

Proof: Let condition (i) be fulfilled. Then $\tilde{N}$ is complemented in $\tilde{X}$, i.e. $\tilde{X} = \tilde{M} \oplus \tilde{N}$, where $\tilde{M} = KerP$ (see [20], theorem 5.1). Consequently, $\{\varphi_n\}_{n\in N}$ is a projective $b_{\tilde{X}}$-frame in Z and by theorem 4.2 it holds a linear expansion algorithm. Take an arbitrary $z \in Z$. Suppose that there exists $\{x_n\}_{n\in N} \in \tilde{X}$, such that $z = \sum_{n=1}^{\infty} x_n \varphi_n$. Let S be a synthesis operator, J be a natural isomorphism between $\tilde{X}$ and $\tilde{Z}$, $Q = I - P$, T be a contraction of J^{-1} on Z. Since $\tilde{X} = \tilde{M} \oplus \tilde{N}$, then $\tilde{x} = \tilde{x}_{\tilde{M}} + \tilde{x}_{\tilde{N}}$. Let $u \in Z$: $\tilde{x}_{\tilde{M}} = T(u)$. We have

$$u = (ST)(u) = S(\tilde{x}_{\tilde{M}}) = S(\tilde{x}) = z .$$

Consequently, $\left\|\{\varphi_n^*(z)\}_{n\in N}\right\|_{\tilde{X}} = \|T(z)\|_{\tilde{X}} = \left\|\tilde{x}_{\tilde{M}}\right\|_{\tilde{X}} = \|Q(\tilde{x})\|_{\tilde{X}} \le \left\|\{x_n\}_{n\in N}\right\|_{\tilde{X}}$.

Let now condition (ii) be fulfilled. Then by theorem 4.2 the $b_{\tilde{X}}$-frame $\{\varphi_n\}_{n\in N}$ is projective and so by theorem 3.3 the subspace $\tilde{N}$ is complemented in $\tilde{X}$, i.e. $\tilde{X} = \tilde{M} \oplus \tilde{N}$. As above, let S be a synthesis operator, J be a natural isomorphism between $\tilde{X}$ and $\tilde{Z} = Z \oplus \tilde{X}$ and $T \in L(Z, \tilde{X})$ be an operator determined by the expression $T(z) = \{\varphi_n^*(z)\}_{n\in N}$.

Denote by Q the operator projecting $\tilde{X}$ on $\tilde{M}$ along $\tilde{N}$ and $P = I - Q$. Take an arbitrary $\tilde{x} \in \tilde{X}$. Obviously, there exists $\tilde{x}_{\tilde{N}} = \tilde{x} - Q(\tilde{x}) \in \tilde{N}$. Since $Q(\tilde{x}) \in \tilde{M}$, then there exists $z \in Z$ such that $T(z) = Q(\tilde{x})$, since $T(Z) = \tilde{M}$. We have

$$z = (ST)(z) = S(Q(\tilde{x})) = S(\tilde{x}) = \sum_{n=1}^{\infty} x_n \varphi_n .$$

According to the condition we get $\left\|\{x_n\}_{n\in N}\right\|_{\tilde{X}} \ge \left\|\{\varphi_n^*(z)\}_{n\in N}\right\|_{\tilde{X}}$. Consequently,

$$\left\|\{x_n\}_{n\in N}\right\|_{\tilde{X}} \ge \|T(z)\|_{\tilde{X}} = \|Q(\tilde{x})\|_{\tilde{X}} ,$$

therefore $\|Q\| = 1$. The theorem is proved.

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