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SOME FIXED POINT THEOREMS FOR δ-COMPATIBLE MAPPINGS IN METRIC SPACES

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ABSTRACT

Some common fixed point theorems for point- valued and set-valued compatible mappings in metric spaces are proved.

***Key words**: Fixed points, δ-compatible mappings.*

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1. INTRODUCTION:

In the last two decades research workers have generalized/extended Banach contraction theorem in various ways. A large number of research papers are available which have generalized the theorem of Banach involving more than one mapping. In these papers they have used the notion of commutativity, weak commutativity and compatibility. For this one can see the papers [3, 4, 5, 6, 8, 9]. It is shown by examples in [3] and [10] that commuting mappings are weakly commuting and weakly commuting mappings are compatible. The converse of these results is not true. Some authors have also made attempts to prove some fixed point theorems for set-valued mappings. Jungck and Rhoades have generalized some results of [1], [2] via compatibility for set-valued mappings.

In this paper we have proved some common fixed point theorems for point valued and set-valued compatible mappings which coincide with the results already proved for fixed point theorems for point-valued mappings.

2. KNOWN DEFINITIONS AND RESULTS:

Let (X,d) be a metric space and let $f,g:(X,d)\to(X,d)$. The mappings f and g are said to be weakly commutative [1] if and only if $d(fgx, gfx) \le d(fx, gx)$ *for* $x\in X$. The mappings f and g are said to be compatible [cf. 6] if and only if whenever $\{x_n\}$ is a sequence in X such that $fx_n, gxn\to t$ then $d(fgx_n, gfxn)\to 0$.

Let $B(X)$ denote the family of bounded subsets of X and let $\delta:B(X)\times B(X)\to[0,\infty)$ by $\delta(A,B)=sup\{d(a,b):a\in A,b\in B\}$. Fisher {[1], [2]} made several discussions on δ and obtained some of its properties. It is to be note that for any three $A,B,C\in B(X)$, $\delta(A,B)\le\delta(A,C)+\delta(C,B)$ and $\delta(A,B)=0$ iff A=B={a}. If $x\in X$, we write $\delta(x,A)$ *for* $\delta(\{x\},A)$.

Let $\{A_n\}$ be a sequence in $B(X)$, we say that $\{A_n\}\to A\subset X$, and we write $A_n\to A$ if and only if

(i) $a\in A$ implies that $a=lim a_n$ for some sequence $\{a_n\}$ in X where $a_n\in A_n$ for $n=1,2,3,\ldots$ and

(ii) for any $\epsilon>0$ there exists $m\in N$, N is a set of natural numbers, such that $A_n\subset A_\epsilon=\{x\in A:d(x, a)<\epsilon$ for some $a\in A$ for $n>m\}$.

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The following lemmas are collected from Fisher {[1], [2]} which we need to prove our theorem.

Lemma: 1 Suppose $\{A_n\}$ and $\{B_n\}$ are sequences in $B(X)$ and (X, d) is a complete metric space. If $A_n \to A \in B(X)$ and $B_n \to B \in B(X)$ then $\delta(A_n, Bn) \to \delta(A, B)$.

Lemma: 2 If $\{A_n\}$ is a sequence of non-empty bounded sets in the complete metric space (X, d) and if $\lim_{n\to\infty} \delta(A_n, \{y\}) = 0$ for some $y \in X$ then $A_n \to \{y\}$.

Let $F: X \to B(X)$. F is said to be continuous at $x \in X$ if $x_n \to x$ implies $Fx_n \to Fx$.
Jungck and Rhoades [6] formulated the notion of compatibility for point valued and set-valued mappings in the following way:

Let $I: X \to X$ and $F: X \to B(X)$. F and I are δ-compatible if and only if $IF(x) \in B(X)$ for $x \in X$ and $\delta(IFx, FIxn) \to 0$ whenever $\{x_n\}$ is a sequence in X such that $Ix_n \to t$ and $Fx_n \to \{t\}$ for some $t \in X$.

Further Jungck and Rhoades [6] proved the following proposition.

Proposition: 1 Let (X,d) be a complete metric space. Suppose I:X→X,F:X→B(X) and I and F are δ-compatible.
(i) Suppose the sequence $\{Fx_n\}$ converges to $\{z\}$ and $\{Ix_n\}$ converges to z. If I is continuous, then $FIx_n \to \{Iz\}$.
(ii) If $\{Iu\}=Fu$ for some $u \in X$, then $FIu=IFu$.

By $\cup FX$ we mean $\cup F(X)=\{y \in X: y \in Fx$ for some $x \in X\}$.

3. THEOREMS AND COROLLARIES:

Theorem: 1 *Let F,G be two set-valued mappings from a complete metric space X into B(X). Let I, J be mappings from X into itself satisfying the inequality*

$$\delta(Fx,Gy) \le \max\{\delta(Ix,Jy),\delta(Ix,Fx),\delta(Jy,Gy)\} - w[\max\{\delta(Ix,Jy),\delta(Ix,Fx),\delta(Jy,Gy)\}]$$

for all x,y∈X. Suppose the mappings F and I are δ-compatible and G and J are δ-compatible and $\cup GX \subset I(X)$ and $\cup FX \subset J(X)$. If one of the mappings I and J is continuous then F, G, I and J have a unique common fixed point z. Moreover Moreover Fz=Gz= {z} and w: [0, ∞) → [0, ∞) such that w is continuous and w(r)<r for r>0.

Proof: Let $x_0 \in X$ be arbitrary. Since J contains the range of F and I contains the range of G, we construct a sequence $\{x_n\}$ inductively in X such that

$$Jx_{2n+1} \in Y_{2n} = Fx_{2n}, Ix_{2n+2} \in Y_{2n+1} \in Gx_{2n+1},\ n=0,1,2,\dots .$$

Let

$$\delta_n = \delta(Fx_n, Gx_{n+1}) \text{ if } n=2m,\ m=0,1,2,\dots .$$

$$= \delta(Gx_n, Fx_{n+1}) \text{ if } n=2m+1.$$

Then

$$\delta_{2m} = \delta(Fx_{2m},Gx_{2m+1}) \le \max\{d(Ix_{2m},Jx_{2m+1}),\delta(Ix_{2m},Fx_{2m}),\delta(Jx_{2m+1},Gx_{2m+1})\} - w[\max\{d(Ix_{2m},Jx_{2m+1}),\delta(Ix_{2m},Fx_{2m}),\delta(Jx_{2m+1},Gx_{2m+1})]$$

i.e.

$$\delta_{2m} \le \max\{\delta(Gx_{2m-1}, Fx2m), \delta(Gx_{2m-1}, Fx2m), \delta(Fx_{2m}, Gx2m_{+1})\} - w[\max\{\delta(Gx_{2m-1},Fx_{2m}),\delta(Gx_{2m-1},Fx_{2m}),\delta(Fx_{2m},Gx_{2m+1})\}$$

$$= \max\{\delta_{2m-1},\delta_{2m+1},\delta_{2m}\} - w[\max\{\delta_{2m-1},\delta_{2m+1},\delta_{2m}\}].$$

If $\delta_{2m} > \delta_{2m-1}$, then $\delta_{2m} \le \delta_{2m} - w\delta_{2m}$, a contradiction. So, $\delta_{2m} \le \delta_{2m-1}$. Then $\delta_{2m} \le \delta_{2m-1} - w(\delta_{2m-1})$. So, we have

$$\delta_n \le \delta_{n-1} - w(\delta_{n-} \qquad (1)$$

i.e. $\quad w(\delta_{n-1}) \le \delta_{n-1} - \delta_n.$

So, $\quad \sum_{i=0}^{n} w(\delta_i) \le \delta_0 - \delta_1 \le \delta_0.$

So, the series of non-negative terms $\sum_{i=0}^{n} w(\delta_i)$ is convergent. Hence, $\lim_{n\to\infty} w(\delta_n) = 0$. From (1) for $n\geq 1$,

we get $\{\delta_n\}$ is a decreasing sequence of non-negative terms which implies that $\lim_{n\to\infty} w\delta_n = a$ for some $a\in R^+$. Since w is continuous $w(a)= \lim_{n\to\infty} w(\delta_n) = 0 =0$. This implies that $a=0$. Hence,

$$\lim_{n\to\infty} \delta_n = 0. \tag{2}$$

We next show that the sequence $\{Fx_0=Y_0,Gx_1=Y_1,Fx_2=Y_2,Gx_3=Y_3,\ldots\}$ is a Cauchy sequence. In order to show that $\{Y_n\}$ is a Cauchy sequence, it is sufficient to show that $\{Y_{2n}\}$ is a Cauchy sequence. For this, there exists a positive number ε such that for each even integer $2k$ there are even positive integer $2m(k)$ and $2n(k)$ such that $2m(k)>2n(k)>2k$ and $\delta(Y_{2m(k)},Y_{2n(k)})<\varepsilon$.

For each even integer $2k$, let $2m(k)$ be the least even integer exceeding $2n(k)$ satisfying the above inequality so that

$$\delta(Y_{2m(k)-2},Y_{2n(k)})\leq\epsilon \text{ and } \delta(Y_{2m(k)},Y_{2n(k)})>\epsilon \tag{3}$$

Then for each even integer $2k$,

$$\delta(Y_{2m(k)},Y_{2n(k)}) \leq \delta(Y_{2n(k)},Y_{2m(k)-2})+\delta(Y_{2m(k)-2},Y_{2m(k)-1})+\delta(Y_{2m(k)-1},Y_{2m(k)})$$

$$= \delta(Y_{2m(k)-2},Y_{2n(k)})+\delta_{2m(k)-2}+\delta_{2m(k)-1}.$$

From (2), (3) and the above inequality

$$\lim_{n\to\infty} \delta(Y_{2m(k)},Y_{2(k)}) = \epsilon \tag{4}$$

On using triangle inequality we get,

$$|\delta(Y_{2m(k)},Y_{2n(k)+1}) -\delta(Y_{2m(k)},Y_{2n(k)})| \leq \delta(Y_{2n(k)},Y_{2n(k)+1})=\delta_{2n(k)},$$

$$|\delta(Y_{2m(k)+1},Y_{2n(k)+1}) -\delta(Y_{2m(k)},Y_{2n(k)+1})| \leq \delta(Y_{2m(k)}, Y_{2n(k)+1})=\delta_{2m(k)},$$

$$|\delta(Y_{2m(k)+1},Y_{2n(k)+2}) -\delta(Y_{2m(k)+1},Y_{2n(k)+1})| \leq \delta(Y_{2n(k)+1}, Y_{2n(k)+2})=\delta_{2n(k)+1}.$$

From (2), (4) and the above inequality passing on limit as $k\to\infty$ we get,

$$\epsilon=\lim_{k\to\infty} \delta(Y_{2m(k)},Y_{2n(k)+1}) = \lim_{k\to\infty} \delta(Y_{2m(k)+1},Y_{2n(k)+1})= \lim_{k\to\infty} \delta(Y_{2m(k)+1},Y_{2n(k)+2}). \tag{A}$$

Now,

$$\delta(Y_{2m(k)+1},Y_{2n(k)+2}) = \delta(Gx_{2m(k)+1},Fx_{2n(k)+2})$$

$$\leq max\{ \delta(Ix_{2n(k)+2},Jx_{2m(k)+1}),\delta(Ix_{2n(k)+2}, Fx_{2n(k)+2}),\delta(Jx_{2m(k)+1},Gx_{2m(k)+1})\}$$
$$- w[max\{ \delta(Ix_{2n(k)+2},Jx_{2m(k)+1}),\delta(Ix_{2n(k)+2},Fx_{2n(k)+2}),\delta(Jx_{2m(k)+1},Gx_{2m(k)+1})\}]$$

$$\leq max\{ \delta(Gx_{2n(k)+1},Fx_{2m(k)}),\delta(Gx_{2n(k)+1}, Fx_{2n(k)+2}),\delta(Fx_{2m(k)},Gx_{2n(k)+1})\}$$
$$- W[max\{ \delta(Gx_{2n(k)+1},Fx_{2m(k)}),\delta(Gx_{2n(k)+1}, Fx_{2n(k)}),\delta(Fx_{2m(k)},Gx_{2n(k)+1})\}]$$

$$= max\{ \delta(Y_{2m(k)},Y_{2n(k)+1}),\delta(Y_{2n(k)+1}, Y_{2n(k)+2}),\delta(Y_{2m(k)},Y_{2m(k)+1})\}$$
$$- w[max\{ \delta(Y_{2m(k)},Y_{2n(k)+1}),\delta(Y_{2n(k)+1},Y_{2n(k)+2}),\delta(Y_{2m(k)+1},Y_{2m(k)+1})\}].$$

Using (A) and passing on limit as $k\to\infty$, we have

$\varepsilon\leq\varepsilon-w(\varepsilon)$ i.e. $w(\varepsilon)\leq 0$ a contradiction and so $\{Y_n\}$ is a Cauchy sequence.

If y_n is an arbitrary point in Y_n for $n=1,2,\ldots$, it follows that $d(y_m,y_n)\leq\delta(Y_m,Y_n)<\varepsilon$ for $m,n>N$. It follows that the sequence $\{Y_n\}$ is a Cauchy sequence in the complete metric space X, the point y_n being independent of the particular choice of Yn and $\{y_n\}$ is Cauchy and because X is a complete metric space let y_n converge to $y\in X$. In particular the sequence $\{Ix_{2n}\}$ and $\{Jx_{2n+1}\}$ will converge to y and further by Fisher [2] the sequence of sets $\{Fx_{2n}\}$ and $\{Gx_{2n+1}\}$ will converge to the set $\{y\}$.

We now suppose that the mapping I is continuous. Then it follows that the sequence $I^2x_{2n}\to Iy$. Since F and I are δ-compatible, by proposition 1 [7], $FIx_n\to \{Iy\}$.

Now,

$$\delta(FIx_{2n},Gx_{2n+1}) \le max\{\ \delta(I^2x_{2n},Jx_{2n+1}),\delta(I^2x_{2n},FIx_{2n}),\delta(Jx_{2n+1},Gx_{2n+1})\}$$
$$-w[max\{\ \delta(I^2x_{2n},Jx_{2n+1}),\delta(I^2x_{2n},FIx_{2n}),\delta(Jx_{2n+1},Gx_{2n+1})\}].$$

Letting n tend to infinity we have

$$d(\mathrm{Iy},y)\le max\{d(Iy,y),d(Iy,Iy),d(y,y)\} - w[max\{dI(Iy,y),d(Iy,Iy)\}]=d(Iy,y)-w(d(Iy,y)),$$

and it follows that $Iy=y$. Further,

$$\delta(fy,Gx_{2n+1})\ \le max\{d(Iy,Jx_{2n+1}),\delta(Iy,Fy),\delta(Jx_{2n+1},Gx_{2n+1})\} - w[max\{d(Iy,Jx_{2n+1}),\delta(Iy,Fy),\delta(Jx_{2n+1},Gx_{2n+1})\}].$$

Letting n tend to infinity we get

$$\delta(Fy,y)\ \le max\{d(y,y),\delta(y,Fy),\delta(y,y)\}\text{-}w[max\{d(y,y),\delta(y,Fy),\delta(y,y)\}].$$

i.e. $\qquad \delta(Fy,y)\ \le \delta(Fy,y) -w(\delta(Fy,y)),$

and so $Fy=\{y\}$. This means that y is in the range of F and since the range of J contains the range of F, there must exist a point y' in X such that $Jy=y$ and so $GJy=Gy$.

Now,

$$\begin{aligned}\delta(y,Gy')\ &= \delta(Fy,Gy')\le max\{d(Iy,\mathrm{J}y'),\delta(Iy,Fy),\delta(Jy',Gy')\} - w[max\{d(Iy,Jy'),\delta(Iy,Fy),\delta(Jy',Gy')\}]\\ &= max\{d(y,y),d(y,y),(y,Gy')\}-w[max\{d(y,y),d(y,y),\delta(y,Gy')\}]\\ &= \delta(y,Gy')-w(\delta(y,Gy')),\end{aligned}$$

and this gives $\delta(y,Gy')=0$, i.e. $Gy'=\{y\}$.

Next,

$$\begin{aligned}\delta(Jy,Gy)\ &\le \delta(Jy,GJy')\\ &\le \delta(Jy,JGy')+\delta(JGy',GJy')\\ &= \delta(Jy,\mathrm{J}Gy'),\ \text{since J and G are compatible}\\ &= d(Jy,Jy)=0.\end{aligned}$$

So $Gy=\{Jy\}$. Next,

$$\begin{aligned}\delta(y,Gy)\ &= \delta(Fy,\ Gy)\\ &\le max\{d(Iy,Jy),\delta(Iy,Fy),\delta(Jy,Gy)\} - w[max\{d(Iy,Jy),\delta(Iy,Fy),\delta(Jy,Gy)\}]\\ &= max\{\delta(y,Gy),d(y,y),d(Jy,Jy)\} - w[max\{d(Iy,Jy),\delta(Iy,Fy),\delta(Jy,Gy)\}]\\ &= \delta(y,Gy) -w\delta(y,Gy).\end{aligned}$$

So, $Gy=\{y\}$. This gives that $\{Jy\} =Gy=\{y\}$.

We have therefore proved that y is a common fixed point of F,G,I and J and that $Fy=Gy=\{y\}$. This result of course holds if J is continuous instead of I.

Now suppose that F, G, I and J have a second fixed point z. Then

$$\delta(Fy,\ Gz)\ \le max\{d(Iy,Jz),\delta(Iy,Fy),\delta(Jz,Gz)\}\text{-}w[max\{d(Iy,Jz),\delta(Iy,Fy),\delta(Jz,Gz)\}]$$

i.e. $d(y,\ z)\ = max\{d(y,z),d(y,y),d(z,z)\} -w[max\{d(y,z),d(y,y),d(z,z)\}]$

i.e. $d(y,\ z)\ \le d(y,z) -w(d(y,z)),$

and so y=z. This means that y is the unique common fixed of F,G,I and J.

This completes the proof of the theorem.

The following Corollary follows immediately.

Corollary: 1 *Let F and G be mappings of a complete metric space (X, d) into B(X) satisfying the inequality*

$$\delta(Fx, Gy) \leq \max\{d(x, y), \delta(x, Fx), \delta(y, Fy)\} - w[\max\{d(x, y), \delta(x, Fx), \delta(y, Fy)\}]$$

for all x, y∈X. Then F and G have unique common fixed point z. Further Fz=Gz= {z} and z is the unique common fixed point of F and G.

Theorem: 2 *Let f, g, I and J be mappings of a complete metric space (X, d) into itself satisfying the inequality*

$$d(fx, gy) \leq \max\{d(Ix, Jy), d(Ix, fx), d(Jy, gy)\} - w[\max\{d(Ix, Jy), d(Ix, fx), d(Jy, gy)\}]$$

for all x, y∈X. If the mappings f and I are compatible and the mappings g and J are compatible, $g(X) \subset I(X), f(X) \subset J(X)$ *and if one of the mappings I and J is continuous then, f, g ,I and J have a unique fixed point z.*

Proof: Define set-valued mappings F and G on X by $Fx = \{fx\}$ and $Gx = \{gx\}$ for all $x \in X$. If I (or J) is continuous then the conditions of Theorem 4 are satisfied and it follows that f, g, I and J have a unique common fixed point z.

Since commuting maps are compatible, we have the following Corollary.

Corollary: 2 Let f, g and h be mappings of a complete metric space into itself, h is continuous, $fh = hf$; $gh = hg$; $f(X) \cup g(X) \subset h(X)$. If f, g and h satisfy $d(fx, gy) \leq maxfd(hx, hy), d(hx, fx), d(hy, gy) - w[max\{d(hx, hy), d(hx, fx), d(hy, gy)]$ for all x, y in X, then f, g and h have a unique common fixed point in X.
Corollary 2 is the theorem of Liu [7].

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