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COMMON FIXED POINT THEOREM FOR FOUR WEAKLY CONTRACTIVE MAPPINGS IN T −ORBITALLY METRIC SPACE

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ABSTRACT

In this paper, we prove the existence of fixed point theorem of a generalized weakly constrictive map in $T-Orbitaly$ *complete metric space.*

Key word: *Metric space, T- Orbitally Metric space, Generalized Weakly Contractive Maps, Fixed Point, Weak Compatible Map.*

AMS Classification: *47H10, 54H25.*

INTRODUCTION:

In 1997, Alber and Guerre- Delabriere [3] introduced the concept of weakly contractive map in Hilbert space and proved the existence of fixed point results. Rhoades [2] extended this concept in Banach space and established the existence of fixed points. Throughout this paper (X, d) is a metric space which we denote simply by X.

we denote, $R^+ = [0, \infty)$ is positive real number, N the set of natural number and R the set of real number.
We write, $\Phi = \{ \psi : R^+ \to R^+ \}$ where ψ setisfies following conditions;
(i) ψ is continuous
(ii) ψ is non decreasing
(iii) $\psi(t) > 0$ for $t > 0$
(iv) $\psi(0) = 0$

Let binary operation $\diamond : R^+ \times R^+ \to R^+$ satisfies the following condition;
(a) $\diamond$ - is positive and commutative
(b) $\diamond$ - is continuous

Five typical examples are as follows, For each $a, b \in R^+$
(i) $a \diamond b = \max\{a, b\}$
(ii) $a \diamond b = a + b$
(iii) $a \diamond b = ab$
(iv) $a \diamond b = ab + a + b$
(v) $a \diamond b = \frac{1}{\max\{a,b,1\}}$

Definition: 1.1 The binary operation $\lozenge -$ is said to satisfy α- property, if there exists a positive real number $\alpha -$ such that, $a \lozenge b \le \max\{a, b\}$ for every $a, b \in R^+$

Definition: 1.2 For any $x_0 \in X$; $O(x_0) = \{ T^n x_0 ; n = 0,1,2,3 \ldots\ldots \}$ is said to the orbit of x_0 where, $T^0 = I$, is the identity map of X.

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A metric space X is said to be $T-$orbitally complete; if every Cauchy sequence Which is contained in $O(x)$ for all $x \in X$ converges to the point of X.

Here we note that every complete metric space is $T-$ orbitally complete for any T, but converges is not true.

Definition: 1.3 Let A and S be the mapping from a metric space X into itself, then the mapping is said to weakly compatible if they are commute at their coincidence points, that is, $Ax = Sx$ implies that $ASx = SAx$.

Definition: 1.4 A self map $T: X \to X$ is said to be generalized weakly contractive map if there exists a $\psi \in \Phi$ such that,
$d(Tx, Ty) \le d(x, y) - \psi.\big(d(x, y)\big)$ with $\lim_{t\to\infty} \psi(t) = \infty$ for all $x, y \in X$

MAIN RESULT:

Theorem: 2.1 Let (X, d) be a $T-$ orbitally complete metric space, satisfying $\alpha-$property, if A, B, S, T be the self mapping of X into itself such that;
(1) $A(X) \subseteq T(X)$ and $B(X) \subseteq S(X)$, $T(X)$ or $S(X)$are closed subset of X.
(2) The pair $(A, S)$and (B, T) are weakly compatible and generalized weakly contractive map.
(3) for all $x, y \in$ closer $O(x_0)$ we define,

$$d(Ax, By) \le M(Ax, By) - \psi(M(Ax, By))$$

Where, $M(Ax, By) = k \max\{ d(Sx, Ty), d(Ax, Sy), d(By, Ty), d(Sx, By), d(Ax, Ty)\}$

Then A, B, S, T have unique fixed point in closer$O(x_0)$.

Proof: We suppose that, $x_0 \in X$ arbitrary and we choose a poin t $x \in X$ such that,

$$y_0 = Ax_0 = Tx_1 \text{ and } y_1 = Bx_1 = Sx_2$$

In general there exists a sequence,

$$y_{2n} = Ax_{2n} = Tx_{2n+1} \text{ and } y_{2n+1} = Bx_{2n+1} = Sx_{2n+2} \text{ for } n = 1,2,3 \ldots\ldots\ldots$$

first we claim that the sequence $\{y_n\}$ is a Cauchy sequence for this from (3) we have,

$$d(y_{2n}, y_{2n+1}) \le M(Ax_{2n}, Bx_{2n+1}) - \psi(M(Ax_{2n}, Bx_{2n+1}))$$

$$d(y_{2n}, y_{2n+1}) < M(Ax_{2n}, Bx_{2n+1})$$

$$d(y_{2n}, y_{2n+1}) < k \max\left\{\begin{matrix} d(Sx_{2n}, Tx_{2n+1}),\ d(Ax_{2n}, Sx_{2n}), \\ d(Bx_{2n+1}, Tx_{2n+1}),\ d(Sx_{2n}, Bx_{2n+1}), \\ d(Ax_{2n}, Tx_{2n+1}) \end{matrix}\right\}$$

$$d(y_{2n}, y_{2n+1}) < k \max\left\{\begin{matrix} d(y_{2n-1}, y_{2n}),\ d(y_{2n-1}, y_{2n+1}), \\ d(y_{2n+1}, y_{2n}),\ d(y_{2n-1}, y_{2n+1}), \\ d(y_{2n}, y_{2n}) \end{matrix}\right\}$$

$$d(y_{2n}, y_{2n+1}) < k \max\left\{\begin{matrix} d(y_{2n-1}, y_{2n}), d(y_{2n+1}, y_{2n}), \\ d(y_{2n-1}, y_{2n+1}) \end{matrix}\right\}$$

$$d(y_{2n}, y_{2n+1}) < k \max\{ d(y_{2n-1}, y_{2n}),\ d(y_{2n+1}, y_{2n})\}$$

We assume that, $d_n = d(y_n, y_{n+1})$ then $d_{2n} < k \max\{d_{2n-1}, d_{2n}\}$

Now if, $d_{2n} \ge d_{2n-1}$ then $d_{2n} \le k\, d_{2n} \le d_{2n}$

which contradiction our hypothesis i. e.

$$d_{2n} \le d_{2n-1}$$

Similarly we write, $d_{2n+1} < d_{2n}$

By follow above, we can show that

$d(y_n, y_{n+1}) \leq k^n \; d(y_0, y_1)$

$\lim_{n\to\infty} d(y_n, y_{n+1}) \to 0$

Our next step is to check $\{y_n\}$ is Cauchy sequence , for this if $m \geq n$ then

$$d(y_n, y_m) \leq \begin{Bmatrix} d(y_n, y_{n+1}) \diamond d(y_{n+1}, y_{n+2}) \diamond \\ d(y_{n+2}, y_{n+3}) \diamond \dots\dots.\, d(y_{m-1}, y_m) \end{Bmatrix}$$

$$d(y_n, y_m) \leq k \max \begin{Bmatrix} d(y_n, y_{n+1}),\; d(y_{n+1}, y_{n+2}), \\ d(y_{n+2}, y_{n+3}), \dots\dots.\, d(y_{m-1}, y_m) \end{Bmatrix}$$

$$\lim_{n\to\infty} d(y_n, y_m) \leq k \lim_{n\to\infty} \begin{Bmatrix} d(y_n, y_{n+1}) \; d(y_{n+1}, y_{n+2}) \\ d(y_{n+2}, y_{n+3}) \; \dots\dots.\, d(y_{m-1}, y_m) \end{Bmatrix}$$

$\lim_{n\to\infty} d(y_n, y_m) \to o$, as $n \to \infty$

It follows that $\{y_n\}$ is a Cauchy sequence and by the completeness of X, $\{y_n\}$ converges to $y \in X$. That is we can write;

$\lim_{n\to\infty} y_n = \lim_{n\to\infty} Ax_{2n} = \lim_{n\to\infty} Tx_{2n+1} = \lim_{n\to\infty} Bx_{2n+1} = \lim_{n\to\infty} Sx_{2n+2} = y$

Now let T(X) is closed subset of X, such that, $Tv = y$

We prove that $Bv = y$ for this again from (3),

$$d(Ax_{2n}, Bv) < k \max \begin{Bmatrix} d(Sx_{2n}, Tv)\,, d(Ax_{2n}, Sx_{2n}), \\ d(Bv, Tv),\; d(Sx_{2n}, Bv), \\ d(Ax_{2n}, Tv) \end{Bmatrix}$$

$$\lim_{n\to\infty} d(Ax_{2n}, Bv) < k \lim_{n\to\infty} \begin{Bmatrix} d(Sx_{2n}, Tv), d(Ax_{2n}, Sx_{2n}), \\ d(Bv, Tv), d(Sx_{2n}, Bv), \\ d(Ax_{2n}, Tv) \end{Bmatrix}$$

$d(y, Bv) < k \; \{d(y, Tv), d(y, y), d(Bv, y), d(y, Bv), 0\}$

$d(y, Bv) < k\,.\, d(y, Bv)$

Which contradiction, $d(y, Bv) > 0$

Hence $Bv = y = Tv$ and that $BTv = TBv$ implies that $By = Ty$.

Now we proof that $By = y$ for this again from (3)

$$d(Ax_{2n}, By) \leq k \max \begin{Bmatrix} d(Sx_{2n}, Ty), d(Ax_{2n}, Bx_{2n}), \\ d(By, Ty), d(Sx_{2n}, By), \\ d(Ax_{2n}, Ty) \end{Bmatrix}$$

$\lim_{n\to\infty} d(Ax_{2n}, By) \leq k\, d(y, By)$

$By = y = Ty$

Since $B(X) \subseteq S(X)$

For, $w \in X$ such that $Sw = y$

Now we show that $Aw = y$

$$d(Aw, By) \le k \max \begin{Bmatrix} d(Sw, Ty), d(Aw, Sw), d(By, Ty) \\ , d(Sw, By), d(Aw, Ty) \end{Bmatrix}$$

It follows that,

$$d(Aw, y) \le k \max \ \{d(y, y), d(Aw, y), d(y, y), d(y, y), d(Aw, y)\}$$

$$d(Aw, y) \le kd(Aw, y)$$

Which contradiction, $d(Aw, y) > 0$ thus $Aw = y = Sw$

Since A and S are weakly compatible, so that $ASw = SAw$ this implies, $Ay = Sy$.

Now we show that, $Ay = y$ for this again from (3),

$$d(Ay, By) \le k \max \begin{Bmatrix} d(Sy, Ty), d(Ay, Sy), d(By, Ty), \\ d(Sy, By), d(Ay, Ty) \end{Bmatrix}$$

It follows that,

$$d(Ay, y) \le k \max \begin{Bmatrix} d(Sy, y), d(Ay, Sy), d(y, y), \\ d(Sy, y), d(Ay, y) \end{Bmatrix}$$

$$d(Ay, y) \le k \max \{d(Sy, y), d(Ay, y)\}$$

$$d(Ay, y) \le k\, d(Ay, y)$$

Which contradiction, $d(Ay, y) > 0$, thus $Ay = y$ and then, we write

$$Ay = Sy = By = Ty = y$$

y is common fixed point of A, B, S, T

If $S(X)$ is closed subset of X then we follows similarly proof.

UNIQUENESS:

We suppose that x, is another fixed point for A, B, S, T then,

$$d(x, y) = d(Ax, By)$$

$$d(Ax, By) \le k \max\{d(Sx, Ty), d(Ax, Sx), dBy, Ty), d(Sx, By), d(Ax, Ty)\}$$

And so

$$d(x, y) \le k \max \ \{d(x, y), d(x, x), d(y, y), d(x, y), d(x, y)\}$$

$$d(x, y) \le k.\, d(x, y)$$

Which contradiction. so that $x = y$

y is unique fixed point of A, B, S, T.

This complete the prove of the theorem.

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