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P_7 - Factorization of complete bipartite multigraphs

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ABSTRACT

P_{2p}-factorization of a complete bipartite graph for p an integer was studied by Wang [1]. Further, Beiliang [2] extended the work of Wang [1], and studied the P_{2k} -factorization of complete bipartite multigraphs. For even value of k in P_k-factorization, the spectrum problem is completely solved [1, 2, 3]. However for odd value of k i.e. P_3, P_5, P_7 and P_9, the path factorization have been studied by a number of researchers [4, 5, 6, 7]. Again, P_3-factorizations of complete bipartite multigraphs and symmetric complete bipartite multi-digraphs were studied by Wang and Beiliang [8]. In the present paper, we study P_7 - factorization of complete bipartite multigraphs and show that the necessary and sufficient conditions for the existence of P_7 - factorization of complete bipartite multigraph are:

(1) $4n \geq 3m$,
(2) $4m \geq 3n$,
(3) $m + n \equiv 0 (\text{mod } 7)$,
(4) $7\lambda mn/[6(m+n)]$ *is an integer.*

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1. INTRODUCTION:

Let $K_{m,n}$ be the complete bipartite graph with two partite set having m and n vertices. The graph $\lambda K_{m,n}$ is disjoint union of λ graphs, each isomorphic to $K_{m,n}$. A subgraph F of $\lambda K_{m,n}$ is called a spanning sub graph of $\lambda K_{m,n}$ if F contains all vertices of $\lambda K_{m,n}$. For positive integer K, a path on K-vertices is denoted by P_k. A P_k-factor of $\lambda K_{m,n}$ is a spanning subgraph F of $\lambda K_{m,n}$ such that every component of F is a P_k, and every pair of P_k has no vertex in common. A P_k-factorization of $\lambda K_{m,n}$ is a set of edge-disjoint P_k-factors of $\lambda K_{m,n}$ which is a partition of the set of edges of $\lambda K_{m,n}$. The multigraph $\lambda K_{m,n}$ is called P_k-factorable whenever it has a P_k-factorization.

In this paper we are discussing the necessary and sufficient conditions for the existence of a P_7-factorization of complete bipartite multigraph $\lambda K_{m,n}$. Let P_7 be the path on seven vertices and $\lambda K_{m,n}$ is $K_{m,n}$ in which every edge is taken λ times. A spanning subgraph F of $\lambda K_{m,n}$ is called a P_7-factor if each component of F is isomorphic to P_7. If $\lambda K_{m,n}$ is expressed as an edge disjoint sum of P_7-factor, then this sum is called a P_7-factorization of $\lambda K_{m,n}$.

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2. MAIN RESULT:

The necessary and sufficient conditions for the existence of a P_7 − factorization of complete bipartite multigraph $\lambda K_{m,n}$ are given in theorem 2.1, below.

Theorem: 2.1 $\lambda K_{m,n}$ has a P_7 − factorization if and only if

(1) $4n \geq 3m,$

(2) $4m \geq 3n,$

(3) $m+n \equiv 0 (\bmod 7),$

(4) $7\lambda mn/[6(m+n)]$ is an integer.

Proof: Let $\lambda K_{m,n}$ is factorized into r number of P_7 − factors, and t be the number of components of each P_7 − factor. Then $t = \frac{(m+n)}{7}$ and $r = \frac{7\lambda mn}{[6(m+n)]}$.

Hence conditions (3) and (4) are necessary.

Among these t components, let x and y be the number of components whose end points are in Y and X, respectively. Then we has $3x+4y=m$ and $4x+3y=n.$ Hence $x = \frac{(4m-3n)}{7}$ and $y = \frac{(4n-3m)}{7}$. From $0 \leq x \leq m$ and $0 \leq y \leq n,$ we have $3n \leq 4m$ and $3m \leq 4n.$ Conditions (1) and (2) are, therefore necessary.

Now we prove the following existence theorem, which is used later in this paper

Theorem: 2.2 If $\lambda K_{m,n}$ has a P_7 − factorization, then $\lambda K_{sm,sn}$ has a P_7 − factorization for every positive integer s.

Proof: Let $K_{s,s}$ is 1-factorable [9], and {H_1, H_2 … H_s} be a 1-factorization of it. For each i with 1 ≤ i ≤ s, replace every edge of H_i with a $\lambda K_{m,n}$ to get a spanning sub graph G_i of $\lambda K_{sm,sn}$ such that the G_i's {1 ≤ i ≤ s } are pair wise edge disjoint and there sum is $\lambda K_{sm,sn}$. Since $\lambda K_{m,n}$ is P_7 − factorable, therefore G_i is also P_7 − factorable, and hence, $\lambda K_{sm,sn}$ is also P_7 − factorable.

Theorem: 2.3 If $\lambda K_{m,n}$ has a P_7 − factorization, then $s\lambda K_{m,n}$ has a P_7 − factorization for every positive integer s.

Proof: Construct a P_7 − factorization of $\lambda K_{m,n}$ repeatedly s number of times. Then we have a P_7 − factorization of $s\lambda K_{m,n}$.

Now we will prove theorem 2.1. There are three cases to consider,

Case: 1 ($(4m = 3n)$: In this case, from theorem 2.2 and theorem 2.3 $\lambda K_{3n,4n}$ has a P_7 − factorization.

Case: 2 $(4n = 3m)$: obviously, $\lambda K_{3m,4m}$ has a P_7 − factorization.

Case: 3 $(4m > 3n$ and $4n > 3m)$: In this case, let $a = \frac{(4n-3m)}{7}$, $b = \frac{(4m-3n)}{7}$, $t = \frac{m+n}{7}$, and $r = \frac{7\lambda mn}{[6(m+n)]}$. Then from conditions (1)-(4) in theorem 2.1, a, b, t and r are integers, and 0 < a < m and 0 < b < n. We have $3a+4b=m$ and $4a+3b=n$. Hence $r = 2\lambda(a+b) + \frac{\lambda ab}{6(a+b)}$. Let $z = \frac{\lambda ab}{6(a+b)}$, which is a positive integer.

And let gcd (3 a , 4b) = d, 3 a = d p , 4b = dq, where gcd (p, q) = 1. Then dq is even and $z = \frac{\lambda dpq}{[6(p+q)]}$.

These equalities imply the following equalities.

$$d=\frac{6(4p+3q)z}{\lambda pq},$$
$$m=\frac{6(p+q)(4p+3q)z}{\lambda pq},$$
$$n=\frac{(16p+9q)(4p+3q)z}{2\lambda pq},$$
$$r=\frac{(p+q)(16p+9q)z}{pq},$$
$$a=\frac{2p(4p+3q)z}{\lambda pq},$$
$$b=\frac{3q(4p+3q)z}{2\lambda pq}.$$

Here,

t = the number of copies of P_7 in any factor,

r = the number of P_7-factor in the factorization,

a = the number of copies of P_7 with its endpoints in Y in a particular P_7- factor (type M),

b = the number of copies of P_7 with its endpoints in X in a particular P_7- factor (type W),

c = the total number of copies of P_7 in the whole factorization.

The following lemma can be verified.

Lemma: 2.1 Let a , b, p and q be positive integers, if gcd (p, q) = 1 then gcd $(p+q, pq)=1$ and if gcd $(ap,bq)=1,$ then gcd $(ap+bq, pq)=1.$

By using p , q, and d, the parameters m and n , satisfying conditions (1)-(4) in theorem 2.1 can be expressed as follows:

Lemma: 2.2

(1) If gcd $(p,9)=1$ and gcd(q, 16) = 1 then there exist a positive integer s such that

$$m=12(p+q)(4p+3q)s/\lambda,\ n=(16p+9q)(4p+3q)s/\lambda,$$
$$a=4p(4p+3q)s/\lambda,\ b=3q(4p+3q)s/\lambda,$$
$$r=2(p+q)(16p+9q)s.$$

(2) If gcd $(p,9)=$ 1 and gcd(q, 16) = 2, Let $q=2q_1$. Then there exist a positive integer s such that

$$m=6(p+2q_1)(2p+3q_1)s/\lambda,\ n=(8p+9q_1)(2p+3q_1)s/\lambda,$$
$$a=2p(2p+3q_1)s/\lambda,\ b=3q_1(2p+3q_1)s/\lambda,$$
$$r=(p+2q_1)(8p+9q_1)s.$$

(3) If gcd $(p,9)=1$ and gcd(q, 16) = 4, Let $q=4q_2$. Then there exist a positive integer s such that

$$m=6(p+4q_2)(p+3q_2)s/\lambda,\ n=2(4p+9q_2)(p+3q_2)s/\lambda,$$
$$a=2p(p+3q_2)s/\lambda,\ b=6q_2(p+3q_2)s/\lambda,$$
$$r=(p+4q_2)(4p+9q_2)s.$$

(4) If $\gcd(p,9)=1$ and gcd(q, 16) = 8, let $q=8q_3$. Then there exist a positive integer s such that

$$m=3(p+8q_3)(p+6q_3)s/\lambda,\ n=2(2p+9q_3)(p+6q_3)s/\lambda,$$
$$a=p(p+6q_3)s/\lambda,\ b=6q_3(p+3q_3)s/\lambda,$$
$$r=(p+8q_3)(2p+9q_3)s.$$

(5) If $\gcd(p,9)=1$ and gcd(q, 16) = 8, let $q=16q_4$. Then there exist a positive integer s such that

$$m=3(p+16q_4)(p+12q_4)s/\lambda,\ n=4(p+9q_4)(p+12q_4)s/\lambda,$$
$$a=p(p+12q_4)s/\lambda,\ b=12q_4(p+12q_4)s/\lambda,$$
$$r=2(p+16q_4)(p+9q_4)s.$$

(6) If $\gcd(p,9)=3$ and gcd(q, 16) = 1, let $p=3p_1$. Then there exist a positive integer s such that

$$m=12(3p_1+q)(4p_1+q)s/\lambda,\ n=3(16p_1+3q)(4p_1+q)s/\lambda,$$
$$a=12p_1(4p_1+q)s/\lambda,\quad b=3q(4p_1+q)s/\lambda,$$
$$r=2(3p_1+q)(16p_1+3q)s.$$

(7) If $\gcd(p,9)=3$ and gcd(q, 16) = 2, let $q=2q_1$ and $p=3p_1$. Then there exist a positive integer s such that

$$m=6(3p_1+2q_1)(2p_1+q_1)s/\lambda,\quad n=3(8p_1+3q_1)(2p_1+q_1)s/\lambda,$$
$$a=6p_1(2p_1+q_1)s/\lambda,\ b=3q_1(2p_1+q_1)s/\lambda,$$
$$r=(3p_1+2q_1)(8p_1+3q_1)s.$$

(8) If $\gcd(p,9)=3$ and gcd(q, 16) = 4, let $q=4q_2$ and $p=3p_1$. Then there exist a positive integer s such that

$$m=6(3p_1+4q_2)(p_1+q_2)s/\lambda,\quad n=6(4p_1+3q_2)(p_1+q_2)s/\lambda,$$
$$a=6p_1(p_1+q_2)s/\lambda,\quad b=6q_2(p_1+q_2)s/\lambda,$$
$$r=(3p_1+4q_2)(4p_1+3q_2)s.$$

(9) If $\gcd(p,9)=3$ and gcd(q, 16) = 4, let $q=8q_3$ and $p=3p_1$. Then there exist a positive integer s such that

$$m=3(3p_1+8q_3)(p_1+2q_3)s/\lambda,\quad n=6(2p_1+3q_3)(p_1+2q_3)s/\lambda,$$
$$a=3p_1(p_1+2q_3)s/\lambda,\quad b=6q_3(p_1+2q_3)s/\lambda,$$
$$r=(3p_1+8q_3)(2p_1+3q_3)s.$$

(10) If $\gcd(p,9)=3$ and gcd (q, 16) = 16, let $q=16q_4$ and $p=3p_1$. Then there exist a positive Integer s such that

$$m=3(3p_1+16q_4)(p_1+4q_4)s/\lambda,\quad n=12(p_1+3q_4)(p_1+4q_4)s/\lambda,$$
$$a=3p_1(p_1+4q_4)s/\lambda,\ b=12q_4(p_1+4q_4)s/\lambda,$$
$$r=2(3p_1+16q_4)(p_1+3q_4)s.$$

(11) If $\gcd(p,9)=9$ and gcd (q, 16) = 1. Let $p = 9p_2$. Then there exist a positive integer s such that

$$m=4(9p_2+q)(12p_2+q)s/\lambda,\ n=3(16p_2+q)(12p_2+q)s/\lambda,$$
$$a=12p_2(12p_2+q)s/\lambda,\ b=q(12p_2+q)s/\lambda,$$

$$r=2(9p_2+q)(16p_2+q)s.$$

(12) If $\gcd(p,9)=9$ and gcd (q, 16) = 2. Let $p = 9p_2$ and $q=2q_1$. Then there exist a positive integer s such that

$$m=2(9p_2+2q_1)(6p_2+q_1)s/\lambda,\ n=3(8p_2+q_1)(6p_2+q_1)s/\lambda,$$

$a = 6p_2(6p_2 + q_1)s/\lambda,\ b = q_1(6p_2 + q_1)s/\lambda,$
$r = (9p_2 + 2q_1)(8p_2 + 3q_1)s.$

(13) If gcd $(p, 9) = 9$ and gcd (q, 16) = 4. Let p = 9p$_2$ and $q = 4q_2$. Then there exist a positive integer s such that
$m = 2(9p_2 + 4q_2)(3p_2 + q_2)s/\lambda,\ n = 6(4p_2 + q_2)(3p_2 + q_2)s/\lambda,$
$a = 6p_2(3p_2 + q_2)s/\lambda,\ b = 2q_2(3p_2 + q_2)s/\lambda,$
$r = (9p_2 + 4q_2)(4p_2 + q_2)s.$

(14) If gcd $(p, 9) = 9$ and gcd (q, 16) = 8. Let p = 9p$_2$ and $q = 8q_3$. Then there exist a positive integer s such that
$m = (9p_2 + 8q_3)(3p_2 + 2q_3)s/\lambda,\ n = 6(2p_2 + q_3)(3p_2 + 2q_3)s/\lambda,$
$a = 3p_2(3p_2 + 2q_3)s/\lambda,\ b = 2q_3(3p_2 + 2q_3)s/\lambda,$
$r = (9p_2 + 8q_3)(2p_2 + q_3)s.$

(15) If gcd $(p, 9) = 9$ and gcd (q, 16) = 16. Let p = 9p$_2$ and $q = 16q_4$. Then there exist a positive integer s such that
$m = (9p_2 + 16q_4)(3p_2 + 4q_4)s/\lambda,\ n = 12(p_2 + q_4)(3p_2 + 4q_4)s/\lambda,$
$a = 3p_2(3p_2 + 4q_4)s/\lambda,\ b = 4p_4(3p_2 + 4q_4)s/\lambda,$
$r = 2(9p_2 + 16q_4)(p_2 + q_4)s.$

Proof: We assume that gcd(p, q) = 1, gcd(p, 9) = 1 and gcd(q, 16) = 1 hold.

Then gcd(16p + 9q, 2) = gcd(4p + 3q, 2) = 1 and gcd(16p, 9q) = gcd(4p, 3q) = 1 hold. From lemma 2.2, we get $n = \dfrac{(16p + 9q)(4p + 3q)z}{2\lambda pq}$, which is an integer.

Therefore $z/2pq$ must be an integer.

Let $s = z/2pq$, then the equalities in (1) hold. Similarly we can prove the other equalities of lemma 2.2.

Below in lemma 2.3 and 2.4, we are giving the direct constructions of graphs for particular values of m and n given in Case 1 and 8 of lemma 2.2. The value of s is taken as 1.

Lemma: 2.3 For any positive integer p and q, let $m = 6(p + 2q)(2p + 3q)/\lambda$ and $n = (8p + 9q)(2p + 3q)/\lambda.$ Then $\lambda K_{m,n}$ has a P_7-factorization.

Proof: Let $a = 2p(2p + 3q)/\lambda$, $b = 3q(2p + 3q)/\lambda$, $r = (p + 2q)(8p + 9q)$ and $r_1 = (p + 2q)$, $r_2 = (8p + 9q)$. Let X and Y be two partite set of $\lambda K_{m,n}$ and set

$$X = \{x_{i,j} : 1 \le i \le r_1, 1 \le j \le m_0\},$$
$$Y = \{y_{i,j} : 1 \le i \le r_2, 1 \le j \le n_0\},$$

where $m_0 = m/r_1 = 6(2p + 3q)/\lambda$ and $n_0 = n/r_2 = (2p + 3q)/\lambda$.

We will construct a P_7-factorization of $\lambda K_{m,n}$. In P_7-factor of $\lambda K_{m,n}$, we have $t = (m + n)/7 = (2p + 3q)^2/\lambda$ number of vertex disjoint copies, where $a = 2p(2p + 3q)/\lambda$, will be of type M and $b = 3q(2p + 3q)/\lambda$ type W. Here type M denotes P_7-factor with its ends point in Y, and type W with its end point in X. For each $1 \le i \le p,$ let

$$E_i = \left\{ x_{i,\, j+(2p+3q)(u-1)/\lambda+3(2p+3q)v/\lambda}\ y_{8(i-1)+4v+u+w,\, j+2i-1/\lambda+w/\lambda} : 1 \le j \le 2p + 3q/\lambda, 1 \le u \le 3, 0 \le v \le 1, 0 \le w \le 1 \ \right\}$$

and for each $1 \le i \le q,$

let

$$E_{p+i} = \Big\{ x_{p+2(i-1)+w+t,\, j+((2p+3q)/\lambda)(v-1)+3((2p+3q)/\lambda)w+((2p+3q)/\lambda)t} \; y_{8p+9(i-1)+3(v-1)+u,\, j+(2p+3(i-1)+u)/\lambda}$$
$$: 1 \le j \le (2p+3q)/\lambda, 0 \le u \le 2, 1 \le v \le 3, 0 \le w \le 1, 0 \le t \le 1 \;\Big\}.$$

Let $F = \bigcup_{1 \le i \le p+q} E_i$, then the graph F is a P_7-factor of $\lambda K_{m,n}$. Define a bijection σ from $X \cup Y$ onto $X \cup Y$, i.e. σ : $X \cup Y \xrightarrow[onto]{} X \cup Y$ such that $\sigma(x_{i,j}) = x_{i+1,j}$ and $\sigma(y_{i,j}) = y_{i+1,j}$ where $i \in (1,2\ldots r_1)$ and $j \in (1,2\ldots r_2)$. Let $F_{\xi,\eta}$ = $\{\sigma^{\xi}(x)\, \sigma^{\eta}(y)$: $x \in X$, $y \in Y$, $xy \in F\}$.

Therefore, the graphs, $F_{\xi,\eta}$ $\{1 \le \xi \le r_1, 1 \le \eta \le r_2\}$, are edge disjoint P_7 - factor of $\lambda K_{m,n}$ and its union is also $\lambda K_{m,n}$. Thus { $F_{\xi,\eta}$; $1 \le \xi \le r_1$, $1 \le \eta \le r_2$} is a P_7 - factorization of $\lambda K_{m,n}$.

Lemma: 2.4 For any positive integer p and q, let $m = 6(3p+4q)(p+q)/\lambda$ and $n = 6(4p+3q)(p+q)/\lambda$. Then $\lambda K_{m,n}$ has a $P_7 -$ factorization.

Proof Let $a = 6p(p+q)/\lambda, b = 6q(p+q)/\lambda$, $r = (3p+4q)(4p+3q)$ and $r_1 = (3p+4q)$, $r_2 = (4p+3q)$. Let X and Y be two partite set of $\lambda K_{m,n}$ and set

$$X = \{x_{i,j} : 1 \le i \le r_1, 1 \le j \le m_0\},$$
$$Y = \{y_{i,j} : 1 \le i \le r_2, 1 \le j \le n_0\}.$$

Where $m_0 = m/r_1 = 6(p+q)/\lambda$ and $n_0 = n/r_2 = 6(p+q)/\lambda$.

We will construct a $P_7 -$ factorization of $\lambda K_{m,n}$. In P_7-factor of $\lambda K_{m,n}$, we have $t = (m+n)/7 = 6(p+q)^2/\lambda$ number of vertex disjoint copies, where $a = 6p(p+q)/\lambda$, will be of type M and $b = 6q(p+q)/\lambda$, type W. Here type M denotes P_7-factor with its ends point in Y, and type W with its end point in X.

For each $1 \le i \le p$, let

$$E_i = \big\{ x_{3(i-1)+u,j}\, y_{4(i-1)+u+v,\, j+6(i-1)/\lambda+2(u-1)/\lambda+v/\lambda} : 1 \le j \le 6(p+q)/\lambda, 1 \le u \le 3, 0 \le v \le 1 \;\big\}.$$

And for each $1 \le i \le q$, let

$$E_{p+i} = \big\{ x_{3p+4(i-1)+u+v,\, j}\; y_{4p+3(i-1)+u,\, 6p/\lambda+j+6(i-1)/\lambda+2(u-1)/\lambda+v} : 1 \le j \le 6(p+q)/\lambda, 1 \le u \le 3, 0 \le v \le 1 \;\big\}.$$

Let $F = \bigcup_{1 \le i \le p+q} E_i$, then the graph F is a P_7-factor of $\lambda K_{m,n}$. Define a bijection σ from $X \cup Y$ onto $X \cup Y$,

i.e. σ : $X \cup Y \xrightarrow[onto]{} X \cup Y$ such that $\sigma(x_{i,j}) = x_{i+1,j}$ and $\sigma(y_{i,j}) = y_{i+1,j}$ where $i \in (1,2\ldots r_1)$ and $j \in (1,2\ldots r_2)$. Let $F_{\xi,\eta}$ = $\{\sigma^{\xi}(x)\, \sigma^{\eta}(y)$: $x \in X$, $y \in Y$, $xy \in F\}$.

Therefore, the graphs $F_{\xi,\eta}$ $\{1 \le \xi \le r_1, 1 \le \eta \le r_2\}$, are edge disjoint P_7-factor of $\lambda K_{m,n}$ and its union is also $\lambda K_{m,n}$.

Thus { $F_{\xi,\eta}$; $1 \le \xi \le r_1, 1 \le \eta \le r_2$ } is a P_7 - factorization of $\lambda K_{m,n}$.

Proof (Theorem (2.1)): By using theorem 2.2 and theorem 2.3 with lemma 2.2 to 2.4, it can be seen that when the parameters *m* and *n* satisfy the conditions (1) – (4) in theorem 2.1, the complete bipartite multigraph $\lambda K_{m,n}$ has P_7-factorization. This completes the proof of theorem 2.1.

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