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FICTITIOUS DOMAIN APPROACH AND LEVEL-SETS METHOD FOR THE STOKES PROBLEM

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ABSTRACT

In this paper, we focus on shape optimization related to the Stokes system. We recall to general framework of classical optimization to compute the shape and topological derivative of a given cost functional. So we combine fictitious domain approach and the two derivatives to propose a numerical scheme (based on level set method) to study the Stokes problem. To end the paper, we give some numerical results for d=2.

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***Keywords and phrases**: Stokes problem, Shape derivative, topological derivative, fictitious domain approach, Xfem method, numerical result.*

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1. INTRODUCTION:

Shape optimization problem is a minimization problem where the unknown variable run over a class of domains; then every shape optimization problem can be written in the form

$$\min\{j(A); A \in \mathcal{A}\}$$

Where $\mathcal{A}$ is a class of admissible domains and j the cost functional. In classical shape optimization, it is the boundary of the initial domain (or a part of the boundary) which moves for reaching the optimal shape. Thus the optimal shape has the same topology as the initial one (for example, if the initial domain is simply connected, the optimal one will be also connected). Unlike the case of classical shape optimization, the topology of the design may change during the optimization process, as for example the inclusion of holes. The physical interpretation of holes depends to the nature of the design. In the case of structural shape optimization, the insertion of a hole means simply removing some material. In the case of fluid dynamics, creating a hole means inserting a small obstacle. The objective is to find an optimal shape without any a priori assumption about the topology.

The goal of this paper is to propose a numerical scheme based on a fictitious domain approach for a Stokes system using level sets method coupling the shape and topological optimization. Topological sensitivity analysis aims at providing an asymptotic expansion of a shape functional acting on the neighborhood of a small hole created inside the domain. The underlying principle is the following see also [5]: For a criterion$j(\Omega) = \mathrm{j}_\Omega(\mathrm{u}_\Omega)$, $\Omega \in \mathbb{R}^d$, and u_Ω is the solution of a boundary value problem defined over Ω. The asymptotic expansion of the cost function $\mathrm{j}(\Omega)$ can be generally written in the form:

$$J(\Omega\backslash\overline{x_0 + \varepsilon\omega}) - J(\Omega) = \rho(\varepsilon)g(x_0) + o\big(\rho(\varepsilon)\big)$$

$$\lim_{\varepsilon\to 0}\rho(\varepsilon) = 0\,, \quad \rho(\varepsilon) > 0.$$

The topological sensitivity $g(\mathrm{x}_0)$ provides information for creating small holes located at x_0. Hence the function g can be used like a descent direction in optimization process.

The fictitious domain approach is the one presented in [3] which is inspired from Xfem principles [6, 12].

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In the numerical part, we consider a channel containing a fluid which satisfies the Stokes system with a given initial velocity. Our aim is to locate the best position of an obstacle in the channel by topological optimization tools and the best shape of this obstacle by regularizing the holes creating during the optimization process by the shape derivative used as velocity in the Hamilton-Jacobi equation.

The paper is organized as follows. In section 2, we present a precise statement of the problem and the objective function. In section 3, we give shape and topological gradient associated to our functional. The main part of this paper is the section 4, in which we propose a numerical method for approximation of the Stokes system, to end this section, we give some numerical result (for *d=2*) in order to illustrate the efficacy of the proposed algorithm.

2. PROBLEM SETTING:

2.1. The Stokes problem:

Let Ω be an open set of$\mathbb{R}^d$; *d=2* or *d=3*. The velocity vector $u=(u_1,\ldots,u_k)$ and the pressure p of a viscous and incompressible fluid F governed by Stokes system in steady regime writes:

$$\begin{cases} -\upsilon\Delta\vec{u}_\Omega + \vec{\nabla}p = \vec{f} & in \quad \Omega \\ \operatorname{div}\vec{u}_\Omega = \quad 0 & \text{in} \quad \Omega \\ \vec{u}_\Omega \quad = \vec{g} & \text{on} \quad \Gamma = \partial\Omega \end{cases} \tag{1}$$

Where ν, is the kinematic viscosity coefficient of F and f a given body force per unit of mass.

It is well known, if Ω is bounded, connected with a Lipchitz continuous boundary Γ, and

$$\vec{f} \in H^{-1}(\Omega)^d \ and \ \vec{g} \in (H^{-\frac{1}{2}}(\Omega))^d$$

respectively, such that

$$\int_\Gamma \vec{g}.\vec{v}d\sigma = 0$$

Then there exists one and only one pair $(\vec{u}, p)$, solution de (1); (see [14] for the proof).

The week formulation of the problem (1) is classically written: Find $\vec{u}_\Omega \in V$ such that:

$$a(\vec{u}_\Omega, \vec{v}) + b(\vec{v}, p) = l(\vec{v}); \quad \forall \vec{v} \in V \tag{2}$$

Where

$$V = \{\vec{u} \in (H_0^1(\Omega), such\ that\ div\,\vec{u} = 0\}$$

$$a(\vec{u}, \vec{v}) = 2\nu \left\{ \sum_{i,j=1}^{d} (D_{i,j}(\vec{u}), (D_{i,j}(\vec{v})) \right\}$$

$$l(\vec{v}) = \langle \vec{f}, \vec{v} \rangle, \quad b(\vec{v}, q) = -(q, div\vec{v}) \ and \ b(\vec{w}, q) = 0, \ \ \forall\, q \in L_0^2(\Omega)$$

2.2. The objective function:

The cost function is the one measuring the outflow rate in a channel for example and can be written in the following form

$$J(\Omega, \vec{u}_\Omega) = \textstyle\sum_{i=1}^{k} \int_{\Gamma_i} |\vec{u}_\Omega.\vec{n}| d\sigma \tag{3}$$

Where Γ_i is an inlet and $\partial\Omega = \Gamma_i \cap \Gamma_0\ with\ \Gamma_0$ is the outlet.

3. Shape optimization result:

The problem consists in calculus of the shape and the topological derivative of the shape functional $J(\Omega, \vec{u}_\Omega)$ given by (3) where $\vec{u}_\Omega$the solution of (1) is in order to propose a numerical method to study the comportment of a fluid governed by the Stokes equation in a channel.

3.1. Shape derivative:

In order to compute shape derivative, we use the approach of Simon and Murat in [10]. We consider a perturbation of the domain Ω in the following sense, for$\theta \in W^{1,\infty}(\mathbb{R}^d \times \mathbb{R}^d), \Omega_\theta = (Id+\theta)(\Omega)$. It is well known that for θ sufficiently small $(Id+\theta)$ is a diffeomorphism in $\mathbb{R}^d$.

Definition: 3.1 *The shape derivative of J(Ω) at Ω is defined as the Frechet derivative in $W^{1,\infty}(\mathbb{R}^d \times \mathbb{R}^d)$ at 0 of the application $\theta \to J(Id+\theta)(\Omega)$, i.e.*

$$J(Id+\theta)(\Omega) = J(\Omega) + J'(\Omega)(\theta) + o(\theta)$$

Where $J'(\Omega)$ is a linear and continuous form on $W^{1,\infty}(\mathbb{R}^d \times \mathbb{R}^d)$.

It follows from the definition and some symbolic calculus, the following result which is somewhat standards in partial differential equations.

Theorem: 3.2 *Let Ω be a smooth bounded open set and$\theta \in W^{1,\infty}(\mathbb{R}^d, \mathbb{R}^d)$. One assumes that the data $\vec{f} \in$ H−1(Ω)d and g∈(H−12(Ω))d and the solution uΩ of (1) are sufficiently smooth. Then the shape derivative of (3) is given by*

$$J'(\Omega)(\theta) = \sum_{i=1}^{k} \int_{\Gamma_i} \left(\frac{\partial |\vec{u}_\Omega . \vec{n}|}{\partial n} + H |\vec{u}_\Omega . \vec{n}| \right) \theta . n ds, \forall\, \theta \in C^1(\mathbb{R}^d, \mathbb{R}^d)$$

Where $\vec{n}$ is the normal derivative on $\partial\Omega$ and H is the main curvature defined on Γ_i.

3.2. Topological derivative:

For a given $x_0 \in \Omega$, we consider the perforated open set $\Omega_\varepsilon = \Omega \backslash \overline{\omega}_\varepsilon, \overline{\omega}_\varepsilon = x_0 + \varepsilon\omega, \omega \in \mathbb{R}^d$ is a fixed reference domain. The small hole $\varepsilon\omega$ can be seen as an obstacle in the viscous fluid. We recall here to the general adjoint method and domain truncation [5] in order to get topological derivative.

Let $\vec{u}_{\Omega_\varepsilon}$ be the solution of the equation in the perturbed domain.

$$\begin{cases} -\upsilon \Delta \vec{u}_{\Omega_\varepsilon} + \vec{\nabla} p_{\vec{u}_{\Omega_\varepsilon}} = f \quad in \quad \Omega_\varepsilon \\ \text{div}\,(\vec{u}_{\Omega_\varepsilon}) = \quad 0 \quad \text{in} \quad \Omega_\varepsilon \\ \vec{u}_{\Omega_\varepsilon} \quad = \vec{g} \;\text{on}\quad \Gamma = \partial\Omega \\ \vec{u}_{\Omega_\varepsilon} \quad = 0 \;\text{on} \quad \partial\Omega_\varepsilon \end{cases} \tag{4}$$

The aim of the topological optimization is to find the asymptotic expansion of $J(\Omega_\varepsilon)$ when ε goes to zeros. For many cases, the asymptotic expansion of the function J can be obtained in the following form:

$$J(\Omega_\varepsilon) = J(\Omega) + \rho(\varepsilon) g(x_0) + o(\rho(\varepsilon)) \tag{5}$$

$$\lim_{\varepsilon \to 0} \rho(\varepsilon) = 0\,, \quad \rho(\varepsilon) > 0.$$

The function $g(x_0)$ is called topological derivative (or topological sensitivity) and provides information for creating a small hole located atx_0. Hence the function g can be used like a descent direction in optimization process. Using general adjoint method, we can derive the following result which gives the topological sensibility.

Theorem: 3.3 *Let $j(\varepsilon) = J(\Omega, \vec{u}_{\Omega_\varepsilon})$ be the cost function given by (3) and $\vec{u}_\Omega$ the solution of the Stokes problem (1) and $\omega = B(0,1)$, then j has the following asymptotic expansion*

For d=3

$$j(\varepsilon) = j(0) + \varepsilon[6\pi\nu \vec{u}_\Omega(x_0). \vec{v}_\Omega(x_0) + \delta J(x_0)] + o(\varepsilon) \tag{6}$$

For d=2

$$j(\varepsilon) = j(0) + \frac{-1}{\log \varepsilon}[4\pi\nu \vec{u}_\Omega(x_0). \vec{v}_\Omega(x_0) + \delta J(x_0)] + o(\frac{-1}{\log \varepsilon}) \tag{7}$$

Where $\vec{v}_\Omega$ is the solution of the adjoint problem

$$\begin{cases} -\upsilon\Delta\vec{v}_\Omega + \vec{\nabla}p = -\overrightarrow{DJ}(\vec{u}_\Omega) & in \quad \Omega \\ div\,\vec{v}_\Omega = \quad 0 & in \quad \Omega \\ \vec{v}_\Omega \quad = \vec{g} & on \quad \Gamma = \partial\Omega \end{cases} \tag{8}$$

Proof: For the proof, we refer the reader to [4].

4. The proposed numerical method:

4.1. Numerical approximation:

In this section we adapt the numerical method proposed in [13] to Stokes problem. This method combines a fictitious domain approach for the approximation of the fluid displacement and the use of both shape and topological derivatives. The fictitious domain approach is the one presented in [3] which is inspired from Xfem principles [6, 12]. In the following, we describe this method adapted to our problem. It is proven in [3] that the approximation of the solution is optimal.

The boundary of the domain being an unknown of the problem, we introduce $\widetilde{\Omega}$ a fixed (in general rectangular or parallelepiped) domain which includes all the potential domainsΩ. This fictitious domain approach requires the introduction of two finite element spaces $\tilde{V}^h \subset H^1(\,\widetilde{\Omega}; \mathbb{R}^d)$ and $\widetilde{W}^h \subset L^2(\,\widetilde{\Omega}; \mathbb{R}^d)$ on the fictitious domain$\widetilde{\Omega}$. As $\widetilde{\Omega}$ can be a rectangular or parallelepiped domain, the ones can be defined on the same structured mesh $\mathcal{T}^h$ (see Fig. 1).

Next we shall suppose that

$$\tilde{V}^h = \left\{v^h \in C\left(\,\overline{\widetilde{\Omega}}; \mathbb{R}^d\right): {v^h}_{|\mathcal{T}} \in P(T))^d;\ \forall\, T \in \mathcal{T}\right\} \tag{9}$$

where $P(T)$ is a finite dimensional space of regular functions such that $P(T) \supseteq P_k(T)$ for some $k \geq 1$, integer. The mesh parameter h stands for $h = \max_{T \in \mathcal{T}^h} h_T$ where h_T is the diameter of$T$. Then we can build

$$V^h = {\tilde{V}^h}_{|\Omega} \quad and \quad W^h = {\widetilde{W}^h}_{|\Gamma_D}$$

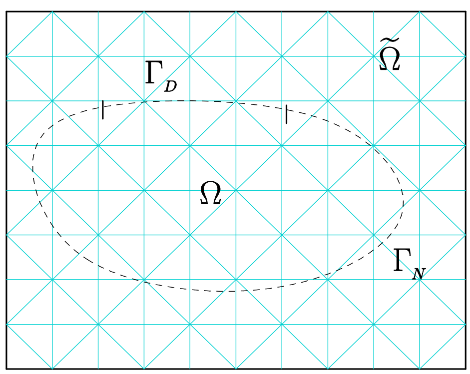

Figure 1: *Example of structured mesh.*

which are natural discretizations of V and $W = L^2(\Gamma_D, \mathbb{R}^d)$, respectively. A mixed approximation of Problem (2) is defined as follows:

$$\begin{cases} Find\ a\ pair\ (\vec{u}^h, p^h) \in V^h \times\ W^h\ such\ that \\ a(\vec{u}^h, \vec{v}^h) - (p^h, div\vec{v}^h) = \langle \vec{f}, \vec{v}^h \rangle, \forall \vec{v}^h \in V^h \\ \qquad (q^h, div\,\vec{u}^h\,) = 0 \quad \forall\, q^h\ \in W^h \end{cases} \tag{10}$$

Similarly to Xfem, where the shape functions of the finite element space are multiplied with a Heaviside function, this corresponds here to the multiplication of the shape functions with the characteristic function ofΩ.

Unfortunately, such a simple formulation leads to a potentially poor approximation of the solution (in $O(\sqrt{h})$ generally, see [3]) due to a possible locking phenomen on the Dirichlet boundary. This is why it is necessary to consider an additional stabilization. Here, we adapt a stabilization technique presented by Barbosa and Hughes in [1, 2] in order to recover an optimal rate of convergence. Note that this stabilization technique can be viewed as a generalization of the former Nitsche's method [7] where the multiplier is eliminated (see [11] for the link between the two stabilization techniques). We present its symmetric version although the nonsymmetric one can be considered in the same way. This technique is based on the addition of a supplementary term involving the normal derivative onΩ_d.

Let us suppose that we have at our disposal an operator

$$R^h : V^h \rightarrow L^2(\Gamma_D)$$

which approximates the normal stress on Γ (i.e. for $v^h \in V^h$ converging to a sufficiently smooth function $v, R^h(v^h)$, tend to $div\vec{v}.\vec{n}$. in an appropriate sense). A first straightforward choice is given by

$$R^h(v^h) = (\lambda div\ v^h)\delta_{ij}$$

In [3], one can see that this gives some satisfactory numerical results in most of the cases except where there is some very small intersection of an element with the real domainΩ. In the latter case, it is proven that a good approximation can be recovered using the extrapolation of $\sigma(v^h)v$ on a neighbor element having a sufficiently large intersection withΩ.

Now, one obtains the stabilized problem by considering the following Lagrangian for$v^h \in V^h$ and $\mu^h \in W^h$:

$$\mathcal{L}_h(v^h, \mu^h) = a(v^h, \mu^h) - l(v^h) + \int_{\Gamma_D} (\mu^h . v^h) ds - \frac{\gamma}{2} \int_{\Gamma_D} || \mu^h + R^h v^h ||^2 ds$$

where for the sake of simplicity $\gamma := h\gamma_0$ is chosen to be a positive constant over Ω (for non-uniform meshes, an element dependent parameter $\gamma = h_T\gamma_0$ is a better choice).

The corresponding discrete problem reads as follows:

$$\begin{cases} \quad Find u^h \in V^h \ and \ \lambda^h \in W^h \ such \ that \\ a(u^h, v^h) + \int_{\Gamma_D} \lambda^h . v^h ds - \gamma \int_{\Gamma_D} \left(\lambda^h + R^h(u^h)\right) . R^h(v^h) ds = l(v^h), \forall v^h \in V^h \\ \\ \quad \int_{\Gamma_D} \mu^h . u^h ds - \gamma \int_{\Gamma_D} \left(\lambda^h + R^h(u^h)\right) . (\mu^h) ds = 0 \ \forall \mu^h \in W^h \end{cases} \tag{11}$$

This formulation is consistent in the sense that the additional term $||\mu^h + R^h(v^h)||^2$ should vanish when h goes to zero since it is well known that in Problem (10) the multiplier μ^h is an approximation of the opposite of the normal stress.

The parameter γ_0 have to be taken sufficiently small such that the coerciveness of the bilinear form is kept. The quality of the approximation is not very sensitive to the parameter γ_0 which can be chosen in a wide range of values.

Now, the shape optimization process needs the description of the boundary ofΩ. As in the framework of Xfem in [12], we chose a level-set approximation of the boundary. This means that the boundary will be represented by the zero level-set of a function approximated on a convenient finite element space.

The advantage of this approach is to obtain both an optimal approximation of the Stokes problem together with an accurate location of boundary of the real domain. Note that to keep the optimality of the approximation, the level-set function has to be approximated at the same order than the displacement$\vec{u}$.

4.2. Optimization algorithm:

The optimization algorithm is summarized in Fig. 2. Since we use the topological gradient to create holes during the optimization process, it is possible to start with a shape containing some initial holes or not. A very small penalization is used when solving the direct problem and the adjoint one to avoid the indeterminacy of the rigid motions of eventual isolated part.

Concerning step 4, new holes of a given radius is created by the simple operation on the level-set function which can be written on each finite element nodex_i.

$$\bar{\psi}(x_i) \coloneqq \max\left(\psi(x_i), \frac{(r^2 - ||x_i||^2)}{2r}\right)$$

where $\psi(x)$ is the level-set function, $\bar{\psi}(x_i)$ its new value, r is the radius of the created hole and c its center.

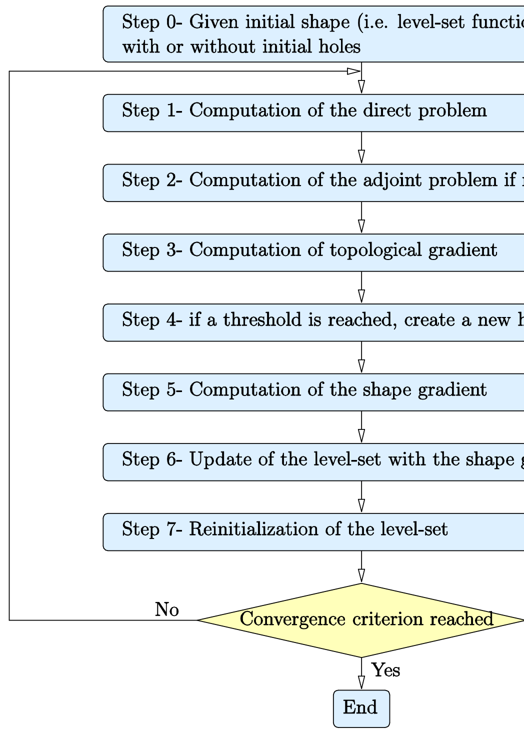

Figure 2: *The proposed algorithm*

At step 6, the update of the level-set is done in a classical way applying the following Hamilton-Jacobi equation on a chosen time interval:

$$\frac{\partial\psi}{\partial t} + v|\nabla\psi| = 0; \;\; in\; \widetilde{\Omega} \tag{12}$$

where the normal velocity v is given by the shape derivative, at least in Ω and its boundary. In our simulations, the gradient is extended by zero on the complement of Ω in$\widetilde{\Omega}$. However, a smoother extension could be considered.

Note that it is convenient to apply a threshold on the gradient to avoid some incoherent values where the shape gradient may have a singularity (corner, transition from Dirichlet to Neumann condition ...).

This Hamilton-Jacobi equation is known to admit multiple non-smooth solutions. Classically, a smooth solution is computed thanks to an upwind scheme. Since the fictitious domain $\widetilde{\Omega}$ can be a rectangular/parallelepiped domain, it is possible to use a classical upwind scheme on a Cartesian grid. However, to keep the possibility of having a non-structured mesh, for instance to proceed to a local refinement, we use a different strategy. Equation (12) is solved on a small time interval $]0,\delta]$ integrating the following equation where the non-linearity is made explicit:

$$\begin{cases} \frac{\partial\bar{\psi}}{\partial t} + v\frac{\nabla\psi^n}{|\nabla\psi^n|}\nabla\bar{\psi} = 0; \;\; in \;\; \widetilde{\Omega}\times]0,\Delta t], \\ \bar{\psi}(x,0) = \psi^n(x). \end{cases} \tag{13}$$

Here ψ^n is the level-set function at the previous time step and ψ^{n+1} is given by $\bar{\psi}(\Delta t, .)$. The problem (13) is a pure convection one. This problem can be solved for instance with the simple Galerkin-Characteristic scheme proposed in [15] (other possibilities: SUPG, discontinuous Galerkin ...). This scheme is unconditionally stable but rather dissipative. The effect is that the level-sets are a little bit smoothed.

In order to regularize the level-set function, the re-initialization step 7 is considered. It consists classically in solving

$$\begin{cases} \dfrac{\partial \psi}{\partial t} + sign(\psi_0)(|\nabla \psi^n| - 1) = 0; \;\; in \;\; \widetilde{\Omega} \times \mathbb{R}_+, \\ \psi(x,0) = \psi_0(x). \;\; in\, \widetilde{\Omega} \end{cases}$$

Whose stationary solution is a signed distance. The same kind of scheme is used than for Equation (12).

4.3. Numerical result:

In numerical application, we suppose that the flow satisfies the following boundary conditions:

- ✓ On the inlets,$\sigma(\vec{u}_\Omega, p).n = \phi \; on \; \Gamma_{in}^i, i = 1,2,3$, where $\sigma(\vec{u}_\Omega, p) = \nu(\nabla \vec{u}_\Omega + \nabla \vec{u}_\Omega^T) - pI$ and I is the identity matrix.
- ✓ And one the outlet (Γ_{out}) we use free boundary condition $\sigma(\vec{u}_\Omega, p).n = 0 \; on \; \Gamma_{out}$
  Namely, we apply the above algorithm to;

$$J(\Omega, \vec{u}_\Omega) = \sum_{i=1}^{2} \int_{\Gamma_k} |\vec{u}_\Omega . \vec{n}| d\sigma$$

Where $\vec{u}_\Omega$ solves

$$\begin{cases} -\upsilon \Delta \vec{u}_\Omega + \vec{\nabla} p = \vec{f} \quad in \quad \Omega \\ \mathrm{div}\, \vec{u}_\Omega \quad = \quad 0 \quad \mathrm{in} \quad \Omega \\ \sigma(\vec{u}_\Omega, p).\vec{n} = \phi \text{ on } \Gamma_{in}^{i}, i = 1,2. \\ \vec{u}_\Omega \quad = \; 0 \;\; \text{on} \quad \Gamma_{out} \end{cases}$$

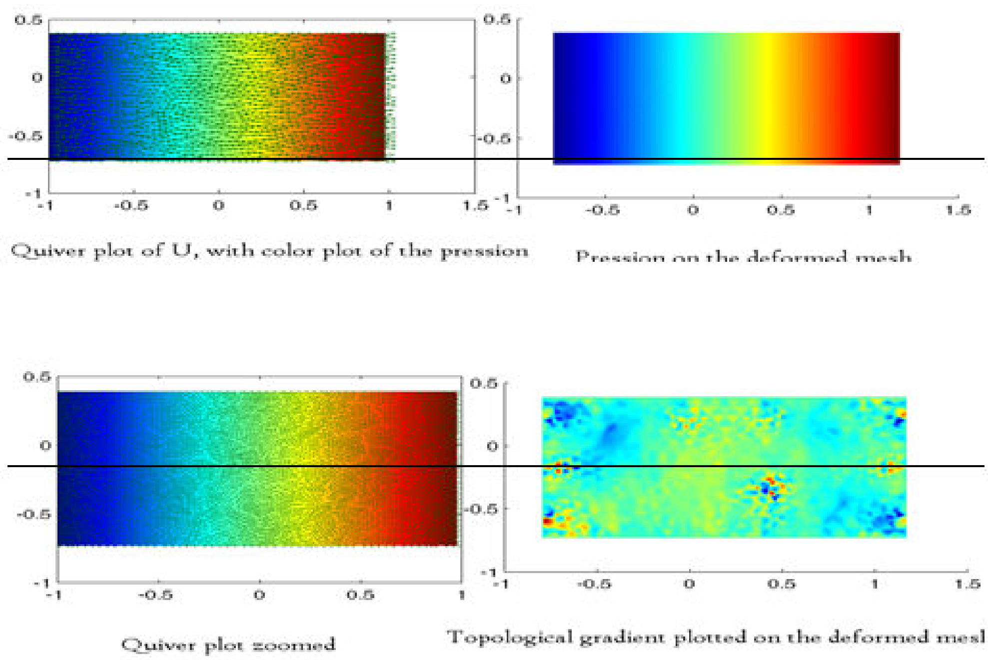

Figure 3: *Topological gradient, velocity and pressure in the initial geometry.*

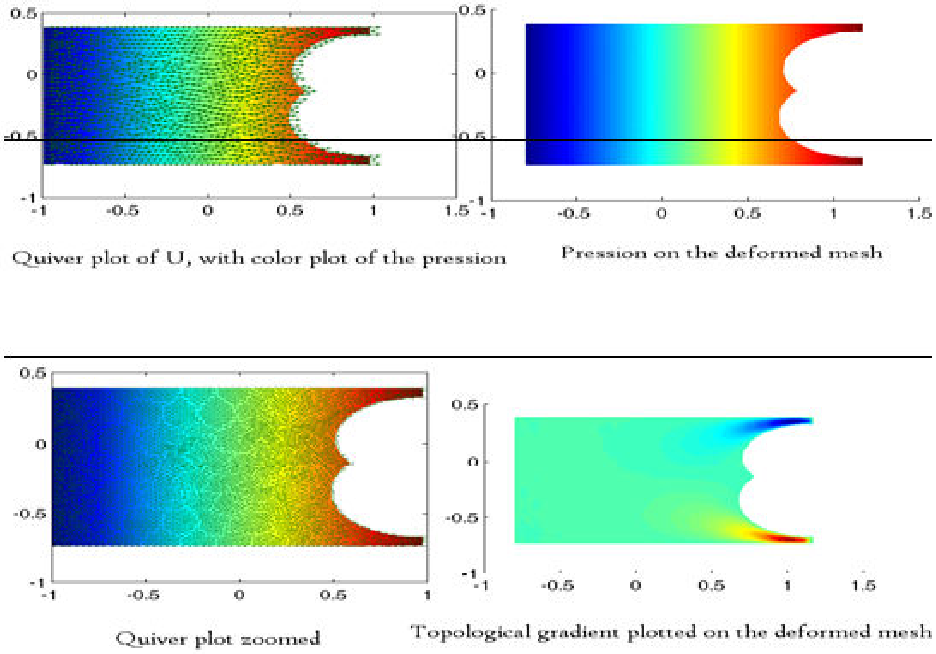

Figure 4: *Topological gradient, velocity and pressure in the final geometry*

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