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An (s, S) Inventory System with Bernoulli Disaster

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ABSTRACT

A single commodity (s, S) inventory system with zero lead time is investigated. Shortages are not permitted. A special type of disaster is considered. Computational methods are developed for the calculation of stationary probabilities of Inventory processes and probability distribution of cycles of replenishments. A cost expression for the Expected Total Cost per Transition E (TCT) is obtained. Results are numerically illustrated.

***Keywords**: Lead Time, Markov renewal process, Semi-Markov Process, Stationary, Probabilities, Expected Total Cost*

INTRODUCTION:

In this model, two types of demands are differentiated. (i) The demands arise due to the disaster upon the unit on exhibition and (ii) the demand for one unit from the items which are under the category of non-exhibited. Of course, a facet assumption is that if there is only one unit on both the categories put together, it may succumb either due to disaster with probability q and with probability 1-q = p of survival from the disaster, or due to a regular demand with probability one whichever occurs first. Sivasamy and Senthamarikannan (1994) embedded process of markov renewal process bulk service ques.

All the four types of transitions between demand epochs (1) and the disaster epochs (2) via $(1 \to 1)$, $(1 \to 2)$, $(2 \to 1)$ and $(2 \to 2)$ are assumed to follow a Markov Renewal Process. Each disaster destroys at most one unit of the inventory which is on exhibition as in Kalpakam and Arivarignan (1985a) and (1985b).

The maximum capacity of the system is S and the stock is brought to S whenever the inventory falls to the reorder point s. Shortages are not permitted and lead time is zero. If there is only one item in the inventory, then it must be on exhibition so that the same may be sold out either by regular demand type or by disaster type which ever occurs first. The times between demand points, disaster to demand and vice versa are assumed to follow a renewal process with a common inter-occurrence density g(t) or distribution function

$$G(t) = \int_0^t g(x)\, dx$$

with a finite mean μ. That is, the transitions between demand epochs and disaster epochs form a Markov chain with '1' denoting demand epoch and '2' denoting a disaster epoch. The unit step transition probability matrix of these MC is represented by $\mathbb{P} = (P_{ij});i, j \in \{1,2\}$. Krishnamoorthy and Varghese (1995) and (1995) deals with an inventory with disaster in which the demand forming a Poisson process, the time between disaster and life times of items are all negative exponential distributions. This paper is restricted to the case of renewal demand cum disasters, Markov dependence among the transitions and each disaster damaging at most one unit at a time.

Notations:

$\mathbb{E}_1 = \{1,2\}$

$\mathbb{E}_2 = \{S, S-1, \cdots, s+2, s+1\}$

$\bar{J} = \{(1,j), (2,j)\}$ for $j \in \mathbb{E}_2$

$\mathbb{E} = \mathbb{E}_1 \times \mathbb{E}_2 = \{\bar{S}, \overline{S-1}, \ldots \overline{s+2}, \overline{s+1}\}$

$m = S - s$

$\mathbb{N} = \{0,1,2,\cdots\}$

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INVENTORY PROCESS:

Let $0 = T_0 < T_1 < T_2 \cdots < T_n < T_{n+1} < \cdots$ be the successive transition epochs [demand epochs plus disaster epochs]. Denoted the type of the transition epoch (1- for demand epochs 2 - for disaster) and the other hand inventory level just after meeting the demand at T_n, $n \in N$ by X_n and Y_n respectively. Thus for I, j $\in E_1$.

$$P\left(X_n = j \,/\, X_{n-1} = i \right) = P_{ij}$$

Assume that the Transition Probability matrix (TPM) P = (Pij) is irreducible and a periodic. Denote its stationary probability vector by $\upsilon = (\upsilon_1, \upsilon_2)$ so that

$$\mu p = \upsilon, \quad \upsilon e = 1$$

It can be shown that the (X_n, Y_n) process is an irreducible and a periodic MC on the state space E with TPM Q, where

$$= P\left[X_{n+1} = j,\ Y_{n+1} = J \,/\, X_n = i,\ Y_n = I \right]$$

$$= \begin{bmatrix} C & D \\ E & F \end{bmatrix}$$

$$C = \begin{array}{cc|cc} & & \overline{(1, S)} & \overline{(2, S)} \\ \overline{S} & (1.S) & 0 & p_{12}p \\ & (2, S) & 0 & p_{22}p \end{array}$$

$$D = \begin{array}{c|cc|cc|c|cc} & \multicolumn{2}{c|}{\overline{S-1}} & \multicolumn{2}{c|}{\overline{S-2}} & & \multicolumn{2}{c}{\overline{s+1}} \\ & (1, S\text{-}1), & (2, S\text{-}1) & (1, S\text{-}1), & (2, S\text{-}1) & \ldots & (1, s+1), & (2, s+2) \\ S\ (1,S) & p_{11} & p_{12q} & 0 & 0 & & 0 & 0 \\ \ (2,S) & p_{21} & p_{22}q & 0 & 0 & & 0 & 0 \end{array}$$

$$E = \begin{array}{cc|cc} & & \multicolumn{2}{c}{\overline{S-1}} \\ & & (1, S\text{-}1), & (2, S\text{-}1) \\ \overline{S-1} & 1, S-1 & 0 & 0 \\ & 2, S-1 & 0 & 0 \\ \overline{S-2} & 1, S-2 & 0 & 0 \\ & 2, S-2 & 0 & 0 \\ & \vdots & \vdots & \vdots \\ \overline{s+1} & 1, s+1 & p_{11} & p_{12q} \\ & 2, s+1 & p_{21} & p_{22}q \end{array}$$

$$F = \begin{array}{cc|cc|cc|cc|c|cc} & & \multicolumn{2}{c|}{\overline{S-1}} & \multicolumn{2}{c|}{\overline{S-2}} & \multicolumn{2}{c|}{\overline{S-3}} & \ldots & \multicolumn{2}{c}{\overline{s+1}} \\ S-1 & 1, S-1 & 0 & p_{12}p & p_{11} & p_{12}q & 0 & 0 & & 0 & 0 \\ & 2, S-1 & 0 & p_{22}p & p_{21} & p_{22}q & 0 & 0 & \cdot\ \cdot & 0 & 0 \\ S-2 & 1, S-1 & 0 & 0 & 0 & p_{12}p & p_{11} & p_{12}q & & 0 & 0 \\ & 2, S-1 & 0 & 0 & 0 & p_{22}p & p_{21} & p_{22}q & \cdot\ \cdot & 0 & 0 \\ & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdot\ \cdot & \vdots & \vdots \\ s+1 & 1, S-1 & 0 & 0 & 0 & 0 & 0 & 0 & & 0 & p_{12}p \\ & 2, S-1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdot\ \cdot & 0 & p_{22}p \end{array}$$

Let $\Pi = (\Pi_S, \Pi_{S-1}, \ldots,\Pi_{s+2}, \Pi_{s+1})$ with $\Pi_j = (\Pi_{1,j}, \Pi_{2,j})$ for j = S, S-1, … s+1 be the stationary probability vector of the **Q**-matrix and $e = (1,1,\ldots 1)^T$ a column vector of unities. Thus Π is the unique solution of the system of equations $\Pi Q = \Pi$ $\Pi e = 1$. This yields the following statements in terms of the above partitioned structure of Π and Q, matrices

$$\Pi_S + (\Pi_{S-1}, \Pi_{S-2}, \ldots, \Pi_{s+1}) E = \Pi_S \tag{1}$$

$$\Pi_S D + (\Pi_{S-1}, \Pi_{S-2}, \ldots, \Pi_{s+1}) F = (\Pi_{S-1}, \Pi_{S-2}, \ldots, \Pi_{s+1}) \tag{2}$$

In order to make the system of equations (1) and (2) as a simultaneous set of equation, equation (1) may be replaced by the following equations

$$\sum_{j=s+1}^{S} \Pi_j = v = (v_1,\ v_2) \tag{3}$$

Now solving (2) and (3), we obtain the following observation (4), for j = S, S-1, S-2, … s+2

$$\Pi_j \begin{pmatrix} p_{11} & p_{12}q \\ p_{21} & p_{22}q \end{pmatrix} = \Pi_{j-1} \begin{pmatrix} 1 & -p_{12}p \\ 0 & 1-p_{22}p \end{pmatrix} \tag{4}$$

Let us assume that

$$\Pi_j = C_1\ (v_1, v_2) \text{ for } j = s+1, s+2\ldots S \tag{5}$$

As a trial solution where C_1 is some real constant. Substituting (5) in (4), we observe that

LHS of (4)	=	$C_1\ (v_1, v_2q)$
RHS of (4)	=	$C_1\ (v_1, v_2 - v_2p_{22}p - v_1p_{12}p\ )$
	=	$C_1\ (v_1, v_2 - [v_1p_{12}p - v_2p_{22}p]\ )$
	=	$C_1\ (v_1, v_2 - v_2p\ )$
	=	$C_1\ (v_1, v_2q)$

Thus (5) satisfies (4). Further substitution of (5) in (3) leads to mc1 $(v_1, v_2) = (v_1, v_2)$

Which is turn implies that $C_1 = \dfrac{1}{m}$ (6)

Thus, $\Pi_j = (1/m)\ (v_1, v_2)$ for j = s+1, s+2, … S (7)

SEMI- MARKOV INVENTORY PROCESS:

The trisequence process $\{(X_n, Y_n)\ , Tn\}$ for $n\in$ N form a MRP on the state space E with an underlying MC $\{X_n, Y_n\}$

If $\overline{Q}(t)$ Stands for the kernel of the MRP, then

$$\overline{Q}(t) = Q\ .G\ (t)$$

Note that $\overline{Q}(\infty) = Q$

Define
$$I(t) = \begin{cases} Y_n & \text{for} \quad T_n \le t < T_{n+1}, n\varepsilon\mathbb{N} \\ 0 & \text{otherwise} \end{cases}$$

and $I\ (0) = Y_0 = S$ (8)

Then $\{I(t), t\in [0, \infty)\ \}$ is a Semi- Markov process on the state space E_2 with an underlying MRP $\{\ (X_n, Y_n), T_n\ )\ n \in N\}$
Letting

$$q\ (i, n; t) = p\ [I\ (t) = n\ /\ Y_0 = S] \tag{9}$$

by the application of Key-Renewal theorem, it may be established that

$\lim_{t\to\infty} q(S,n;t) = q_n$ (Say) exists and

$$q_n = \sum_{j=1}^{2} \pi_{(j,n)} \quad n \in E_2 \tag{10}$$

Thus the average inventory at a random point of time is obtained by

$$E(I) = \sum_{n \in E_2} n q_n = (s+S+1)/2 \tag{11}$$

and the average inventory just after a transition epoch is given by

$$E(I^+) = \sum_{j=1}^{2} \sum n \quad \pi_{(j,n)} = (s+S+1)/2 = E(I) \tag{12}$$

CYCLE OF REPLENISHMENTS:

If τ is the shortest time taken during the successive visits between $\overline{S}$ and $\overline{S}$ levels by the MRP Q then
$\tau = \inf \{ t: (X_{n+t}, Y_{n+t}) = (J, S) / (X_n, Y_n) = (i, s) \} \qquad n \in N$

This may be called as (i/j) cycle as the same cycle is initiated by as i-type transition and terminated by a j-type transition. If this cycle consists of K - number of transitions with k-th epoch being the replenishment epoch for the next cycle, then the conditional distribution of (K = k), k = 1,2,… in the (i/j) cycle is given by

$$\beta_{i,j}(K) = P_{i \quad i_1} P_{i_1 \quad i_2} \cdots P_{i_{k-1} \quad i_k} \tag{13}$$

Where $i_k = j$. If, for I, j $\in E_1$ $\beta(k) = (\beta_{ij}(k))$ is a square matrix of order 2 then by the direct application of filtering technique due to Cinlar (1969), it is established that

$$\beta(k) = \begin{cases} C & k = 1 \\ D\,F(k-2)\,E & k = 2,3.\cdots \end{cases} \tag{14}$$

Let $\beta = \sum_{k=1}^{\infty} \beta(k)$. Then $\beta e = e$. From the structures of C,D,E and F, it is noted that D F^{K-2} E= 0 for k = 2,3,… m-1, m E**e** = (I - F) **e** and D**e** = (I -F)**e**. Using these facts, conditional distribution of number of transitions in any cycle of replenishment may be obtained as below.

During any cycle of replenishment, there must be at least 'm' transitions. Let this conditional distribution in a (i/j) cycle containing k-transitions be

$$\overline{\beta}(k) = (\overline{\beta}_{ij}(k))\; k = m, \quad m+1,\ldots \tag{15}$$

Further $\sum_{k=1}^{\infty} \beta(k)\,\mathbf{e} = \mathbf{e}$ implies that

$$Ce + D\,F^{m-2} \; e = e \text{ or } D\,F^{m-2}\, e = (I - C)\, e$$

i.e
$$(I - C)^{-1} \; D\,F^{m-2} \; e = e \tag{16}$$

Thus
$$\overline{\beta}(K) = (I - C)^{-1} \; D \; F^{K-2}\, E \qquad K = m, \quad m+1,\ldots \tag{17}$$

Let $\overline{\beta} = \sum_{k=m}^{\infty} \overline{\beta}(k)$. Then

$$\overline{\beta} = \sum_{k=m}^{\infty} \overline{\beta}(k) \quad (I-C)^{-1} \; D \quad F^{m-2} \, (I-F)^{-1} \; E \tag{18}$$

Note that the matrix $\overline{\beta}$ forms a TPM i.e., $\overline{\beta}$ e = e.

The expected number of transitions in an (i/j) Cycle is given by

$$E(k) = \begin{pmatrix} E_1 & (k) \\ E_2 & (k) \end{pmatrix} = \sum_{k=m}^{\infty} k\overline{\beta}(k)\,\mathbf{e}$$

$$= (I-C)^{-1} D \{(m F^{m} - F^{m+1} (I-F)^{-1} Ee + \overline{\beta}\mathbf{e} \tag{19}$$

Since $\overline{\beta}$ is the TPM of the (i/j) - cycle, its stationary probability vector $\beta^* = (\beta_1^*, \; \beta_2^*)$, is obtained by solving the system $\beta^* \overline{\beta} = \beta^*$ and $\beta^*\mathbf{e}=1$

Thus,

$$\left.\begin{array}{l} \beta_1^* = \overline{\beta}_{21} / (\overline{\beta}_{12} + \overline{\beta}_{21}) \\ \beta_2^* = \overline{\beta}_{12} / (\overline{\beta}_{12} + \overline{\beta}_{21}) \end{array}\right] \tag{20}$$

COST ANALYSIS:

Assume the following cost considerations for a (./j) cycle as
H_j - the unit cost per transition
L_j - fixed ordering cost
C_j - unit variable cost

Then the conditional expected cost per transition in a (./j) cycle is given by

$$E_j\,(TCT) = \frac{L_j}{E_j(k)} + \frac{(S-s)}{E_j(k)} C_j + H_j\, E(I^+) \tag{21}$$

$$= T\,J, \text{ say}$$

Then the expected total cost per transition in any cycle is given by

$$E\,(TCT) = \sum_{j=1}^{2} T_j \beta_j^* \tag{22}$$

OPTIMIZATION PROBLEMS:

Let L_j. C_j and H_j be the set up cost, variable cost per unit time and holding cost per unit time when the cycle of replenishment is terminated by a 'j' type demand j = 1,2.

Numerical values and graphs of E (TCT) which is a function of v_1 and 'q' are obtained for the following costs, P matrices and 'q' values.

$L_1 = 15, C_1 = 10 \quad H_1 = 5$

$L_2 = 20, C_2 = 20 \quad H_2 = 8$

$$p = \left\{ \begin{pmatrix} .1 & .9 \\ .3 & .7 \end{pmatrix}, \; \begin{pmatrix} .5 & .5 \\ .3 & .7 \end{pmatrix}, \; \begin{pmatrix} .9 & .1 \\ .3 & .7 \end{pmatrix} \right\}$$

$q = \{.1, .3, .5, .7, .9\}$

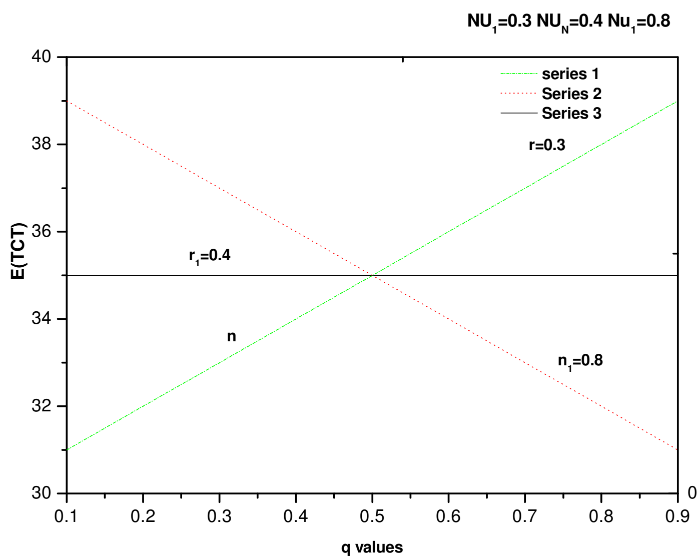
For example, the matrix $\beta^* = \beta_1^*, \; \beta_2^*,$ and E (k) vector are given below for the case of

$$p = \begin{pmatrix} .1 & .9 \\ .3 & .7 \end{pmatrix} \text{ and } q = .3$$

$$\overline{\beta} = \begin{pmatrix} .5263 & .4737 \\ .5263 & .4737 \end{pmatrix}, \quad \beta_1^* = .5263, \quad \beta_2^* = .4737$$

$$E(k) = \begin{pmatrix} 6. & 0434 \\ 6. & 0439 \end{pmatrix}$$

The relationship between E (TCT) versus v_1 and 'q' values is represented as below



The graph of E (TCT) = f (v, q) is exhibited, by taking E (TCT) on the Y- axis and 'q' on the x-axis. For v_1 = q = 0.5, the value of E (TCT) is a constant. for v_1 < 0.5 with variation in 'q' the curve is upward moving straight line. For v_1 > 0.5 with variation in 'q' the curve is downward moving to the right. At 'q' = 0.5 all the three straight lines intersect.

These facts ensure that the E (TCT) is minimum corresponding to the pair (v_1, q) in which v_1 and q values are the highest non-negative fractions.

REFERENCES

[1] Kalpakam, S and Arivarignan. G (1985a): Analysis of an Exhibiting Inventory System. *Stock, Analy Appl.,* 3 (4), pp 447-465.

[2] Kalpakam, S and Arivarignan. G (1985b): A continues review Inventory system with Arbitrary Inter arrival times between Failure, *Opsearch* 22, No.3, pp. 153-93.

[3] Krishnamoorthy, A and Varghese T.V. (1995): Inventory with Disaster, *Optimization* 35, pp.85-93.

[4] Krishnamoorthy, A and Varghese T.V. (1995): On a Two commodity. Inventory Problem with Markov- Shift in demand *OPSEARCH*, Vol. 32, No1, pp 44-56.

[5] Cinlar, E., (1969): Introduction to Stochastic Process, Prenticed-Hall Inc. Englewood Clis, New Jersey.

[6] Sivasamy, R. and Senthamarikannan, K., (1994): Embedded processes of a Markov renewal bulk service queue, Asia Paci_c Journal of Operational Reserch, Vol.11, pp.51-65,

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