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ON SOLVING PELL MEANS

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ABSTRACT

Pell sequence is associated with Fibonacci sequence. Many identities have been already derived for Pell sequence in books and mathematical journals. This paper will present formulas in solving Pell means as well as solving the sequence itself.

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Key Words and Phrases: *Pell sequence, Pell-like sequence, Pell mean, missing terms.*

INTRODUCTION:

Pell Sequence is one of many sequences which share basic properties and identities of Fibonacci sequence as well as with Fibonacci's associated sequence, Lucas sequence [1]. By allowing an extension to the Pell sequence with negative subscripts, it can be defined as

$\{P_n\}$:	…	P_{-4}	P_{-3}	P_{-2}	P_{-1}	P_0	P_1	P_2	P_3	P_4	P_5	…
	…	-12	5	-2	1	0	1	2	5	12	29	…

in which

$$P_0 = 0, \qquad P_1 = 1, \qquad P_{n+2} = 2P_{n+1} + P_n$$

and

$$P_{-n} = (-1)^{n+1}P_n,$$

as shown by Horadam in [2].

In mathematical statement, Pell sequence is a succession of numbers that are obtained by getting the sum of twice the previous number and the number before that. On the other hand, Fibonacci sequence (1, 1, 2, 3, 5, 8, 13, 21….) is a succession of numbers that are obtained through adding the two preceding numbers.

All integer sequences that satisfy Fibonacci-like sequence, $g(n+2) = g(n) + g(n+1)$, can be solved given two initial values using the equation

$$g(n) = F(n)g(1) + F(n-1)g(0),$$

where F(n) and F(n-1) are corresponding terms of Fibonacci sequence.

The nth term of Pell sequence can be solved using the explicit formula

$$P_n = \frac{(1+\sqrt{2})^n - (1-\sqrt{2})^n}{2\sqrt{2}}.$$

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Also, it can be solved using the binomial sums

$$P_n = \sum_{k=0}^{[n-2/2]} \binom{n}{2k+1} 2^k.$$

A well known counterpart of Pell sequence is called as Pell-Lucas sequence where the n^th Pell-Lucas number can be solved by the binomial sum

$$Q_n = 2\sum_{k=0}^{n/2} \binom{n}{2k} 2^k$$

or by the Binet-type formula

$$Q_n = (1-\sqrt{2})^n + (1+\sqrt{2})^n.$$

As seen from the formula above, terms in Pell Sequence are proportional to silver ratio $(1+\sqrt{2})$ which is analogous to the terms in Fibonacci sequence that are proportional to golden ratio $(1+\sqrt{5})$. Many Pell sequence identities have been presented by Horadam in [2] and the author will add some properties by studying Pell means. Pell means are terms between any two terms of Pell or Pell-like sequence.

Definition: 1.1 *If $a, x_1, x_2, x_3, \dots, x_{n-1,} x_{n,} b$ is a Pell or Pell-like sequence, then $x_1, x_2, x_3, \dots, x_{n-1,} x_n$ are called Pell means between a and b.*

Inserting arithmetic means, geometric means and harmonic means has been very well solved in many college and high school mathematics books. Recently, a formula for solving Fibonacci means based on the number of missing terms has been proposed and derived in [3]. This formula is derived to find the first missing term of any Fibonacci-like sequence with n missing terms in any arbitrary constant first term *a* and last term *b*. It has the form of

$$x_1 = \frac{b - F_n a}{F_{n+1}}.$$

Since it is known that Pell sequence is a derivative of Fibonacci sequence, it is not farfetched to think that solving Pell means is related to that formula.

This paper will present deriving formulas in solving Pell means.

THE FORMULA:

Deriving the formula for the first missing term of any Pell-like sequence could be found using a recognizable pattern. Just like in Fibonacci sequence, it can be noted that all the missing terms in a Pell or Pell-like sequence could be solved by finding the first missing term x_1 given arbitrary constant *a* and *b*; where *a* is the first term given and *b* is the last term given in this sequence.

From the definition of Pell sequence, the general formula for x_1 will make the solving easy for the other missing terms. Like the approach in [3], x_1 can be solved for Pell or Pell-like sequence with

a. One missing term, the sequence is a, x_1, b

$$a + 2x_1 = b$$

then

$$x_1 = \frac{b-a}{2}.$$

b. Two missing terms, the sequence is a, x_1, x_2, b

$$a + 2x_1 = x_2$$

$$x_1 + 2x_2 = b$$

then

$$x_1 = \frac{b-2a}{5}.$$

c. Three missing terms, the sequence is a, x_1, x_2, x_3, b

$$a + 2x_1 = x_2$$

$$x_1 + 2x_2 = x_3$$

$$x_2 + 2x_3 = b$$

then

$$x_1 = \frac{b-5a}{12}$$

d. Four missing terms, the Pell like sequence is a, x_1, x_2, x_3, x_4, b

$$a + 2x_1 = x_2$$

$$x_1 + 2x_2 = x_3$$

$$x_2 + 2x_3 = x_4$$

$$x_3 + 2x_4 = b$$

then

$$x_1 = \frac{b-12a}{29}$$

A clear observation could be seen in the formula for x_1 that the numerical coefficient of *a* in numerator and the denominator of the formulas were following the Pell sequence as shown in Table 1. The formula can be illustrated as

$$x_1 = \frac{b-P_n a}{P_{n+1}}.$$

Table- 1: Relationship of number of missing terms with numerator and denominator of formulas.

Number of Missing Term	Coefficient of a in Numerator	Coefficient of Denominator
1	1	2
2	2	5
3	5	12
4	12	29
.	.	.
.	.	.
.	.	.
N	$\frac{(1+\sqrt{2})^n-(1-\sqrt{2})^n}{2\sqrt{2}}$	$\frac{(1+\sqrt{2})^{n+1}-(1-\sqrt{2})^{n+1}}{2\sqrt{2}}$

Therefore for Pell mean, the formula for x_1 is

$$x_1 = \frac{b-\left[\frac{(1+\sqrt{2})^n-(1-\sqrt{2})^n}{2\sqrt{2}}\right]a}{\frac{(1+\sqrt{2})^{n+1}-(1-\sqrt{2})^{n+1}}{2\sqrt{2}}}$$

It can also be written as

$$x_1 = \frac{2b\sqrt{2}-\left[(1+\sqrt{2})^n-(1-\sqrt{2})^n\right]a}{(1+\sqrt{2})^{n+1}-(1-\sqrt{2})^{n+1}}$$

where x_1 $is\ the\ first\ missing\ term\ in\ Pell\ like\ sequence$
$a\ is\ the\ first\ term\ given$
$b\ is\ the\ last\ term\ given$
$n\ is\ the\ number\ of\ missing\ terms$.

These formulas could now be used to find all the Pell means in the sequence.

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