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THE FURTHER UNIFIED THEORY FOR MODIFICATIONS OF $\hat{g}$-CLOSED SETS

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ABSTRACT

In this paper, the notion of mn $\hat{g}$-closed which is defined on a set with two families of sets satisfying some minimal conditions is introduced and investigated. This set enables us to unify certain kind of modifications of $\hat{g}$-closed sets due to Veerakumar [26]

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1. INTRODUCTION:

In 1970, Levine [10] introduced the notion of generalized closed (briefly, g-closed) sets and a separation axiom called $T1_{/2}$ between T_1 and T_0 in topological spaces. Recently, many modifications of g-closed sets were defined and investigated. They are applied to introduce several low separation axioms. Especially, Takashi Noiri investigated g-closed sets and introduced mg-closed sets [17], mng-closed sets [18] and locally m-closed sets [17] under minimal conditions. Veerakumar [26] (= Sheik John [23]) introduced $\hat{g}$-closed (= ω-closed) sets in topological spaces.

In this paper, in order to unify a certain kind of modifications of $\hat{g}$-closed sets, we introduce the notion of mn $\hat{g}$-closed sets in bi m-space. In Section 3, we deal with minimal structures. In Section 4, we introduce the notion of mn $\hat{g}$-closed sets and obtain several basic characterizations of mn $\hat{g}$-closed sets. In Section 5, we introduce the notion of locally (m, n)-closed sets and show that a subset is n_x-closed if and only if it is mn $\hat{g}$-closed and locally (m, n)-closed. In Section 6, we introduce the notion of *(m, n)-normal spaces and obtain their characterizations.

2. PRELIMINARIES:

Let (X,τ) be a topological space and A a subset of X. The closure of A and the interior of A are denoted by Cl(A) and Int(A), respectively.

Definition: 2.1 Let (X, τ) be a topological space. A subset A of X is said to be
(1) α-open [16] if $A \subset Int(Cl(Int(A)))$,
(2) semi-open [9] if $A \subset Cl(Int(A))$.

The family of all α-open (resp. Semi-open) sets in (X,τ) is denoted by α(X) (resp. SO(X)).

Definition: 2.2 Let (X,τ) be a topological space. A subset A of X is said to be α-closed [1] (resp. semi-closed [5]) if the complement of A is α-open (resp. semi-open).

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Remark: 2.3 Let (X, τ) be a topological space and A a subset of X. The intersection of all α-closed (resp. semi-closed) sets of X containing A is called the α-closure [1] (resp. semi-closure [5]) of A and is denoted by αCl(A) (resp. sCl(A)).

Remark: 2.4 Let (X, τ) be a topological space and A a subset of X . The union of all α-open (resp. semi-open) sets contained in A is called the α-interior (resp. semi-interior) and is denoted by αInt(A) (resp. sInt(A)).

Definition: 2.5 Let (X,τ) be a topological space and A a subset of X. A point $x \in X$ is called a Θ-cluster (resp. δ-cluster) point of A if $Cl(V) \cap A \neq \varphi$ (resp. $Int(Cl(V)) \cap A \neq \varphi$) for every open set V containing x. The set of all Θ-cluster (resp. δ-cluster) points of A is called the Θ-closure (resp. δ-closure) of A and is denoted by $Cl_\Theta(A)$ (resp. $Cl_\delta(A)$) [28]. A subset A is said to be Θ-closed (resp. δ-closed) if $Cl_\Theta(A) = A$ (resp. $Cl_\delta(A) = A$). The complement of a Θ-closed (resp. δ-closed) set is said to be Θ-open (resp. δ-open). The union of all Θ-open (resp. δ-open) sets contained in the subset A is called the Θ-interior (resp. δ-interior) of A and is denoted by $Int_\Theta(A)$ (resp. $Int_\delta(A)$).

The collection of all Θ-open (resp. δ-open) sets in X is denoted by τ_Θ (resp. τ_δ).

Definition: 2.6 Let (X,τ) be a topological space. A subset A of X is said to be λ-closed [20] if A is the intersection of a closed set and a Λ-set where Λ-set is a set L which is equal to the intersection of open supersets of L.

The complement of λ-closed set is said to be λ-open.

The collection of all λ-open sets in X is denoted by λO(X).

Definition: 2.7 Let (X,τ) be a topological space. A subset A of X is said to be
(1) $\hat{g}$ **-**closed [26] if $Cl(A) \subset U$ whenever $A \subset U$ and $U \in SO(X)$,

The complement of $\hat{g}$ **-**closed set is said to be $\hat{g}$ **-**open.

The collection of all $\hat{g}$ **-**open sets in X is denoted by τ_ω.

(2) δ $\hat{g}$ **-**closed [25] if $Cl_\delta(A) \subset U$ whenever $A \subset U$ and $U \in \tau_\omega$,
(3) Θω**-**closed [6] if $Cl_\Theta(A) \subset U$ whenever $A \subset U$ and $U \in SO(X)$,
(4) α $\hat{g}$ **-**closed [1] if $\alpha Cl(A) \subset U$ whenever $A \subset U$ and $U \in \tau_\omega$,
(5) Λ $\hat{g}$ **-**closed [20] if $Cl(A) \subset U$ whenever $A \subset U$ and $U \in \lambda O(X)$.

Definition: 2.8 Let (X,τ) be a topological space. A subset A of X is said to be
(1) a generalized closed (briefly, g-closed) [10] if $Cl(A) \subset U$ whenever $A \subset U$ and U is open,
(2) a semi-generalized closed (briefly, sg-closed) [3] if $sCl(A) \subset U$ whenever $A \subset U$ and U is semi-open,
(3) a generalized semi-closed (briefly, gs-closed) [2] if $sCl(A) \subset U$ whenever $A \subset U$ and U is open,
(4) an α -generalized closed (briefly, α g-closed) [14] if $\alpha Cl(A) \subset U$ whenever $A \subset U$ and U is open,
(5) a generalized α -closed (briefly, g α -closed) [13] if $\alpha Cl(A) \subset U$ whenever $A \subset U$ and U is α -open.

3. MINIMAL STRUCTURES:

Definition: 3.1 [12] Let X be a nonempty set and $\wp(X)$ the power set of X. A subfamily m_x of $\wp(X)$ is called a minimal structure (briefly, m-structure) on X if $\varphi \in m_x$ and $X \in m_x$.

A set X with an m-structure m_x (resp. two m-structures m_x and n_x) is called an m-space (resp. bi m-space) and is denoted by (X, m_x) (resp. (X, m_x, n_x)). Each member of m_x is said to be m_x-open and the complement of an m_x-open set is said to be m_x-closed.

Definition: 3.2 [21] Let X be a nonempty set and m_x an m-structure on X. For a subset A of X, the m_x-closure of A and the m_x-interior of A are defined as follows:
(1) $m_x\text{-}Cl(A) = \cap\{ F : A \subset F , X - F \in m_x \}$,
(2) $m_x\text{-}Int(A) = \cup\{ U : U \subset A , U \in m_x \}$.

Remark: 3.3 [18] Let (X,τ) be a topological space and A a subset of X. If $m_x = \tau$ (resp. SO(X), α(X)), then we have
(1) m_x-Cl(A) = Cl(A) (resp. sCl(A), αCl(A)),
(2) m_x-Int(A) =Int(A) (resp. sInt(A), αInt(A)).

Lemma: 3.4 [12] Let X be a nonempty set and m_x a minimal structure on X. For subsets A and B of X, the following properties hold :
(1) m_x-Cl(X − A) = X − m_x-Int(A) and m_x-Int(X − A) = X − m_x-Cl(A),
(2) If (X − A) $\in m_x$, then m_x-Cl(A) = A and if A $\in m_x$, then m_x-Int(A) = A,
(3) m_x-Cl(φ) = φ, m_x-Cl(X) = X, m_x-Int(φ) = φ and m_x-Int(X) = X,
(4) If $A \subset B$, then m_x-Cl(A) $\subset$ m_x-Cl(B) and m_x-Int(A) $\subset$ m_x-Int(B),
(5) A $\subset$ m_x-Cl(A) and m_x-Int(A) $\subset$ A,
(6) m_x-Cl(m_x-Cl(A)) = m_x-Cl(A) and m_x-Int(m_x-Int(A)) = m_x-Int(A).

Lemma: 3.5 [21] Let X be a nonempty set with a minimal structure m_x and A a subset of X. Then x$\in$ m_x-Cl(A) if and only if $U \cap A \neq \varphi$ for every $U \in m_x$ containing x.

Definition: 3.6 [18] A minimal structure m_x on a nonempty set X is said to have property B if the union of any family of subsets belonging to m_x belongs to m_x.

Notice that SO(X) and α(X) have property B.

Lemma: 3.7 [18] Let X be a nonempty set and m_x a minimal structure on X satisfying property B. For a subset A of X, the following properties hold:
(1) A $\in m_x$ if and only if m_x-Int(A) = A,
(2) A is m_x-closed if and only if m_x-Cl(A) = A,
(3) m_x-Int(A) $\in m_x$ and m_x-Cl(A) is m_x-closed.

Definition: 3.8 [15] Let (X, m_x) be an m-space. A subset A of X is said to be m_x-semi-open if A $\subset$ m_x-Cl(m_x-Int(A)). The family of all m_x-semi-open sets in X is denoted by m_x-SO(X).

The complement of m_x-semi-open set is said to be m_x-semi-closed.

Notice that m_x-SO(X) have property B.

Remark: 3.9 [15] Let (X, m_x) be an m-space. For a subset A of X, the m_x-semi-closure of A and the m_x-semi-interior of A, denoted by m_x-sCl(A) and m_x-sInt(A), respectively are defined as follows:

(1) m_x-sCl(A) = $\cap$ {F : A $\subset$ F, F is m_x-semi-closed in X},
(2) m_x-sInt(A) = $\cup$ {U : U $\subset$ A, U is m_x-semi-open in X}.

Lemma: 3.10 [15] Let X be a nonempty set and m_x a minimal structure on X. For subsets A and B of X, the following properties hold :
(1) m_x-sCl(X − A) = X − m_x-sInt(A) and m_x-sInt(X − A) = X − m_x-sCl(A),
(2) (X − A) $\in$ m_x-SO(X) if and only if m_x-sCl(A) = A and A $\in$ m_x-SO(X) if and only if m_x-sInt(A) = A,
(3) m_x-sCl(φ) = φ, m_x-sCl(X) = X, m_x-sInt(φ) = φ and m_x-sInt(X) = X,
(4) If A $\subset$ B, then m_x-sCl(A) $\subset$ m_x-sCl(B) and m_x-sInt(A) $\subset$ m_x-sInt(B),
(5) A $\subset$ m_x-sCl(A) and m_x-sInt(A) $\subset$ A,
(6) m_x-sCl(m_x-sCl(A)) = m_x-sCl(A) and m_x-sInt(m_x-sInt(A)) = m_x-sInt(A),
(7) m_x-sCl(A) is m_x-semi-closed.

Theorem: 3.11 [22] Let X be a non-empty set and m_x a minimal structure on X. Then every mx-open set is mx-semi-open but not conversely.

Example: 3.12 [22] Let X ={a, b, c}and m_x ={φ, X, {a}, { b}}. Then {a, c} is m_x-semi-open set but it is not m_x-open.

Lemma: 3.13 [22] Let X be a nonempty set with a minimal structure m_x and A a subset of X. Then x$\in$ m_x-sCl(A) if and only if $U \cap A \neq \varphi$ for every U $\in$ m_x-SO(X) containing x.

4. mn $\hat{g}$ -CLOSED SETS:

Definition: 4.1 Let (X, m_x, n_x) be a bi m-space. A subset A of X is said to be mn $\hat{g}$ -closed if n_x-Cl(A) $\subset$ U whenever A $\subset$ U and U is m_x-semi-open. The complement of an mn $\hat{g}$ -closed set is said to be mn $\hat{g}$ -open.

Remark: 4.2 Let (X, τ) be a topological space.
(1) If $n_x = \tau$ and m_x-SO(X) = τ (resp. SO(X), λO(X)), then, an mn $\hat{g}$ -closed set is a g-closed (resp. $\hat{g}$ -closed, $\Lambda\hat{g}$ -closed) set.
(2) If $n_x = \alpha(X)$ (resp. τ_δ) and m_x-SO(X) = τ_ω), then an mn $\hat{g}$ -closed set is a $\alpha\hat{g}$ -closed (resp. $\delta\hat{g}$ -closed) set.
(3) If $n_x = \tau_\theta$ (resp. SO(X)) and m_x-SO(X) = SO(X), then an mn $\hat{g}$ -closed set is $\theta\omega$-closed (resp. sg-closed) set.
(4) If n_x = SO(X) (resp. α(X)) and m_x-SO(X) = τ, then an mn $\hat{g}$ -closed set is a gs-closed (resp. αg-closed) set.
(5) If $n_x = \alpha(X)$ = m_x-SO(X), then an mn $\hat{g}$ -closed set is gα-closed set.

In this section, let (X, m_x, n_x) be a bi m-space. We obtain some characterizations of mn $\hat{g}$ -closed sets and mn $\hat{g}$ -open sets.

Proposition: 4.3 For subsets A, B of X, the following properties hold:
(1) if A is n_x-closed, then A is mn $\hat{g}$ -closed.
(2) if n_x has property B and A is mn $\hat{g}$ -closed and m_x-semi-open set, then A is n_x-closed.
(3) if A is mn $\hat{g}$ -closed and A $\subset$ B $\subset$ n_x-Cl(A), then B is mn $\hat{g}$ -closed.

Proof: (1) Let A be n_x-closed and A $\subset$ U $\in$ m_x-SO(X). Then by Lemma 3.4 A = n_x-Cl(A) $\subset$ U and hence, A is mn $\hat{g}$ -closed.
(2) Let A be mn $\hat{g}$ -closed and m_x-semi-open set. Then n_x-Cl(A) = A. Since n_x has property B, by Lemma 3.7. A is n_x-closed.
(3) Let B $\subset$ U and U $\in$ m_x-SO(X). Then A $\subset$ U and A is mn $\hat{g}$ -closed. Hence, n_x-Cl(A) $\subset$ U. By Lemma 3.4, n_x-Cl(A) $\subset$ n_x-Cl(B) $\subset$ n_x-Cl(n_x-Cl(A)) and hence n_x-Cl(A) = n_x-Cl(B), and hence, n_x-Cl(B) $\subset$ U. Therefore B is mn $\hat{g}$ -closed.

Proposition: 4.4 For each x$\in$ X, either {x} is m_x-semi-closed (or) mn $\hat{g}$ -open.

Proof: Suppose that {x} is not m_x-semi-closed. Then X − {x} is not m_x-semi-open and the only m_x-semi-open set containing X − {x} is the space X itself. Therefore, n_x-Cl(X − {x}) $\subset$ X and hence, X − {x} is mn $\hat{g}$ -closed. Thus, {x} is mn $\hat{g}$ -open.

Proposition: 4.5 A subset A of X is mn $\hat{g}$ -open if and only if F$\subset$ n_x-Int(A) whenever F $\subset$ A and F is m_x-semi-closed.

Proof Necessity: Suppose that A is mn $\hat{g}$ -open. Let F $\subset$ A and F be m_x-semi-closed. Then X − A $\subset$ X − F $\in$ m_x-SO(X) and X − A is mn $\hat{g}$ -closed . Therefore, we obtain X − n_x-Int(A) = n_x-Cl(X − A) $\subset$ X − F and hence F $\subset$ n_x-Int(A).

Sufficiency: Let X − A $\subset$ G and G $\in$ m_x-SO(X). Then X − G $\subset$ A and X − G is m_x-semi-closed. By the hypothesis, we have X − G $\subset$ n_x-Int(A) and hence, n_x-Cl(X − A) = X − n_x-Int(A) $\subset$ G. Therefore, X − A is mn $\hat{g}$ -closed and A is mn $\hat{g}$ -open .

Corollary: 4.6 For subsets A , B of X, the following properties hold:

(1) if A is n_x-open , then A is mn $\hat{g}$ -open .
(2) if n_x has property B and A is mn $\hat{g}$ -open and m_x-semi-closed, then A is n_x-open .
(3) if A is mn $\hat{g}$ -open and n_x-Int(A) $\subset$ B $\subset$ A , then B is mn $\hat{g}$ -open.

Proof: This follows from Propositions 4.3 and 4.5

Definition: 4.7 Minimal structures m_x and n_x on a nonempty set X are said to have property $\mathcal{C}$ [18] if $M \in m_x$ and $N \in n_x$ imply $M \cup N \in m_x$.

Theorem: 4.8 Let n_x have property B, m_x-SO(X) and n_x-SO(X) have property $\mathcal{C}$. Then for a subset A of X, the following properties are equivalent. However, (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (3) hold even if n_x, m_x-SO(X) and n_x-SO(X) do not have these properties.

(1) A is mn $\hat{g}$ -closed.

(2) n_x-Cl(A) – A does not contain any nonempty m_x-semi-closed set.

(3) n_x-Cl(A) – A is mn $\hat{g}$ -open.

Proof: (1) $\Rightarrow$ (2): Suppose that A is mn $\hat{g}$ -closed. Let $F \subset n_x$-Cl(A) –A and F be m_x-semi-closed. Then $A \subset X - F \in m_x$-SO(X) and hence, n_x-Cl(A) $\subset$ X – F. Therefore, we have $F \subset X - n_x$-Cl(A) and hence, $F \subset n_x$-Cl(A) $\cap$ (X – n_x-Cl(A)) = φ

(2) $\Rightarrow$ (3): Let $F \subset n_x$-Cl(A) – A and F be m_x-semi-closed. By (2), we have F = φ and $F \subset n_x$-Int(n_x-Cl(A) – A). It follows from Proposition 4.5 that n_x-Cl(A) – A is mn $\hat{g}$ -open.

(3) $\Rightarrow$ (1): Suppose that $A \subset U$ and $U \in m_x$ -SO(X). Then, n_x-Cl(A) – U $\subset$ n_x-Cl(A) – A. By (3), n_x-Cl(A) – A is mn $\hat{g}$ -open. Since n_x has property B, n_x-Cl(A) is n_x-closed by Lemma 3.7. Therefore n_x-Cl(A) is n_x-semi-closed by Theorem 3.11. Since m_x -SO(X) and n_x-SO(X) have property $\mathcal{C}$, n_x-Cl(A) – U is m_x-semi-closed. By Proposition 4.5. we have n_x-Cl(A) – U $\subset$ n_x-Int(n_x-Cl(A) – A) = φ. Therefore, we have n_x-Cl(A) $\subset$ U and hence, A is mn $\hat{g}$ -closed. Now, the proof of n_x-Int(n_x-Cl(A) – A) = φ is given as follows. Suppose that n_x-Int(n_x-Cl(A) – A) ≠ φ. There exists $x \in n_x$-Int(n_x-Cl(A) – A). Then, there exists $G \in n_x$ such that $x \in G \subset n_x$-Cl(A) – A. Since $G \subset X - A$, we have G $\cap$ A = φ and $G \in n_x$. By Lemma 3.5, G $\cap$ n_x-Cl(A) = φ and $G \subset X - n_x$-Cl(A). Therefore, we obtain $G \subset n_x$-Cl(A) $\cap$ (X – n_x-Cl(A)) = φ. This is a contradiction.

Proposition: 4.9 A subset A of X is mn $\hat{g}$ -closed if and only if n_x-Cl(A) $\cap$ F = φ whenever A $\cap$ F = φ and F is m_x-semi-closed.

Proof Necessity: Suppose that A is mn $\hat{g}$ -closed. Let A $\cap$ F = φ and F be m_x-semi-closed. Then $A \subset X - F \in m_x$-SO(X) and n_x-Cl(A) $\subset$ X – F. Therefore we have n_x-Cl(A) $\cap$ F= φ.

Sufficiency Let $A \subset U$ and $U \in m_x$-SO(X). Then A $\cap$ (X – U) = φ and X – U is m_x-semi-closed . By the hypothesis, n_x-Cl(A) $\cap$ (X – U) = φ and hence, n_x-Cl(A) $\subset$ U. Therefore, A is mn $\hat{g}$ -closed .

Theorem: 4.10 A subset A of X is mn $\hat{g}$ -closed if and only if m_x-sCl({x}) $\cap$ A ≠ φ for each $x \in n_x$-Cl(A).

PROOF Necessity: Suppose that A is mn $\hat{g}$ -closed. Let m_x-sCl({x}) $\cap$ A = φ for some $x \in n_x$-Cl(A). Then m_x-sCl({x}) is m_x-semi-closed and hence, $A \subset X - m_x$-sCl({x}) $\in m_x$-SO(X). Since A is mn $\hat{g}$ -closed, n_x-Cl(A) $\subset X - m_x$-sCl({x}) $\subset$ X – {x}. This contradicts that $x \in n_x$-Cl(A).

Sufficiency Suppose that A is not mn $\hat{g}$ -closed. Then φ ≠ nx-Cl(A) – U for some $U \in m_x$-SO(X) containing A. There exists $x \in n_x$-Cl(A) – U. Since $x \notin U$, by Lemma 3.13 m_x-sCl({x}) $\cap$ U = φ and hence, m_x-sCl({x}) $\cap$ A $\subset$ m_x-sCl({x}) $\cap$ U = φ. This shows that m_x-sCl({x}) $\cap$ A = φ for some $x \in n_x$-Cl(A).

We conclude this section with a characterization of mn $\hat{g}$ -open sets.

Theorem: 4.11 Let n_x have property B and m_x-SO(X) and n_x-SO(X) have property $\mathcal{C}$. A subset A of X is mn $\hat{g}$ -open if and only if G = X whenever G is m_x-semi-open and n_x-Int(A) $\cup$ (X – A) $\subset$ G.

Proof Necessity: Suppose that A is mn $\hat{g}$ -open and let G be m_x-semi-open and n_x-Int(A) $\cup$ (X –A) $\subset$ G. Then X – G $\subset$ n_x-Cl(X – A) – (X – A). Since X – A is mn $\hat{g}$ -closed and X – G is m_x-semi-closed, by Theorem 4.8.

X – G = φ and hence, G = X.

Sufficiency: Suppose that $F \subset A$ and F is m_x-semi-closed. Since n_x have property B. By Lemma 3.7. n_x-Int(A) is n_x-open. By Theorem 3.11, n_x-Int(A) is n_x-semi-open. Moreover, since m_x-SO(X) and n_x-SO(X) have property $\mathcal{C}$, we have $n_x\text{-Int}(A) \cup (X - A) \subset n_x\text{-Int}(A) \cup (X - F) \in m_x\text{-SO}(X)$. By the hypothesis, X = nx-Int(A) ∪ (X − F). Hence $F = F \cap (n_x\text{-Int}(A) \cup (X - F)) = F \cap n_x\text{-Int}(A) \subset n_x\text{-Int}(A)$. It follows from Proposition 4.5. that A is mn $\hat{g}$ -open.

Definition: 4.12 Let (X, m_x) be an m-space and A a subset of X. The set $\wedge_m \hat{g}$ (A) is defined as follows: $\wedge_m \hat{g}(A) = \cap \{ U : A \subset U \in m_x\text{-SO}(X)\}$.

Proposition: 4.13 A subset A of X is mn $\hat{g}$ -closed if and only if $n_x\text{-Cl}(A) \subset \wedge_m \hat{g}(A)$.

Proof Necessity: Suppose that A is mn $\hat{g}$ -closed. Let U be any m_x-semi-open set containing A. Then $n_x\text{-Cl}(A) \subset U$ and hence , we obtain $n_x\text{-Cl}(A) \subset \wedge_m \hat{g}(A)$.

Sufficiency: Suppose $n_x\text{-Cl}(A) \subset \wedge_m \hat{g}(A)$. Let $A \subset U$ and U is m_x-semi-open. Then $n_x\text{-Cl}(A) \subset \wedge_m \hat{g}(A) \subset U$. This shows that A is mn $\hat{g}$ -closed.

Definition: 4.14 A bi m-space (X, m_x, n_x) is said to be mn-semi-$T_{1/2}$ if every singleton of X is n_x-semi-open or m_x-semi-closed.

Theorem: 4.15 A bi m-space (X, m_x, n_x) is mn-semi-$T_{1/2}$ if and only if every mn $\hat{g}$ -closed of X is n_x-semi-closed.

Proof Necessity: Let (X, m_x, n_x) be mn-semi-$T_{1/2}$. Suppose that there exists an mn $\hat{g}$ -closed set A of X which is not n_x-semi-closed. Then there exists $x \in n_x\text{-sCl}(A) - A$. In case {x} is n_x-semi-open, since $x \in n_x\text{-sCl}(A)$, by Lemma 3.13, $x \in A$. This is a contradiction. In case {x} is m_x-semi-closed, we have $x \in X - A$ and
$A \subset X - \{x\} \in m_x\text{-SO}(X)$. Since A is mn $\hat{g}$ -closed, we have $n_x\text{-sCl}(A) \subset n_x\text{-Cl}(A) \subset X - \{x\}$. This is also a contradiction. Therefore, every mn $\hat{g}$ -closed set is n_x-semi-closed.

Sufficiency: Suppose that {x} is not m_x-semi-closed. Then, by Proposition 4.4, {x} is mn $\hat{g}$ -open and hence, {x} is n_x-semi-open. Therefore, (X, m_x, n_x) is mn-semi-$T_{1/2}$.

5. LOCALLY (m, n)-CLOSED SETS:

In this section, we introduce the notion of locally (m,n)-closed sets in a bi m-space (X, m_x, n_x) and show that a subset is n_x-closed if and only if it is locally (m, n)-closed and mn $\hat{g}$ -closed.

Definition: 5.1 A subset A of a bi m-space (X, m_x, n_x) is said to be locally (m, n)-closed if $A = U \cap F$, where U is m_x-open and F is n_x-closed .

Remark: 5.2 Let (X, τ) be a topological space. If $m_x = n_x = \tau$ (resp. SO(X) , $\alpha(X)$, τ_ω), then, a locally (m, n)-closed set is said to be locally closed [7] (resp. semi-locally closed [4], α-locally closed [8], $\hat{g}$ -locally closed [27]).

Theorem: 5.3 Let (X, m_x, n_x) be a bi m-space and n_x have property B. Let m_x have property B and m_x and n_x have property $\mathcal{C}$. For a subset A of X, the following properties are equivalent.:
(1) A is locally (m, n)-closed
(2) $A = U \cap n_x\text{-Cl}(A)$ for some $U \in m_x$
(3) $n_x\text{-Cl}(A) - A$ is m_x- closed.
(4) $A \cup (X - n_x\text{-Cl}(A)) \in m_x$
(5) $A \subset m_x\text{-Int}(A \cup (X - n_x\text{-Cl}(A)))$.

Proof: (1) $\Rightarrow$ (2). If A is locally (m, n)-closed, then there exist an m_x-open set U and a n_x-closed set F in (X, m_x, n_x) such that $A = U \cap F$. Since $A \subset F$, we have $n_x\text{-Cl}(A) \subset n_x\text{-Cl}(F) = F$. Since $A \subset U$, $A \subset U \cap n_x\text{-Cl}(A) \subset U \cap F = A$. Therefore, we obtain $A = U \cap n_x\text{-Cl}(A)$ for some $U \in m_x$.

(2) $\Rightarrow$ (3). Suppose that A = U ∩ n_x-Cl(A) for some U ∈ m_x. Then, n_x-Cl(A) − A = n_x-Cl(A) ∩ [X − (U ∩ n_x-Cl(A))] = n_x-Cl(A) ∩ (X − U). Since m_x and n_x have property $\mathcal{C}$, n_x-Cl(A) ∩ (X − U) is m_x-closed and hence, n_x-Cl(A) − A is m_x-closed.

(3) $\Rightarrow$ (4). we have X − (n_x-Cl(A) − A) = (X− n_x-Cl(A)) ∪ A and hence, by (3) we obtain A ∪ (X − n_x-Cl(A)) ∈ m_x.

(4) $\Rightarrow$ (5). By (4), A ⊂ A ∪ (X − n_x-Cl(A)) = m_x-Int(A ∪ (X − n_x-Cl(A))).

(5) $\Rightarrow$ (1). We put U = m_x-Int[A ∪ (X − n_x-Cl(A))]. Then, U∈ m_x and A = A ∩ U ⊂ U ∩ n_x-Cl(A) ⊂ [A∪ (X− n_x-Cl(A))] ∩ n_x-Cl(A) = A ∩ n_x-Cl(A) = A. Therefore, we obtain A = U ∩ n_x-Cl(A), where U ∈ m_x and n_x-Cl(A) is n_x-closed.

Theorem: 5.4 Let (X, m_x, n_x) be a bi m-space and m_x have property B. Let n_x have property B and n_x and m_x have property $\mathcal{C}$. Then a subset A of X is n_x-closed if and only if A is mn $\hat{g}$ -closed and locally (m, n)-closed.

Proof Necessity: Let A be n_x-closed. By Proposition 4.3 (1), A is mn $\hat{g}$ -closed. Since X ∈ m_x and A = X ∩ A, A is locally (m, n)-closed.

Sufficiency: Suppose that A is mn $\hat{g}$ -closed and locally (m, n)-closed. Since A is locally (m, n)-closed, by Theorem 5.3, we have A ⊂ m_x-Int(A ∪ (X − n_x-Cl(A))). By Lemma 3.7, m_x-Int(A ∪ (X − n_x-Cl(A))) ∈ m_x ⊂ m_x-SO(X) and A is mn $\hat{g}$ -closed. Therefore, we obtain n_x-Cl(A) ⊂ m_x-Int(A ∪ (X − n_x-Cl(A))) ⊂ (A ∪ (X − n_x-Cl(A))) and hence n_x-Cl(A) ⊂ A. Thus, n_x-Cl(A) = A and by Lemma 3.7, A is n_x-closed.

Corollary: 5.5 Let (X, τ) be a topological space and A a subset of X. Then,
(1) A is closed if and only if it is g-closed and locally closed,
(2) A is semi-closed if and only if it is sg-closed and semi locally closed,
(3) A is α-closed if and only if it is gα-closed and α-locally closed,

Proof: This is an immediate consequence of Theorem 5.4.

6. *(m, n)-NORMAL SPACES:

In this section, as an application of mn $\hat{g}$ -closed sets, we introduce the notion of *(m,n)-normality in a bi m-space (X, m_x, n_x). This notion enables us to unify several modifications of normal spaces.

Definition: 6.1 A bi m-space (X, m_x, n_x) is said to be *(m,n)-normal if for disjoint m_x-semi-closed sets F_1 and F_2, there exist U_1 , U_2 ∈ n_x such that F_1⊂ U_1 , F_2 ⊂ U_2 and U_1 ∩ U_2 = φ.

Remark: 6.2 Let (X, τ) be a topological space.
(1) If m_x-SO(X) = τ and n_x = τ then *(m,n)-normality coincides with normality [11].
(2) If m_x-SO(X) = SO(X) = n_x then *(m, n)-normality coincides with semi-normality [19].
(3) If m_x-SO(X) = m_x then *(m,n)-normality coincides with (m, n)-normality [18].

Theorem: 6.3 Let (X, m_x, n_x) be a bi m-space and n_x have property B. Then, for the bi m-space (X, m_x, n_x), the following properties are equivalent:
(1) (X, m_x, n_x) is *(m, n)-normal.
(2) for any disjoint m_x-semi-closed sets F_1, F_2, there exist mn $\hat{g}$ -open sets V_1, V_2 such that F_1 ⊂ V_1 , F_2 ⊂ V_2 and V_1∩V_2 = φ,
(3) for any m_x-semi-closed set F and any m_x-semi-open set U containing F, there exists an mn $\hat{g}$ -open set V such that F ⊂ V ⊂ n_x-Cl(V) ⊂ U,
(4) for any m_x-semi-closed set F and any m_x-semi-open set U containing F, there exists an n_x-open set G such that F ⊂ G ⊂ n_x-Cl(G) ⊂ U,
(5) for any disjoint m_x-semi-closed sets F_1, F_2, there exists an mn $\hat{g}$ -open set V such that F_1 ⊂ V and n_x-Cl(V) ∩ F_2 = φ,
(6) for any disjoint m_x-semi-closed sets F_1, F_2, there exists an n_X-open set G such that F_1 ⊂ G and n_x-Cl(G) ∩ F_2 = φ.

Proof: (1) $\Rightarrow$ (2). By Corollary 4.6, every n_x-open set is mn $\hat{g}$ -open.

(2) $\Rightarrow$ (3). Let F be an m_x-semi-closed set and U any m_x-semi-open set containing F. Then F, X – U are disjoint m_x-semi-closed sets and by (2) there exist mn $\hat{g}$ -open sets V_1, V_2 such that $F \subset V_1$, $X - U \subset V_2$ and $V_1 \cap V_2 = \varphi$. Since V_2 is mn $\hat{g}$ -open , by Proposition 4.5, $X - U \subset n_x\text{-Int}(V_2)$ and $n_x\text{-Int}(V_2)$ is n_x-open by Lemma 3.7. Hence, by Lemma 3.5, $n_x\text{-Cl}(V_1) \cap n_x\text{-Int}(V_2) = \varphi$.
Therefore, we obtain $F \subset V_1 \subset n_x\text{-Cl}(V_1) \subset X - n_x\text{-Int}(V_2) \subset U$. Put $V = V_1$, then we obtain $F \subset V \subset n_x\text{-Cl}(V) \subset U$.

(3) $\Rightarrow$ (4). Let F be a m_x-semi-closed set and U any m_x-semi-open set containing F. Then by (3) there exists an mn $\hat{g}$ -open set V such that $F \subset V \subset n_x\text{-Cl}(V) \subset U$. By Proposition 4.5, $F \subset n_x\text{-Int}(V)$. Put $G = n_x\text{-Int}(V)$. Since n_x has property B, G is n_x-open. Furthermore, we obtain $F \subset G \subset n_x\text{-Cl}(G) \subset n_x\text{-Cl}(V) \subset U$.

(4) $\Rightarrow$ (5). Let F_1, F_2 be any disjoint m_x-semi-closed sets. Since $X - F_2$ is a m_x-semi-open set containing F_1, by (4) there exists an n_x-open V such that $F_1 \subset V \subset n_x\text{-Cl}(V) \subset X - F_2$. By Corollary 4.6 V is mn $\hat{g}$ -open. Furthermore, we have $F_1 \subset V$ and $n_x\text{-Cl}(V) \cap F_2 = \varphi$.

(5) $\Rightarrow$ (6). Let F_1, F_2 be any disjoint m_x-semi-closed sets. Then there exists an mn $\hat{g}$ -open set V such that $F_1 \subset V$ and $n_x\text{-Cl}(V) \cap F_2 = \varphi$. By Proposition 4.5, $F_1 \subset n_x\text{-Int}(V)$. Set $G = n_x\text{-Int}(V)$. Since n_x has property B . $G \in n_x$, $F_1 \subset G$ and $n_x\text{-Cl}(G) \cap F_2 = \varphi$.

(6) $\Rightarrow$ (1). Let F_1, F_2 be any disjoint m_x-semi-closed sets. Then by (6) there exists $G \in n_x$ such that $F_1 \subset G$ and $n_x\text{-Cl}(G) \cap F_2 = \varphi$. Now, put $U_1 = G$ and $U_2 = X - n_x\text{-Cl}(G)$. Then U_1 and U_2 are disjoint n_x-open sets, $F_1 \subset U_1$ and $F_2 \subset U_2$. This show that (X, m_x, n_x) is *(m, n)-normal.

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