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SOME RESULTS OF FIXED POINT THEOREMS IN L–SPACE

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ABSTRACT

There are several Theorems are prove in L- Space, using various type of mappings .In this paper, we prove some fixed point theorem and common fixed point Theorems, in L- space using different, symmetric rational mappings.

Keywords: *Fixed point, Commmon Fixed point, L-Space, Continuous Mapping, Self Mapping, Weakly Compatible Mappings.*

2000 Mathematics Subject Classification: *47H10, 54H25.*

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1. INTRODUCTION:

It was shown by S. Kasahara [13] in 1976, that several known generalization of the Banach Contraction Theorem can be derived easily from a Fixed Point Theorem in an L-Space. Iseki [10] has used the fundamental idea of Kasahara to investigate the generalization of some known Fixed Point Theorem in L-Space.

Let N be the set of natural numbers and X be a nonempty set. Then L-Space is defined to be the pair $(X,\to)$ of the set X and a subset $\to$ of the set $X^N \times X$, satisfying the following conditions;

$\mathbf{L_1} -$ if $x_n = x \in X$ for all $n \in N$, then $(\{x_n\}_{n\in N}, x) \in \to$ $\mathbf{L_2} -$ if $(\{x_n\}_{n\in N}, x) \in \to$, then $\left(\{x_{n_i}\}_{i\in N}, x\right)$

for every subsequence $\{x_{n_i}\}_{i\in N}$ of $\{x_n\}_{n\in N}$ In what follows instead of writing $(\{x_n\}_{n\in N}, x) \in \to$, we write $\{x_n\}_{n\in N} \to x$ or $x_n \to x$ and read $\{x_n\}_{n\in N}$ converges to x. Further we give some definitions regarding L-Space.

Definition: 1 Let $(X,\to)$ be an L-Space. It is said to be 'separated' if each sequence in X converges to at most one point of X.

Definition: 2 A mapping f on $(X,\to)$ into an L-Space $(X',\to')$ is said to be continuous f: $x_n \to x$ implies $f(x_n) \to' f(x)$ for some subsequence $\{x_n\}_{i\in N}$ for $\{x_n\}_{n\in N}$

Definition: 3 Let d- be a non negative extended real valued function on $X \times X$: $0 \le d(x,y) \le \infty$, for all x, $y \in X$ The L-Space is said to be d- complete if each sequence $\{x_n\}_{n\in N}$ in X with $\sum_{i=0}^{\infty} d(x_i, x_{i+1}) < \infty$ converges to the atmost one point of X.

In this context Kasahara S. proved a lemma, which as follows,

Lemma (S. Kasahara): Let $(X,\to)$ be an L- space which is d- complete for a non negative real valued function d on $X \times X$. if $(X,\to)$ is separated then, $d(x,y) = d(y,x) = 0$ implies, $x = y$ for all $x, y \in X$ During the past few years many great mathematicians Yeh[19], Singh[18], Pathak, and Dubey[14], Sharma, and Agrawa[17], Patel,Sahu, and Sao[15], Patel and Patel[15], worked for L- Space. In this chapter, we similar investigation for the study of Fixed Point

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Theorems in L- Space are worked out. We find some more Fixed Point Theorem and Common Fixed Point Theorem in L- Sapce.

2. MAIN RESULT:

Theorem: 2.1 Let $(X, \rightarrow)$ be a separated L-space, which is d- complete for a non negative real valued function d on $X \times X$ with $d(x,x) = 0$, for each x in X. Let E, F and T be three continuous self mapping of X into itself, satisfying the following condition;

$1c_1{:}- E(X) \subset T(X)$ and $F(X) \subset T(X), ET = TE, FT = TF.$

$1c_2{:}- d(Ex, Fy) \leq \alpha \frac{d(Tx,Ty).[d(Tx,Fy)+d(Ty,Ex)]}{d(Tx,Ex)+d(Ty,Fy)}$

For all x, y in X, where non negative α, such that $0 \leq \alpha < 1$, with $Tx \neq Ty$. Then E, F, T have unique common fixed point.

Proof: Let $x_0 \in X$, since $E(X) \subset T(X)$ we can choose a point $x_1 \in X$, such that $Tx_1 = Ex_0$, also $F(X) \subset T(X)$, we can choose $x_2 \in X$ such that $x_2 = Fx_1$ In general we can choose the point;

$$Tx_{2n+1} = Ex_{2n} \tag{1.1}$$

$$Tx_{2n+2} = Fx_{2n+1} \tag{1.2}$$

Now consider,

$$d(Tx_{2n+1}, Tx_{2n+2}) = d(Ex_{2n}, Fx_{2n+1})$$

From $1c_2$

$$d(Ex_{2n}, Fx_{2n+1}) \leq \alpha \frac{d(Tx_{2n}, Tx_{2n+1}).[d(Tx_{2n}, F x_{2n+1}) + d(T x_{2n+1}, Ex_{2n})]}{d(Tx_{2n}, Ex_{2n}) + d(T x_{2n+1}, F x_{2n+1})}$$

$$d(Tx_{2n+1}, Tx_{2n+2}) \leq \alpha \frac{d(Tx_{2n}, Tx_{2n+1}).[d(Tx_{2n}, T x_{2n+2}) + d(T x_{2n+1}, Tx_{2n+1})]}{d(Tx_{2n}, Tx_{2n+1}) + d(T x_{2n+1}, T x_{2n+2})}$$

$$d(Tx_{2n+1}, Tx_{2n+2}) \leq \alpha\, d(Tx_{2n}, Tx_{2n+1})$$

For $n = 1, 2, 3, \ldots \ldots ....$

Whether, $d(Tx_{2n+1}, Tx_{2n+2}) = 0$ or not,

Similarly, we have

$d(Tx_{2n+1}, Tx_{2n+2}) \leq \alpha^n.\ d(Tx_0, Tx_1)$

For every positive integer n, this means that,

$$\sum_{i=0}^{\infty} d(Tx_{2i+1}, Tx_{2i+2}) < \infty$$

Thus the d- completeness of the space implies that, the sequence $\{T^n x_0\}_{n\in N}$ converges to some u in. so by (1.1) and (1.2);$\{E^n x_0\}_{n\in N}$ and $\{F^n x_0\}_{n\in N}$ also converges to the some point u, respectively.

Since E, F, T are continuous, there is a subsequence t of $\{T^n x_0\}_{n\in N}$ such that,

$$E(T(t)) \rightarrow E(u),\ T(E(t)) \rightarrow T(u),$$

$$F(T(t)) \rightarrow F(u),\ T(F(t)) \rightarrow T(u),$$

,

By $(11\ c_1)$ we have,

$$E(u) = F(u) = T(u) \tag{1.3}$$

Thus,

$$T(Tu) = T(Eu) = E(Tu) = E(Eu) = E(Fu) = T(Fu) = F(Tu) = F(Eu) = F(Fu) \tag{1.4}$$

By $1c_2$, (1.3) and (1.4), we have,

If $E(u) \neq F(Eu)$

$$d(Eu, F(Eu)) \leq \alpha \frac{d(Tu,T(Eu)).[d(Tu,F(Eu))+ d(T(Eu),Eu)]}{d(Tu,Eu)+d(T(Eu),F(Eu))}$$

$$d(Eu, F(Eu)) \leq 0$$

Which contradiction

Hence; $Eu = F(Eu)$ (1.5)

From (1.4) and (1.5) we have

$$Eu = F(Eu) = T(Eu) = E(Eu)$$

Hence Eu is a common fixed point of E, F, and T.

Uniqueness:

Let v is another fixed point of E, F, and T different from u, then by $1c_2$ we have,

$$d(u, v) = d(Eu, Fv)$$

$$d(Eu, Fv) \leq \alpha \frac{d(Tu, Tv). [d(Tu, Fv) + d(Tv, Eu)]}{d(Tu, Eu) + d(Tv, Fv)}$$

$$d(Eu, Fv) \leq 0$$

Which contradiction.

Therefore u is unique fixed point of E, F, and T in X.

Theorem : 2.2 Let $(X, \rightarrow)$ be a separated L-space, which is d- complete for a non negative real valued function d on $X \times X$ with $d(x, x) = 0$, for each x in X. Let A, F and T be three self mapping of X into itself, satisfying the following condition;

$2c_1: -\ A(X) \subseteq F(X) \cap T(X)$ and $AT = TA,\ FA = AF.$

$2c_2: -\ d(Fx, Ty) \leq \alpha \frac{d(Ax,Fx)[1+d(Ay,Ty)]}{1+d(Ax,Ay)} + \beta \frac{d(Ax,Ay)[d(Ax,Ty)+d(Ay,Fx)]}{d(Ax,Fx)+d(Ay,Ty)} + \gamma\, d(Ax, Ay)$

For all x, y in X, where non negative α, β, γ such that $\alpha + \beta + \gamma < 1$,

Then A, F, T have unique common fixed point.

Proof: for any arbitrary x_0 in X, we define a sequence $\{x_n\}$ of elements of X such that,

$$Ax_{2n+1} = Fx_{2n} \tag{2.1}$$

$$Ax_{2n+2} = Tx_{2n+1} \tag{2.2}$$

For all $n = 0,1,2,3 \ldots \ldots \ldots$

Now consider,

$$d(Ax_{2n+1}. Ax_{2n+2}) = d(Fx_{2n}. Tx_{2n+1})$$

From $2c_2$

$$d(Fx_{2n}, Tx_{2n+1}) \leq \alpha \frac{d(Ax_{2n}, Fx_{2n})[1+d(Ax_{2n+1}, Tx_{2n+1})]}{1+d(Ax_{2n}, Ax_{2n+1})} + \beta \frac{d(Ax_{2n}, Ax_{2n+1})[d(Ax_{2n}, Tx_{2n+1})+d(Ax_{2n+1}, Fx_{2n})]}{d(Ax_{2n}, Fx_{2n})+d(Ax_{2n+1}, Tx_{2n+1})} + \gamma\, d(Ax_{2n}, Ax_{2n+1})$$

$$d(Ax_{2n+1}. Ax_{2n+2}) \leq \alpha \frac{d(Ax_{2n}, Ax_{2n+1})[1+d(Ax_{2n+2}, Ax_{2n+1})]}{1+d(Ax_{2n+1}, Ax_{2n+2})} + \beta \frac{d(Ax_{2n}, Ax_{2n+1})[d(Ax_{2n}, Ax_{2n+2})+d(Ax_{2n+1}, Ax_{2n+1})]}{d(Ax_{2n}, Ax_{2n+1})+d(Ax_{2n+1}, Ax_{2n+2})} + \gamma\, d(Ax_{2n}, Ax_{2n+1})$$

$$d(Ax_{2n+1}. Ax_{2n+2}) \leq (\alpha + \beta + \gamma)\, d(Ax_{2n}, Ax_{2n+1})$$

Where $q = (\alpha + \beta + \gamma) < 1$;
Processing the same way we get,

$d(Ax_{2n+1}. Ax_{2n+2}) \leq q^n \, d(Ax_{2n}, Ax_{2n+1})$

For $n = 1, 2, 3, \ldots \ldots \ldots$

Whether, $d(Ax_{2n+1}, Ax_{2n+2}) = 0$ or not,

Similarly, we have

$$d(Ax_{2n+1}, Ax_{2n+2}) \leq q^n. \; d(Ax_0, Ax_1)$$

For every positive integer n, this means that,

$$\sum_{i=0}^{\infty} d(Ax_{2i+1}, Ax_{2i+2}) < \infty$$

Thus the d- completeness of the space implies that, the sequence$\{A^n x_0\}_{n\in N}$ converges to some u in. so by (2.1) and (2.2); $\{F^n x_0\}_{n\in N}$ and $\{T^n x_0\}_{n\in N}$ also converges to the some point u, respectively.

Since A, F, T are continuous, there is a subsequence t of $\{A^n x_0\}_{n\in N}$ such that,

$$F(A(t)) \to F(u), \qquad A(F(t)) \to A(u),$$

$T(A(t)) \to T(u)$, $A(T(t)) \to A(u)$,

By $(2c_1)$ we have,

$$A(u) = F(u) = T(u) \tag{2.3}$$

u is common fixed point of A, F and T.

Uniqueness:

Let us assume that w is another fixed point of A, F and T different from u $u \neq w$ then

$$d(Au, Aw) = d(Fu, Tw)$$

from $12c_2$

$(Fu, Tw) \leq \alpha \dfrac{d(Au,Fu)[1+d(Aw,Tw)]}{1+d(Au,Aw)} + \beta \dfrac{d(Au,Aw)[d(Au,Tw)+d(Aw,Fu)]}{d(Au,Fu)+d(Aw,Tw)} + \gamma \, d(Au, Aw)$

$d(Au, Aw) \leq \gamma \, d(Au, Aw)$

Which contradiction.

Hence u is unique common fixed point of A, F, T. in X.

Theorem : 3 Let $(X, \to)$ be a separated L-space, which is d- complete for a non negative real valued function d on $X \times X$ with $d(x, x) = 0$, for each x in X. Let E, F and T be three continuous self mapping of X into itself, satisfying the following condition;

$3c_1$: $- E(X) \subset T(X)$ and $F(X) \subset T(X)$, $ET = TE$, $FT = TF$.
$3c_2$: $- d(Ex, Fy) \leq \alpha . \{d(Tx, Ex). d(Ty, Fy). d(x, y)\}^{\frac{1}{3}}$

For all x, y in X, where non negative α, β, γ, such that

$0 \leq \alpha + \beta + \gamma < 1$, with $Tx \neq Ty$. Then E, F , T have unique common fixed point .

Proof: Let $x_0 \in X$, since $E(X) \subset T(X)$ we can choose a point $x_1 \in X$, such that $Tx_1 = Ex_0$, also $F(X) \subset T(X)$, we can choose $x_2 \in X$ such that $Tx_2 = Fx_1$ In general we can choose the point;

$$Tx_{2n+1} = Ex_{2n} \tag{3.1}$$

$$Tx_{2n+2} = Fx_{2n+1} \tag{3.2}$$

For every $n \in N$, we have

$$d(Tx_{2n+1}, Tx_{2n+2}) = d(Ex_{2n}, Fx_{2n+1})$$

From $3c_2$

$$d(Ex_{2n}, Fx_{2n+1}) \leq \alpha . \{d(Tx_{2n}, Ex_{2n}). d(Tx_{2n+1}, Fx_{2n+1}). d(x_{2n}, x_{2n+1})\}^{\frac{1}{3}}$$

$$d(Tx_{2n+1}, Tx_{2n+2}) \leq \alpha . \{d(Tx_{2n}, Tx_{2n+1}). d(Tx_{2n+1}, Tx_{2n+2}). d(x_{2n}, x_{2n+1})\}^{\frac{1}{3}}$$
$$d(Tx_{2n+1}, Tx_{2n+2}) \leq \alpha^{\frac{3}{2}} \ d(Tx_{2n}, Tx_{2n+1})$$

Let $q = \alpha^{\frac{3}{2}} < 1$

$$d(Tx_{2n+1}, Tx_{2n+2}) \leq q \ d(Tx_{2n}, Tx_{2n+1})$$

For $n = 1, 2, 3, \dots \dots ....$

Whether, $d(Tx_{2n+1}, Tx_{2n+2}) = 0$ or not,

Similarly, we have

$$d(Tx_{2n+1}, Tx_{2n+2}) \leq q^n. \ d(Tx_0, Tx_1)$$

For every positive integer n, this means that,

$$\sum_{i=0}^{\infty} d(Tx_{2i+1}, Tx_{2i+2}) < \infty$$

Thus the d- completeness of the space implies that, the sequence $\{T^n x_0\}_{n\in N}$ converges to some u in. so by (3.1) and (3.2);$\{E^n x_0\}_{n\in N}$ and $\{F^n x_0\}_{n\in N}$ also converges to the some point u, respectively.

Since E, F, T are continuous, there is a subsequence t of $\{T^n x_0\}_{n\in N}$ such that,

$$E\big(T(t)\big) \to E(u),\ T\big(E(t)\big) \to T(u),$$
$$F\big(T(t)\big) \to F(u),\ T\big(F(t)\big) \to T(u),$$

By $(3c_1)$ we have,

$$E(u) = F(u) = T(u) \tag{3.3}$$

Thus,

$$T(Tu) = T(Eu) = E(Tu) = E(Eu) = E(Fu) = T(Fu) = F(Tu) = F(Eu) = F(Fu) \tag{3.4}$$

By $3c_2$, (3.3) and (3.4) we have,

If $E(u) \neq F(Eu)$

$$d(Eu, F(Eu)) \leq \alpha . \{d(Tu, Eu). d(T(Eu), F(Eu)). d(u, (Eu))\}^{\frac{1}{3}}$$

$$d(Eu, F(Eu)) \leq 0$$

Which contradiction

Hence; $Eu = F(Eu)$ (3.5)

From (3.4) and (3.5), we have

$Eu = F(Eu) = T(Eu) = E(Eu)$

Hence Eu is a common fixed point of E, F, and T.

Uniqueness:

Let v is another fixed point of E, F, and T different from u, then by $3c_2$ we have,

$$d(u,v) = d(Eu, Fv)$$
$$d(Eu, Fv) \leq \alpha . \{d(Tu, Eu). d(Tv, Fv). d(u, v)\}^{\frac{1}{3}}$$
$$d(Eu, Fv) \leq 0$$

Which contradiction.

Therefore u is unique fixed point of E, F, and T in X.

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